数理解析研究所講究録
第2042巻 2017年 72-74



# NOTES ON Bl-AD$_{\omega_1}$

DAISUKE IKEGAMI

ABSTRACT. In this note, we show that the axiom Bl-AD$_{\omega_1}$ is inconsistent under ZF + AC$_\omega(\mathbb{R})$. This answers the question of Löwe in [3, Question 52].

## 1. INTRODUCTION AND BASIC DEFINITIONS

In this note, we will prove the following:

**Theorem 1.1** (ZF + AC$_\omega(\mathbb{R})$). The axiom Bl-AD$_{\omega_1}$ is inconsistent.

The axiom Bl-AD$_{\omega_1}$ is a natural strengthening of the Axiom of Blackwell determinacy (Bl-AD). For the background of this research topic, one can refer to the survey paper on Blackwell determinacy [3]. We mostly follow the standard notations and we use basic notions from the Jech's textbook on set theory [1].

First, let us define what Bl-AD$_{\omega_1}$ is.

Let $X$ be a set with more than one element and assume $\mathsf{AC}_\omega({}^\omega X)$. By Prob$(X)$, we denote the set of all Borel probability measures on $X$ with a countable support, i.e., the set of all Borel probability measures $p$ such that there is a countable set $C \subseteq X$ with $p(C) = 1$.[1] From now on, we regard $X$ as a discrete topological space and topologize ${}^\omega X$ as the product space. For any finite sequence $s$ of elements in $X$, let $[s]$ be the basic open set generated by $s$, i.e., $[s] = \{x \in {}^\omega X \, ; \, s \subseteq x\}$.

Let $X^{\mathrm{Even}}$ ($X^{\mathrm{Odd}}$) be the set of finite sequences in $X$ with even (odd) length. We call a function $\sigma\colon X^{\mathrm{Even}} \to \mathrm{Prob}(X)$ a *mixed strategy for player I* and a function $\tau\colon X^{\mathrm{Odd}} \to \mathrm{Prob}(X)$ a *mixed strategy for player II*. Given mixed strategies $\sigma$ and $\tau$ for players I and II respectively, let $\nu(\sigma,\tau)\colon {}^{<\omega}X \to \mathrm{Prob}(X)$ as follows: For each finite sequence $s$ of elements in $X$,

$$\nu(\sigma,\tau)(s) = \begin{cases} \sigma(s) & \text{if } \mathrm{lh}(s) \text{ is even,} \\ \tau(s) & \text{if } \mathrm{lh}(s) \text{ is odd,} \end{cases}$$

where $\mathrm{lh}(s)$ is the length of $s$. Since some of the calculations in this paper require a lot of parentheses, let us reduce their number by convention. If $(x_0, ..., x_n)$ is a finite sequence, we

2010 *Mathematics Subject Classification.* Primary 03E60; Secondary 03E15, 91A05, 91A44.
*Key words and phrases.* Infinite games, Blackwell determinacy, descriptive set theory.
The author would like to thank Teruