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On local reflection of the properties of graphs with uncountable characteristics

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Abstract

We study the relationships between the properties of graphs: "of coloring number $> \mu$" and "of chromatic number $> \mu$" for a regular cardinal μ in terms of set-theoretic reflection of these properties.

We show that under certain conditions the non-reflection of the property "of coloring number $> \mu$" of graphs of bounded cardinality implies the non-reflection of the property "of chromatic number $> \mu$". The implication is proved by interpolating it by non-reflection of the properties which

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An updated and extended version of this paper with more details and proofs is going to be downloadable as:

http://fuchino.ddo.jp/papers/RIMS16-uncountable-reflection-x.pdf are related to generalized and/or modified forms of Fodor-type Reflection Principle, Strong Chang's Conjecture, Rado's Conjecture and Galvin's Conjecture.

As an application of this result we show a non reflection theorem on chromatic number $> \mu$ which partially covers the results in Shelah [11].

Further results in this line will be presented in Fuchino, Ottenbreit and Sakai [9].

1 Introduction

For regular cardinal μ and cardinals κ and λ with $\mu^+ < \kappa \leq \lambda$, let

$\mathsf{REFL}_{\mathsf{col}}(\mu, <\kappa, \lambda)$: For any graph G of cardinality λ, if G has coloring number $> \mu$ then there is a subgraph H of G of cardinality $< \kappa$ such that H also has coloring number $> \mu$.

Similarly, let

$\mathsf{REFL}_{\mathsf{chr}}(\mu, <\kappa, \lambda)$: For any graph G of cardinality λ, if G has chromatic number $> \mu$ then there is a subgraph H of G of cardinality $< \kappa$ such that H also has chromatic number $> \mu$.

In this note we shall give a proof of the following Theorem:

Theorem 1.1 *Suppose that μ, κ, λ are cardinals such that*

(1.1) $\quad \mu^{<\mu} = \mu,\ \mu^+ < \kappa,\ \kappa^{<\mu} = \kappa$ *and* $\lambda < \kappa^{+\omega}$.

If $\mathsf{REFL}_{\mathsf{col}}(\mu, <\kappa, \lambda^\mu)$ does not hold then $\mathsf{REFL}_{\mathsf{chr}}(\mu, <\kappa, \lambda^\mu)$ does not hold.

Note that the condition $\mu^{<\mu} = \mu$ implies that μ is a regular cardinal since, by Kőnig's Theorem, we have $\kappa^{cf(\kappa)} > \kappa$ for any cardinal κ.

Stress is put here on the cases where μ is uncountable. For $\mu = \omega$, $\mu^{<\mu} = \mu$ and $\kappa^{<\mu} = \kappa$ hold automatically and the condition $\lambda < \kappa^{+\omega}$ can also be dropped from the assertion of Theorem 1.1. For more see [8].

Further results in this line for uncountable μ will be presented in Fuchino, Ottenbreit and Sakai [9].

2 Preliminary

In this note (and in the further article [9] in preparation) the Roman letters G and V are not going to denote a generic set and the ground model as it is usual the case in set theory but rather they are used to denote a graph and its set of vertices. We shall use $\mathcal{V}$ and $\mathbb{G}$ instead for the ground model and a generic set respectively.

We consider a graph as a structure of the form $G = \langle V, \mathcal{E} \rangle$ where $\mathcal{E} \subseteq V^2$ and $\mathcal{E}$ is thought to be the symmetrical and non reflective binary relation representing the adjacency of the graph G. We often identify G with the underlying set V of G and even write $G = \langle G, \mathcal{E} \rangle$. For $X \subseteq G$ and $p \in G$ (i.e. for $X \subseteq V$ and $p \in V$), we denote

(2.1) $\quad \mathcal{E}^p_X = \{q \in X \,:\, q \,\mathcal{E}\, p\}.$

Recall that the *coloring number* of a graph $G = \langle G, \mathcal{E} \rangle$ is defined as the minimal cardinal μ with the property that

(2.2) there is a well-ordering $\lhd$ on G such that for any $p \in G$, denoting the initial segment below p with respect to the ordering $\lhd$ by $I^{\lhd}_p = \{q \in G \,:\, q \lhd p\}$, we have $|\,\mathcal{E}^p_{I^{\lhd}_p}\,| < \mu$.

The coloring number of a graph G is denoted by $\mathsf{col}(G)$.

For a graph $G = \langle V, \mathcal{E} \rangle$ and $X \subseteq V$, $G \restriction X$ denotes the induced subgraph $\langle X, \mathcal{E} \cap X^2 \rangle$ of G. For a set X a sequence $\mathcal{F} = \langle X_\alpha \,:\, \alpha < \delta \rangle$ of subsets of X is said to be a *filtration* of X if $\mathcal{F}$ is continuously increasing (with respect to $\subseteq$), $|\,X_\alpha\,| < |\,X\,|$ for all $\alpha < \delta$ and $\bigcup_{\alpha<\delta} X_\alpha = X$. Note that, for any set X, we have a strictly increasing filtration of X of length $\delta = cf(|\,X\,|)$.

For a graph $G = \langle V, \mathcal{E} \rangle$ a sequence $\langle G_\alpha \,:\, \alpha < \delta \rangle$ of induced subgraphs of G with $G_\alpha = G \restriction V_\alpha$ for $\alpha < \delta$ is said to be a *filtration* of G if $\langle V_\alpha \,:\, \alpha < \delta \rangle$ is a filtration of V.

The following Lemma is proved easily by (simultaneous) induction on the cardinality of G:

Lemma 2.1 (Erdős and Hajnal [3] see also [5]) (1) *If $\mu = \mathsf{col}(G)$ there is a well-ordering $\lhd$ on G of order type $|\,G\,|$ witnessing this.*
(2) *For any graph G, $\mathsf{col}(G) \leq \mu$ if and only if there is a filtration $\langle G_\alpha \,:\, \alpha < \delta \rangle$ of G such that*

(2.3) $\quad \mathsf{col}(G_\alpha) \leq \mu$ *and*

(2.4) $\quad |\,\mathcal{E}^p_{G_\alpha}\,| < \mu$ *for all $p \in G_{\alpha+1} \setminus G_\alpha$*

for all $\alpha < \delta$. □

Recall that, for a graph $G = \langle G, \mathcal{E} \rangle$, the *chromatic number* $\mathsf{chr}(G)$ of G is the minimal cardinal μ such that G can be partitioned into μ many pairwise non adjacent (i.e. independent) subgraphs.

Lemma 2.2 *Suppose that "inv" is one of "col" or "chr". If $\mathsf{REFL}_{\mathsf{inv}}(\mu, <\kappa, \lambda)$ holds, $\kappa' \geq \kappa$ and $\lambda' \leq \lambda$, then $\mathsf{REFL}_{\mathsf{inv}}(\mu, <\kappa', \lambda')$ holds.*

Proof. Suppose that $\mathsf{REFL}_{\mathsf{inv}}(\mu, <\kappa', \lambda')$ does not hold and let $G = \langle G, \mathcal{E}\rangle$ be a graph of cardinality λ' which is a witness of the failure of $\mathsf{REFL}_{\mathsf{inv}}(\mu, <\kappa', \lambda')$. Thus $\mathsf{inv}(G) > \mu$ but $\mathsf{inv}(G_0) \leq \mu$ for all $G_0 \in [G]^{<\kappa'}$.

Let G' be a set of cardinality λ disjoint from G and let $G_1 = G \cup G'$. Then the graph $G_1 = \langle G_1, \mathcal{E}\rangle$ is of cardinality λ. $\mathsf{inv}(G_1) > \mu$ but $\mathsf{inv}(G_0) \leq \mu$ for all $G_0 \in [G_1]^{<\kappa'}$. (and hence this holds for all $G_0 \in [G_1]^{<\kappa}$). Thus G_1 is a witness of the failure of $\mathsf{REFL}_{\mathsf{inv}}(\mu, <\kappa, \lambda)$.

$\square$ (Lemma 2.2)

3 Reflection properties related to generalized Fodor-type Reflection Principles

For regular cardinals μ, κ, λ with $\mu^+ < \kappa \leq \lambda$, let $\mathsf{FRP}(\mu, <\kappa, \lambda)$ be the following assertion:

$\mathsf{FRP}(\mu, <\kappa, \lambda)$: For any stationary $S \subseteq E^\lambda_\mu$ and $g : S \to [\lambda]^\mu$ there is an $\alpha^* \in E^\lambda_{>\mu} \cap E^\lambda_{<\kappa}$ such that α^* is closed with respect to g and $\{x \in [\alpha^*]^\mu : \sup(s) \in S,\ g(\sup(x)) \cap \sup(x) \subseteq x\}$ is stationary in $[\alpha^*]^\mu$.

Using this notation, the Fodor-type Reflection principle (FRP) introduced in [4] can be formulated as

FRP: $\mathsf{FRP}(\aleph_0, <\aleph_2, \lambda)$ holds for all regular uncountable λ.

FRP is known to be equivalent (over ZFC) to many known mathematical reflection statements like the one saying that a locally compact Hausdorff space X is metrizable if and only if all subspace of X of size $\leq \aleph_1$ are metrizable. FRP also implies many interesting consequences like SCH while it does not restrict the size of the continuum unlike many other reflection principles like Rado's Conjecture we are going to discuss below which imply that the continuum is less than or equal to $\aleph_2$ or even CH.

By the following result of Hiroshi Sakai, this principle cannot be consistently generalized by taking an uncountable μ in place of $\aleph_0$ in $\mathsf{FRP}(\aleph_0, <\aleph_2, \lambda)$.

A cardinal κ is said to be *λ-inaccessible* if $\mu^\lambda < \kappa$ holds for all $\mu < \kappa$. Similarly we shall also say that κ is *$<\lambda$-inaccessible* if $\mu^{<\lambda} < \kappa$ holds for all $\mu < \kappa$.

Theorem 3.1 (H. Sakai [9]) *Let λ be a singular cardinal, and let μ and λ be regular cardinals with $\mu^+ < \kappa \leq \lambda$ Suppose that every regular cardinal ν with $\mu < \nu < \kappa$ is $cf(\lambda)$-inaccessible. Then* $\mathsf{FRP}(\mu, <\kappa, \lambda^+)$ *fails.*

This delimitation set by the theorem above explains the releance of the conditions on cardinals in the following Proposition.

In spite of Theorem 3.1, we can modify the property $\mathsf{FRP}(\mu, <\kappa, \lambda)$ to obtain a reasonable generalization of FRP for higher cardinals. This will be discussed in [9].

Proposition 3.2 *For any cardinals μ, κ and λ^* such that*

(3.1) $\quad \mu^+ < \kappa \leq \lambda^* < \kappa^{+\omega}$,

if $\mathsf{FRP}(\mu, <\kappa, \lambda)$ *holds for all* $\lambda < \lambda^*$ *then* $\mathsf{REFL}_{\mathsf{col}}(\mu, <\kappa, \lambda)$ *holds for all* $\lambda < \lambda^*$.

Proof. By induction of λ^*. If $\lambda^* \leq \kappa$ then $\mathsf{REFL}_{\mathsf{col}}(\mu, <\kappa, \lambda)$ trivially holds for all $\lambda < \lambda^*$.

We assume that the Proposition holds for all $\kappa \leq \lambda_0^* < \lambda^*$ and show that the Proposition also holds for λ^*. By (3.1), there is $\lambda_0 < \lambda^*$ such that $(\lambda_0)^+ = \lambda^*$. Thus it is enough to show that $\mathsf{REFL}_{\mathsf{col}}(\mu, <\kappa, \lambda_0)$ holds.

Suppose that this is not the case. Then there is a graph G of cardinality λ_0 such that $\mathsf{col}(G) > \mu$ but all subgraphs H of G of cardinality $< \kappa$ have coloring number $\leq \mu$. Without loss of generality $G = \langle \lambda_0, \mathcal{E} \rangle$ for some adjacency relation $\mathcal{E}$.

Let $\langle \eta_\alpha : \alpha < \lambda_0 \rangle$ be a continuously and strictly increasing sequence of ordinals cofinal in λ_0 and $\xi_\alpha \in \eta_{\alpha+1} \setminus \eta_\alpha$ for $\alpha < \lambda_0$ are such that, for all $\alpha < \lambda_0$,

(3.2) $\quad$ If $|\,\mathcal{E}^{\xi}_{\eta_\alpha}\,| \geq \mu$ for some $\xi \in \lambda_0 \setminus \eta_\alpha$, then $|\,\mathcal{E}^{\xi_\alpha}_{\eta_\alpha}\,| \geq \mu$.

By induction hypothesis we have $\mathsf{col}(G \restriction \eta_\alpha) \leq \mu$ for all $\alpha < \lambda_0$. Thus

(3.3) $\quad S = \{\alpha < \lambda_0 \,:\, |\,\mathcal{E}^{\xi_\alpha}_{\eta_\alpha}\,| \geq \mu\}$ is stationary

(since otherwise we would obtain $\mathsf{col}(G) \leq \mu$ by Lemma 2.1. This is a contradiction to the assumption on G).

Claim 3.2.1 *$S_1 = S \cap E^{\lambda_0}_{\mu}$ is stationary.*

$\vdash$ Suppose that S_1 were non stationary. Then, at least one of $S_0 = S \cup E^{\lambda_0}_{<\mu}$ and $S_2 = S \cup E^{\lambda_0}_{>\mu}$ would be stationary. Suppose that $i \in \{0, 2\}$ is such that S_i is stationary. Then for each $\alpha \in S_i$ there is $\nu_\alpha < \eta_\alpha$ such that $|\,\mathcal{E}^{\xi_\alpha}_{\nu_\alpha} \geq \mu\,|$. By Fodor's Lemma, there is a stationary $S_4 \subseteq S_i$ and $\nu^* < \lambda_0$ such that $\nu_\alpha = \nu^*$ for all $\alpha \in S_4$. It follows that $E^{\lambda_0}_{\mu} \setminus \sup(\nu^*) \subseteq S$. This is a contradiction since the left side of the inclusion is stationary and it is thus a subset of $S_1 = S \cap E^{\lambda_0}_{\mu}$. $\dashv$ (Claim 3.2.1)

For each $\alpha \in S_1$, let $s_\alpha \in [\mathcal{E}^{\xi_\alpha}_{\eta_\alpha}]^{\mu}$ and let $g : S_1 \to [\lambda_0]^{\mu}$ be the defined by $g(\alpha) = s_\alpha \cup \{\xi_\alpha\}$ for $\alpha \in S_1$.

By $\mathsf{FRP}(\mu, <\kappa, \lambda_0)$, there is $\alpha^* \in E^{\lambda_0}_{>\mu} \cap E^{\lambda_0}_{<\kappa}$ such that α^* is closed with respect to g and

$$\{x \in [\alpha^*]^{\mu} \,:\, \sup(x) \in S_1 \text{ and } g(\sup(x)) \cap \sup(x) \subseteq x\}$$

is stationary. It follows that there is an $I \in [\alpha^*]^{cf(\alpha^*)}$ with a filtration $\langle I_\xi : \xi < cf(\alpha^*)\rangle$ such that each of I_ξ, $\xi < cf(\alpha^*)$ is closed with respect to g and

$$S = \{\xi < cf(\alpha^*) \,:\, \sup(I_\xi) \in S_1 \text{ and } g(\sup(I_\xi)) \cap \sup(I_\xi) \subseteq I_\xi\}$$

is stationary. But then, by Lemma 2.1, we must conclude $\mathsf{col}(G \restriction I) > \mu$. This is a contradiction to the choice of G. □ (Proposition 3.2)

4 Reflection principles related to a variant of Strong Chang's Conjecture

(4.1) Let θ be a regular cardinal large enough (compared with λ below). Let $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset\rangle$ where $\sqsubset$ is a fixed well-ordering of $\mathcal{H}(\theta)$.

The well-ordering $\sqsubset$ is included in the structure $\mathcal{M}$ here because of the built-in Skolem functions it introduces.

The following principle $\mathsf{CC}^{\downarrow}(\mu, <\kappa, \lambda)$ is a generalized version of a principle considered in [8]. These principles are inspired by a variant of Strong Chang's Conjecture in Doebler [2]; the Strong Chang's Conjecture in its original form was introduced in Todorčević [15].

For a regular cardinal μ and cardinals κ, λ with $\mu^{<\mu} = \mu$ and $\mu^+ < \kappa \leq \lambda$, let $\mathsf{CC}^{\downarrow}(\mu, <\kappa, \lambda)$ be the assertion defined as follows:

$\mathsf{CC}^{\downarrow}(\mu, <\kappa, \lambda)$: For any $M \in [\mathcal{M}]^\mu$ with $M \prec \mathcal{M}$, $\mu \subseteq M$ μ, κ, $\lambda \in M$ and $[M]^{<\mu} \subseteq M$, we have that

(4.2) for any $\alpha \in \lambda$, there is M^* with $M \prec M^* \prec \mathcal{M}$ and $\alpha^* \in \lambda \setminus \alpha$ such that $\mu < cf(\alpha^*) < \kappa$ and $\alpha^* = \min(\lambda \cap M^* \setminus \sup(\lambda \cap M))$.

Proposition 4.1 *Suppose that μ, κ, λ are cardinals such that*

(4.3) *μ and λ are regular;*

(4.4) *$\mu^{<\mu} = \mu$;*

(4.5) *$\mu^+ < \kappa \leq \lambda$ and*

(4.6) *λ is $<\mu$-inaccessible.*

Then $\mathsf{CC}^{\downarrow}(\mu, <\kappa, \lambda)$ *implies* $\mathsf{FRP}(\mu, <\kappa, \lambda)$.

Proof. Assume $\mathsf{CC}^{\downarrow}(\mu, <\kappa, \lambda)$ and suppose that $S \subseteq E^\lambda_\mu$ is stationary and $g : S \to [\lambda]^\mu$. Let $M \in [\mathcal{M}]^\mu$ be such that

(4.7) $M \prec \mathcal{M}$;

(4.8) μ, κ, λ, S, $g \in M$;

(4.9) $\mu \subseteq M$;

(4.10) $[M]^{<\mu} \subseteq M$;

(4.11) M is closed with respect to g;

(4.12) $\sup(\lambda \cap M) \in S$ and $g(\sup(M)) \cap \sup(M) \subseteq M$.

Note that there is such an M by (4.4) and (4.6). By $\mathsf{CC}^{\downarrow}(\mu, < \kappa, \lambda)$ there are $\alpha^* \in \lambda$ and $M^* \prec \mathcal{M}$ such that

(4.13) $M \prec M^*$;

(4.14) $\mu < cf(\alpha^*) < \kappa$; and

(4.15) $\alpha^* = \min(\lambda \cap M^* \setminus \sup(\lambda \cap M))$.

We show that this α^* witnesses $\mathsf{FRP}(\mu, < \kappa, \lambda)$ for our S and g.

α^* is closed with respect to g since it is closed with respect to g in M^* by (4.15). Thus it is enough to show that

(4.16) $Z = \{x \in [\alpha^*] \,:\, \sup(x) \in S \text{ and } g(\sup(x)) \cap \sup(x) \subseteq x\}$

is stationary. By elementarity, it is enough to show that Z intersects with all club sets of $[\alpha^*]^{\mu}$ in M^*.

Suppose that $C \in M^*$ is a club subset of $[\alpha^*]^{\mu}$ and let $h \in M^*$ be such that $h : {}^{\omega>}\alpha^* \to \alpha^*$ and

(4.17) $C \supseteq C_h = \{x \in [\alpha^*]^{\mu} \,:\, \mu \subseteq x \text{ and } x \text{ is closed with respect to } h\}$.

Then we have

(4.18) $\alpha^* \cap M \in Z \cap C_h$

[$\alpha^* \cap M \in Z$ by (4.12) and $\alpha^* \cap M \in C_h$ by (4.9)]. Thus $Z \cap C \neq \emptyset$. □ (Proposition 4.1)

For a regular cardinal λ, a mapping $f : \lambda \to \lambda$ is said to be *regressive* if $f(\alpha) < \alpha$ holds for all $\alpha < \lambda$. We denote with ${}^{\lambda\downarrow}\lambda$ the set $\{f \in {}^{\lambda}\lambda \,:\, f \text{ is regressive}\}$.

The game $G^{\downarrow}_{\mu}(< \kappa, \lambda)$ for Players I and II is defined as follows: A match in $G^{\downarrow}_{\mu}(< \kappa, \lambda)$ is a sequence of length μ of the form:

$$\mathfrak{M}: \quad \begin{array}{c|cccccccc} I & f_0 \in {}^{\lambda\downarrow}\lambda & & f_1 \in {}^{\lambda\downarrow}\lambda & & \cdots & f_\xi \in {}^{\lambda\downarrow}\lambda & & \cdots \\ \hline II & & \delta_0 \in \lambda & & \delta_1 \in \lambda & \cdots & & \delta_\xi \in \lambda & \cdots \end{array} \quad (\xi < \mu)$$

II wins in a match $\mathfrak{M}$ of $G^{\downarrow}_{\mu}(< \kappa, \lambda)$ as above if

(4.19) $B_{\mathfrak{M}} = \{\alpha \in E^{\lambda}_{>\mu} \cap E^{\lambda}_{<\kappa} \,:\, f_\xi(\alpha) < \sup\{\delta_i \,:\, i < \mu\} \text{ for all } \xi < \mu\}$

is unbounded in λ.

Let us denote with $\mathsf{WS}_I(G^{\downarrow}_{\mu}(< \kappa, \lambda))$ ($\mathsf{WS}_{II}(G^{\downarrow}_{\mu}(< \kappa, \lambda))$, resp.) the assertion "the player I (the player II, resp.) has a winning strategy in the game $G^{\downarrow}_{\mu}(< \kappa, \lambda)$.

Proposition 4.2 *Suppose that*

(4.20) $\mu^{<\mu} = \mu$

holds. Then, for any cardinals κ, λ *with* $\mu^+ < \kappa \leq \lambda$, $\mathsf{WS}_{II}(G^{\downarrow}_{\mu}(<\kappa,\lambda))$ *holds if and only if* $\mathsf{CC}^{\downarrow}(\mu, <\kappa, \lambda)$ *holds.*

Proof. Suppose first that $\mathsf{WS}_{II}(G^{\downarrow}_{\mu}(<\kappa,\lambda))$ holds. Let $\mathcal{M}$ be defined as in (4.1) and let $M \prec \mathcal{M}$ be such that

(4.21) $|\,M\,| = \mu$;

(4.22) $[M]^{<\mu} \subseteq M$;

(4.23) μ, κ, $\lambda \in M$ and $\mu \subseteq M$.

Note that there is such M by (4.20).

Let $\sigma \in M$ be a wining strategy of the player II in $G^{\downarrow}_{\mu}(<\kappa,\lambda)$. Let $\mathfrak{M} = \langle f_\xi, \delta_\xi \,:\, \xi < \mu\rangle$ be a match in $G^{\downarrow}_{\mu}(<\kappa,\lambda)$ such that

(4.24) $\langle f_\xi, \delta_\xi \,:\, \xi < \gamma\rangle \in M$ for all $\gamma < \mu$;

(4.25) II plays according to σ in $\mathfrak{M}$;

(4.26) $\langle f_\xi \,:\, \xi < \mu\rangle$ enumerates ${}^{\lambda\downarrow}\lambda \cap M$.

Note that (4.24) is possible because of (4.22).

Since II wins in the match $\mathfrak{M}$, there is α^* such that

(4.27) $\sup(\lambda \cap M) < \alpha^*$;

(4.28) $\alpha^* \in E^{\lambda}_{>\mu} \cap E^{\lambda}_{<\kappa}$ and

(4.29) $f_\xi(\alpha^*) < \sup\{\delta_\xi \,:\, \xi < \mu\} \leq \sup(\lambda \cap M)$.

Since all Skolem function f in M with parameters from M such that $f \restriction \lambda$ is a regressive function from λ to λ are among f_ξ, $\xi < \mu$, it is readily seen that $M^* = sk_{\mathcal{M}}(M \cup \{\alpha^*\})$ is as desired.

Suppose now that $\mathsf{CC}^{\downarrow}(\mu, <\kappa, \lambda)$ holds. In a match $\mathfrak{M}$, the player II can choose a continuously increasing sequence $\langle M_\xi \,:\, \xi < \delta\rangle$ of elementary submodels of $\mathcal{M}$ such that, for all $\xi < \mu$,

(4.30) $|\,M_{\xi+1}\,| = \mu$;

(4.31) $[M_{\xi+1}]^{<\delta} \subseteq M_{\xi+1}$

(4.32) $\delta_\xi = \sup(\lambda \cap M_\xi)$.

Then $M = \bigcup_{\xi<\mu} M_\xi$ is an elementary submodel of $\mathcal{M}$ of cardinality μ with $[M]^{<\mu} \subseteq M$ and $\sup(\{\delta_\xi \,:\, \xi < \mu\}) = \sup(\lambda \cap M)$. Thus each α^* as in the definition of $\mathsf{CC}^{\downarrow}(\mu, <\kappa, \lambda)$ for this M is in $B_{\mathfrak{M}}$. □ (Proposition 4.2)

Let θ and $\mathcal{M}$ be as in (4.1). For $M \in [\mathcal{M}]^{\mu}$ such that

(4.33) $M \prec \mathcal{M}$;

(4.34) $\mu \subseteq M$, μ, κ, $\lambda \in M$; and

(4.35) $[M]^{<\mu} \subseteq M$,

let

(4.36) $D_M = \{\alpha \in E^{\lambda}_{>\mu} \cap E^{\lambda}_{<\kappa} \,:\, f(\alpha) < \sup(\lambda \cap M) \text{ for all } f \in {}^{\lambda\downarrow}\lambda \cap M\}$.

Clearly we have $D_M \supseteq \sup(\lambda \cap M) \cap (E^{\lambda}_{>\mu} \cap E^{\lambda}_{<\kappa})$.

Let

(4.37) $\mathcal{B} = \{M \in [\mathcal{M}]^{\mu} \,:\, M \prec \mathcal{M},$
$M \models (4.34), (4.35) \text{ and } D_M \text{ is bounded in } \lambda\}$.

The following is immediate from the definition of D_M.

Lemma 4.3 *$\alpha^* < \lambda$ is an upper bound of D_M if and only if, for any $\alpha \in (E^{\lambda}_{>\mu} \cap E^{\lambda}_{<\kappa}) \setminus \alpha^*$ there is some $f \in {}^{\lambda\downarrow}\lambda \cap M$ such that $f(\alpha) \geq \sup(\lambda \cap M)$.* □

The following characterization of $\mathsf{WS}_{II}(G^{\downarrow}_{\mu}(<\kappa, \lambda))$ is going to play an important role in the next section.

Lemma 4.4 *Suppose that $\mu^{<\mu} = \mu$ and $\mu^+ < \kappa \leq \lambda$. Then, for any cardinal λ, $\mathsf{WS}_{II}(G^{\downarrow}_{\mu}(<\kappa, \lambda))$ holds if and only if $\mathcal{B}$ is non-stationary in $[\mathcal{M}]^{\mu}$.*

Proof. If $cf(\lambda) \leq \mu$ this is clear. Under this condition $\mathsf{WS}_{II}(G^{\downarrow}_{\mu}(<\kappa, \lambda))$ holds since the player II can choose her moves δ_ξ, $\xi < \mu$ such that $\{\delta_\xi \,:\, \xi < \mu\}$ is cofinal in λ. $\mathcal{B}$ is non-stationary since there are end-segment many $M \in [\mathcal{M}]^{\mu}$ with (4.33), (4.34) and (4.35) such that $\lambda \cap M$ is cofinal in λ.

Thus we may assume $cf(\lambda) > \mu$. Suppose first that $\mathsf{WS}_{II}(G^{\downarrow}_{\mu}(<\kappa, \lambda))$ holds. We show that $\mathcal{C} = \{M \in [\mathcal{M}]^{\mu} \,:\, M \prec \mathcal{M}\}$ is disjoint from $\mathcal{B}$. Suppose $M \in \mathcal{B} \cap \mathcal{C}$. By the assumption, there is a wining strategy $\sigma \in M$ of the player II in $G^{\downarrow}_{\mu}(<\kappa, \lambda)$. Let $\mathfrak{M} = \langle f_\xi, \delta_\xi \,:\, \xi < \mu\rangle$ be a match in $G^{\downarrow}_{\mu}(<\kappa, \lambda)$ satisfying (4.24), (4.25) and (4.26). Since the player II wins in $\mathfrak{M}$

(4.38) $\{\alpha \in E^{\lambda}_{>\mu} \cap E^{\lambda}_{<\kappa} \,:\, f_\xi(\alpha) < \sup(\{\delta_i \,:\, i < \xi\}) \text{ for all } \xi < \mu\}$

is unbounded. By (4.26) it follows that D_M is unbounded. This is a contradiction to $M \in \mathcal{B}$.

Suppose now that $\mathcal{B}$ is non-stationary. Let $\mathcal{C} \subseteq [\mathcal{M}]^{\mu}$ be a club disjoint from $\mathcal{B}$. We may assume that

(4.39) $M \prec \mathcal{M}$ holds for all $M \in \mathcal{C}$.

In a match $\mathfrak{M} = \langle f_\xi, \delta_\xi : \xi < \mu \rangle$, the player II can choose her moves δ_ξ, $\xi < \mu$ in such a way that, along with her moves, she also chooses elements M_ξ of $\mathcal{C}$ which should build a continuously increasing sequence $\langle M_\xi : \xi < \mu \rangle$ and such that

(4.40) $\mu \subseteq M_0$, μ, κ, $\lambda \in M_0$;

(4.41) $\{f_\eta : \eta \leq \xi\} \subseteq M_{\xi+1}$ for all $\xi < \mu$;

(4.42) $[M_\xi]^{<\mu} \subseteq M_{\xi+1}$ for all $\xi < \mu$; and

(4.43) $\delta_\xi = \sup(\lambda \cap M_\xi)$ for all $\xi < \mu$.

Since $\mathcal{C}$ is a club $M = \bigcup_{\xi<\mu} M_\xi \in \mathcal{C}$ and hence $M \notin \mathcal{B}$. By (4.39), (4.40) and (4.42), M satisfies (4.33), (4.34) and (4.35). Thus

(4.44) D_M is unbounded.

Since $\{f_\xi : \xi < \mu\} \subseteq M$ by (4.41), it follows from (4.44) that the player II wins in all such matches $\mathcal{M}$. □ (Lemma 4.4)

5 Reflection properties related to a generalization of Rado's Conjecture

In the following we assume that a tree is a partial ordering $T = \langle T, <_T \rangle$ such that $\{s \in T : s <_T\}$ is well-ordered by $<_T$. In particular, we do not assume that a tree has a single root. By this convention any subset of a tree T can be considered as a subtree of T.

A tree T is said to be *μ-special* if T can be partitioned into $\leq \mu$ subsets T_α, $\alpha < \delta$ ($\delta \leq \mu$) such that each T_α is pairwise incomparable (i.e. each T_α is an antichain).

The following reflection property is related to a generalization of Rado's Conjecture:

$\mathsf{RC}(\mu, <\kappa, \lambda)$: For any tree T of cardinality λ, if T is not μ-special, then there is a subtree T' of T of size $< \kappa$ such that T' is not μ-special.

Note that, using this notation, Rado's Conjecture (RC) can be reformulated as:

RC: $\mathsf{RC}(\aleph_0, <\aleph_2, \lambda)$ holds for all cardinal λ.

The following Proposition 5.4 for $\mu = \aleph_0$ together with Proposition 4.2 for $\mu = \aleph_0$ and a slight extension of Proposition 4.1 for $\mu = \aleph_0$ proves that Rado's Conjecture implies Fodor-type Reflection Principle (see Fuchino, Sakai, Torres and Usuba [8]).

In contrast to FRP the straightforward generalization of Rado's Conjecture to uncountable cardinals:

RC_μ: $\mathsf{RC}(\mu, < \mu^{++}, \lambda)$ holds for all cardinal λ

is consistent. We shall discuss more about this generalizaion in [9].

As mentioned before RC implies that the cardinality of the continuum to be $\leq \aleph_2$. Starting from a super compact cardinal we can force RC together with each of $2^{\aleph_0} = \aleph_1$ or $2^{\aleph_0} = \aleph_2$ (see Todorčević [13], [15]).

Let us begin with some tools we need for the proof of Proposition 5.4. A tree is said to be $\leq \mu$-Baire if the intersection $\bigcap \mathcal{D}$ of any open dense subsets of cardinality $\leq \mu$ is again open dense where $D \subseteq T$ is said to be open dense if it is upward closed and for any $t \in T$ there is $t' \in T$ with $t <_T t'$ (i.e. D is open dense in the forcing poset obtained by putting T upside down).

The following is easy to prove:

Lemma 5.1 (1) *Let T be a tree without maximal elements. If T is $\leq \mu$-Baire then T is not μ-special.*

(2) *If a tree T is of height $< \mu^+$ then T is μ-special.*

(3) *Any tree T is not μ-special if T has a branch of length $\geq \mu^+$. In particular, any tree of height $> \mu^+$ is not μ-special.* □

For a subtree T_0 of a tree T, a mapping $f : T_0 \to T$ is said to be regressive if $f(t) <_T t$ holds for all $t \in T_0$ which is not minimal in T.

Todorčević [12] proves the following Theorem only for the case $\mu = \aleph_0$ but the general case given below can be proved with exactly the same proof.

Theorem 5.2 (Pressing Down Lemma for Trees, Todorčević [12]) *Suppose that $f : T \to T$ is regressive and $f^{-1}{}''\{t\}$ is μ-special for all $t \in T$ then T is μ-special.* □

For a tree T, let $Lim(T) = \{t \in T : ht_T(t) \text{ is a limit ordinal}\}$.

Corollary 5.3 *Suppose that $f : Lim(T) \to T$ is regressive and $f^{-1}{}''\{t\}$ is μ-special for all $t \in T$ then T is μ-special.*

Proof. Let $\bar{f} : T \to T$ be defined by

$$(5.1) \qquad \bar{f}(t) = \begin{cases} f(t \restriction \alpha); & \text{if } \alpha \text{ is the largest limit ordinal below } ht_T(t+1) \\ \text{the minimal element below } t; & \text{it there is no such } \alpha. \end{cases}$$

Then $\bar{f}$ is regressive. For $t \in T$,

$$(5.2) \qquad \bar{f}^{-1}{}''\{t\} = \bigcup_{n\in\omega} \{u \in T : u \text{ is an } n\text{'th successor of an element of } f^{-1}{}''\{t\}\}$$

is μ-special since $f^{-1}{}''\{t\}$ is μ-special. □ (Corollary 5.3)

Proposition 5.4 *Suppose that μ, κ, λ are cardinals such that $\mu^{<\mu} = \mu < \mu^+ < \kappa \leq cf(\lambda)$. If $\mathsf{WS}_{II}(G^{\downarrow}_{\mu}(<\kappa, \lambda))$ does not hold then $\mathsf{RC}(\mu, <\kappa, \lambda^{\mu})$ does not hold.*

Proof. Assume that $\mathsf{WS}_{II}(G^{\downarrow}_{\mu}(<\kappa, \lambda))$ does not hold. By Lemma 4.4, $\mathcal{B}$ in (4.37) is stationary in $[\mathcal{M}]^{\mu}$.

Let

(5.3) $\quad \mathcal{T} = \{\langle M_\xi : \xi \leq \delta\rangle :$ (a) $\delta < \mu^+$, (b) $\langle M_\xi : \xi \leq \delta\rangle$ is a continuously increasing sequence of elementary submodels of $\mathcal{M}$ of cardinality μ, (c) $M_\xi \in \mathcal{B}$ for all successor $\xi \leq \delta$ and for all limit $\xi \leq \delta$ of cofinality μ, (d) $M_\xi \in M_{\xi+1}$for all $\xi < \delta\}$.

For t, t', let $t <_{\mathcal{T}} t' \Leftrightarrow t$ is an initial segment of t'.

We show that the tree $\mathcal{T} = \langle \mathcal{T}, <_{\mathcal{T}}\rangle$ witnesses the non reflection of non μ-specialness[1].

Claim 5.4.1 *All $T \in [\mathcal{T}]^{<\kappa}$ are μ-special.*

$\vdash$ For $t \in \mathcal{T}$ with $t = \langle M_\xi : \xi \leq \delta\rangle$ we denote $\ell_0(t) = \delta$ while the length of the sequence t is $\delta + 1$. M_t denotes the last component M_δ of the sequence t and $M_{t,\xi}$ the ξ'th component M_ξ for $\xi \leq \delta$. Let

(5.4) $\quad d(t) = \lambda \cap M_t$

and

(5.5) $\quad d(T) = \bigcup\{d(t) : t \in T\}$

for $T \subseteq \mathcal{T}$. Note that $d(T) \in [\lambda]^{<\kappa}$ for $T \in [\mathcal{T}]^{<\kappa}$. In particular, by the assumption on λ, $d(T)$ for such T is bounded in λ.

We show by induction on $\eta < \kappa$ that

(5.6) $\quad$ if $T \in [\mathcal{T}]^{<\kappa}$ and $otp(d(T)) = \eta$ then T is a μ-special tree

holds for all $\eta < \kappa$. Clearly this implies the claim.

Suppose (5.6) holds for all $\eta_0 < \eta$.

Case I: $\eta < \mu^+$. Suppose that $T \in [\mathcal{T}]^{<\kappa}$ and $otp(d(T)) = \eta$. By (5.3), (d), we have $ht(T) < \mu^+$. Thus T is μ-special by Lemma 5.1, (2).

Case II: η is a successor ordinal. This case cannot occur by definition of $d(T)$.

Case III: η is a limit ordinal of cofinality $\leq \mu$. Let $\delta = cf(\eta) \leq \mu$. Suppose that $T \in [\mathcal{T}]^{<\kappa}$ and $otp(d(T)) = \eta$. Then we can find an increasing sequence

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[1] This tree is not yet the final witness of the negation of $\mathsf{RC}(\mu, <\kappa, \lambda^{\mu})$ we are looking for since it has cardinality $\geq \theta >> \lambda$.

$\langle \xi_\alpha : \alpha < \delta \rangle$ of ordinals with $\sup(\{\xi_\alpha : \alpha < \delta\}) = \sup(d(T))$. Let $T_\alpha = \{t \in T : d(T) \subseteq \xi_\alpha\}$ for $\alpha < \delta$. Each T_α is μ-special by induction hypothesis. Hence $T = \bigcup_{\alpha<\delta} T_\alpha$ is also μ-special.

Case IV: η is a limit ordinal of cofinality $> \mu$. Suppose that $T \in [\mathcal{T}]^{<\kappa}$ and $otp(d(T)) = \eta$. Note that by the assumption on λ we have $\sup(d(T)) < \lambda$. Let

(5.7) $\quad T_0 = T \setminus \{t \in T : t \text{ is maximal in } T\}.$

Since $\{t \in T : t \text{ is maximal in } T\}$ is an antichain in T it is enough to show that T_0 is μ-special.

If $otp(d(T_0)) < otp(d(T))$ then by induction hypothesis T_0 is special. Hence we may assume that $otp(d(T_0)) = otp(d(T))$ (and so $\sup(d(T_0)) = \sup(d(T)))$. Let $\nu = \sup(d(T_0))$ $(= \sup(d(T)))$ and $\delta = cf(\nu)$. We have $\mu < \delta < \kappa$. Let $\langle \nu_\beta : \beta < \delta \rangle$ be a continuously and strictly increasing sequence of ordinals cofinal in ν. Note that $\nu > \sup(D_{M_t})$ holds for all $t \in Lim(T_0)$ by the definition of T_0 and (5.3), (d). Thus, for all $t \in Lim(T_0)$ there is $f_t \in {}^{\lambda\downarrow}\lambda \cap M_t$ such that $f_t(\nu) \geq \sup(\lambda \cap M_t)$ by Lemma 4.3.

Noting that $\ell_0(t)$ defined at the beginning of the proof is a limit ordinal for $t \in Lim(T_0)$ and hence we have $M_t = \bigcup_{\xi<\ell_0(t)} M_{t,\xi}$, let

(5.8) $\quad h(t) = t \restriction (\xi_0 + 1)$ where $\xi_0 = \min\{\xi < \ell_0(t) : f_t \in M_{t,\xi}\}$.

Then we have $h : Lim(T_0) \to T_0$ and h is regressive.

Subclaim 5.4.1.1 *$h^{-1}{}''\{u\}$ is μ-special for all $u \in T_0$.*

$\vdash$ Suppose $u \in T_0$. Since M_u is of cardinality μ, it is enough to show that $T_f = \{t \in h^{-1}{}''\{u\} : f_t = t\}$ is μ-special for each $f \in {}^{\lambda\downarrow}\lambda \cap M_u$. Since $f(\nu) < \nu$, there is $\beta^* < \delta$ such that $f(\nu) < v_{\beta^*}$. For any $t \in T_f$, we have $\sup(d(t)) \leq f_t(\nu) = f(\nu) \leq \nu_{\beta^*}$. Thus $T_f \subseteq \{t \in T_0 : d(t) \subseteq \nu_{\beta^*}\}$. The subtree of $\mathcal{T}$ on the right side of the inclusion is μ-special by the induction hypothesis. Hence T_f is also μ-special.

$\dashv$ (Subclaim 5.4.1.1)

By Corollary 5.3 it follows that T_0 is μ-special. $\dashv$ (Claim 5.4.1)

Claim 5.4.2 *$\mathcal{T}$ is $\leq \mu$-Baire. Hence it is not μ-special by Lemma 5.1,* (1).

$\vdash$ Suppose that D_m, $m < \mu$ are open dense subsets of $\mathcal{T}$ and $t \in \mathcal{T}$. We have to show that there is $t' \in \mathcal{T}$ such that $t <_{\mathcal{T}} t'$ and $t' \in \bigcap_{m<\mu} D_m$. Let $\tilde{\mathcal{M}}$ be the expansion of the structure $\mathcal{M}$ obtained by adding the unary relations D_m, $m < \mu$. Let $M \prec \tilde{\mathcal{M}}$ be such that

(5.9) $\quad t \in M$; and

(5.10) $\quad M \in \mathcal{B}$.

There is such M since $\mathcal{B}$ is stationary by Lemma 4.4.

Let x_m, $m < \mu$ be an enumeration of M. Since M satisfies (4.35), $\langle x_m : m < \mu_0 \rangle \in M$ for all $\mu_0 < \mu$.

Let $\langle t_m : m < \mu \rangle$ be a continuously increasing sequence in $\mathcal{T}$ such that

(5.11) $t_m \in M$ for all $m \in \mu$;

(so by the same reasoning as above $\langle t_m : m < \mu_0 \rangle \in M$ for all $\mu_0 < \mu$)

(5.12) $t_0 = t$;

(5.13) $t_{m+1} \in D_m \cap M$ for all $m < \mu$

(this is possible since D_m is open dense and by the elementarity of M);

and

(5.14) $x_m \in M_{t_{m+1}}$ for all $m < \mu$.

Let

(5.15) $t' = \bigcup \{t_m : m < \mu\} ^\frown \langle M \rangle$.

Then $t' \in \mathcal{T}$ by (5.14) and (5.10). $t \leq_{\mathcal{T}} t'$ by (5.12). $t' \in D_m$ for all $m < \mu$ by (5.13). $\dashv$ (Claim 5.4.2)

Let $\mathcal{N} \prec \mathcal{M}$ be such that

(5.16) $\lambda \subseteq \mathcal{N}$, $|\mathcal{N}| = \lambda^{<\mu}$;

(5.17) $[\mathcal{N}]^{<\mu} \subseteq \mathcal{N}$.

Let $\mathcal{T}_0 = \mathcal{T} \cap \mathcal{N}$. Then $|\mathcal{T}_0| \leq \lambda^\mu$ and the proofs of Claim 5.4.1 and Claim 5.4.2 also apply to $\mathcal{T}_0$. Thus $\mathcal{T}_0$ witnesses that $\mathsf{WS}_{II}(G^{\downarrow}_{\mu}(<\kappa, \lambda))$ does not hold.

$\square$ (Proposition 5.4)

6 Reflection properties related to generalizations of Galvin's Conjecture

To interpolate the implication from $\mathsf{REFL}_{\mathsf{chr}}(\mu, <\kappa, \lambda)$ to $\mathsf{RC}(\mu, <\kappa, \lambda)$ we would like to prove in this section, we introduce yet another reflection property which stands in connection with a generalization of Galvin's Conjecture.

For a partial ordering $P = \langle P, <_P \rangle$ a subordering $P' = \langle P', <_{P'} \rangle$ of P with $P' \subseteq P$ and $<_{P'} \; = \; <_P \cap (P')^2$ is said to be a *chain* if $<_{P'}$ linearly orders P'.

$\mathsf{GC}(\mu, <\kappa, \lambda)$: For any partial ordering P of cardinality λ, if P is not a union of $\leq \mu$-many chains, then there is a subordering P' of P of size $< \kappa$ such that P' is not a union of $\leq \mu$-many chains.

Galvin's Conjecture ([15]) can be formulated as:

GC: $\mathsf{GC}(\aleph_0, <\aleph_2, \lambda)$ holds for all cardinal λ.

Unlike Rado's Conjecture, the consistency of Galvin's Conjecture is a long standing open question.

Proposition 6.1 *Suppose that μ, κ, λ are cardinals such that $\mu^+ < \kappa \leq \lambda$.*

(1) $\mathsf{GC}(\mu, <\kappa, \lambda)$ *implies* $\mathsf{RC}(\mu, <\kappa, \lambda)$.

(2) $\mathsf{REFL}_{\mathsf{chr}}(\mu, <\kappa, \lambda)$ *implies* $\mathsf{GC}(\mu, <\kappa, \lambda)$.

Proof (Todorčević [16]). (1): Suppose that $\mathsf{RC}(\mu, <\kappa, \lambda)$ does not hold and let $T = \langle T, <_T \rangle$ be a tree of size λ witnessing this. Let $\lhd$ be an arbitrary linear ordering on T and let $\lhd_T$ be the binary relation on T defined by

(6.1) $t_0 \lhd_T t_1 \Leftrightarrow$ t_0 and t_1 are incomparable with respect to $<_T$ and $t'_0 \lhd t'_1$ where t'_0 and t'_1 are minimal elements below t_0 and t_1 respectively with respect to $<_T$ such that t'_0 and t'_1 are incomparable

It is easy to see that $\lhd_T$ is a partial ordering on T. By the definition of $\lhd_T$, we have that, for any $X \subseteq T$,

(6.2) X is a chain in $\langle T, \lhd_T \rangle \Leftrightarrow X$ is an antichain in $\langle T, <_T \rangle$.

Thus $\langle T, \lhd_T \rangle$ is a counterexample to $\mathsf{GC}(\mu, <\kappa, \lambda)$.

(2): Suppose that $\mathsf{GC}(\mu, <\kappa, \lambda)$ does not hold and let $\langle P, <_P \rangle$ be a partial ordering of size λ which is a counterexample to $\mathsf{GC}(\mu, <\kappa, \lambda)$. Let $\mathcal{E}_P$ be the adjacency relation on P defined by

(6.3) $p \, \mathcal{E}_P \, q \Leftrightarrow p$ and q are incomparable with respect to $<_P$.

Then, for any $X \subseteq P$, we have

(6.4) X is a chain in P (with respect to $<_P$) $\Leftrightarrow$ elements of X are pairwise non adjacent.

Thus $\langle P, \mathcal{E}_P \rangle$ is a counterexample to $\mathsf{REFL}_{\mathsf{chr}}(\mu, <\kappa, \lambda)$. $\square$ (Proposition 6.1)

7 A proof of Theorem 1.1 and some applications

We can now put together the propositions we proved in the previous sections to obtain a proof of Theorem 1.1.

Proof of Theorem 1.1: Suppose that $\mathsf{REFL}_{\mathsf{col}}(\mu, <\kappa, \lambda)$ does not hold. Let $\lambda^* \leq \lambda$ be such that

(7.1) $\mathsf{REFL}_{\mathsf{col}}(\mu, <\kappa, \lambda_0)$ holds for all $\lambda_0 < \lambda^*$ but $\mathsf{REFL}_{\mathsf{col}}(\mu, <\kappa, \lambda^*)$ does not.

By (the proof of) Proposition 3.2, $\mathsf{FRP}(\mu, <\kappa, \lambda^*)$ does not hold. By Proposition 4.1 it follows that $\mathsf{CC}^{\downarrow}(\mu, <\kappa, \lambda^*)$ does not hold. By Proposition 4.2, this is equivalent to the assertion that $\mathsf{WS}_{II}(G^{\downarrow}_{\mu}(<\kappa, \lambda^*))$ does not hold. Proposition 5.4 now implies that $\mathsf{RC}(\mu, <\kappa, (\lambda^*)^{\mu})$ does not hold. Thus, by Proposition 6.1, $\mathsf{REFL}_{\mathsf{chr}}(\mu, <\kappa, (\lambda^*)^{\mu})$ does not hold. Since $(\lambda^*)^{\mu} \leq \lambda^{\mu}$, it follows by Lemma 2.2, that $\mathsf{REFL}_{\mathsf{chr}}(\mu, <\kappa, \lambda^{\mu})$ does not hold. □ (Theorem 1.1)

A stationary subset S of a cardinal λ is said to be *non-reflecting* if $S \cap \delta$ is not stationary for any $\delta \in Lim(\lambda)$.

Lemma 7.1 *Let μ, λ be regular cardinals with $\mu^+ < \lambda$. Suppose that there is a non-reflecting stationary $S \subseteq E^{\lambda}_{\mu}$. Then there is a graph $G = \langle \lambda, \mathcal{E} \rangle$ such that* (a) $\mathsf{col}(G) = \mu^+$ *but* (b) $\mathsf{col}(G \restriction X) \leq \mu$ *for all $X \in [\lambda]^{<\lambda}$. In particular $\mathsf{REFL}_{\mathsf{col}}(\mu, <\lambda, \lambda)$ does not hold.*

Proof. Without loss of generality, we may assume that $S \subseteq Lim(\lambda)$. Let $\langle c_\xi : \xi \in S\rangle$ be such that, for all $\xi \in S$,

(7.2) $c_\xi \subseteq \xi \setminus Lim(\xi)$ and c_ξ is cofinal in ξ;

(7.3) $otp(c_\xi) = \mu$.

Let

(7.4) $\mathcal{E} = \{\langle \alpha, \beta\rangle, \langle \beta, \alpha\rangle : \alpha \in S, \beta \in c_\alpha\}$.

Since λ is regular, the following Claim implies (b).

Claim 7.1.1 *For any $\eta \in Lim(\lambda)$, $\mathsf{col}(G \restriction \eta) \leq \mu$.*

$\vdash$ Let $C \subseteq \eta$ be a club subset of η such that $C \cap S = \emptyset$. Let $\lhd$ be a well-ordering on η such that $\lhd \cap C^2 = \in \cap c^2$ and, for any α, $\beta \in C$ with $(\alpha, \beta) \cap C = \emptyset$ [2]; (α, β) is also an open interval between α and β with respect to $\lhd$; and $S \cap (\alpha, \beta)$ is an initial segment of (α, β) with respect to $\lhd$.

Then $\lhd$ witnesses that $\mathsf{col}(G \restriction \eta) \leq \mu$. $\dashv$ (Claim 7.1.1)

By the definition of $\mathcal{E}$ (7.4), it is clear that $\mathsf{col}(G) \leq \mu^+$. So the following Claim implies (a) and finishes the proof.

[2] We denote here with (α, β) the open interval $\{\xi < \eta : \alpha < \xi < \beta\}$.

Claim 7.1.2 $\mathsf{col}(G) \geq \mu^+$.

$\vdash$ Suppose that $\lhd$ is an arbitrary well-ordering of G of order type λ (see Lemma 2.1, (1)). By the stationarity of S there is $\xi^* \in S$ such that ξ^* is an initial segment with respect to $\lhd$. But then $\mathcal{E}_{\xi^*}^{\xi^*} = c_{\xi^*}$ and hence $|\mathcal{E}_{\xi^*}^{\xi^*}| = \mu$. Thus there is no well-ordering of G confirming that $\mathsf{col}(G) \leq \mu$. $\dashv$ (Claim 7.1.2)

$\square$ (Lemma 7.1)

The following Theorem covers some of the instances of the results in [11].

Theorem 7.2 *If μ and λ are cardinals such that $\mu^{<\mu} = \mu$, $\mu^+ < \lambda < \mu^{+\omega}$ and there is a non reflecting stationary set $S \subseteq E_\mu^\lambda$ then $\mathsf{REFL}_{\mathsf{chr}}(\mu, <\lambda, \lambda^\mu)$ does not hold. That is, there is a graph $G = \langle G, \mathcal{E}\rangle$ of cardinality λ^μ such that $\mathsf{chr}(G) > \mu$ but $\mathsf{chr}(G \restriction X) \leq \mu$ for all $X \in [G]^{<\lambda}$.*

Proof. By Lemma 7.1 $\mathsf{REFL}_{\mathsf{col}}(\mu, <\lambda, \lambda)$ does not hold. Hence, by Theorem 1.1, $\mathsf{REFL}_{\mathsf{chr}}(\mu, <\lambda, \lambda^\mu)$ does not hold. $\square$ (Theorem 7.2)

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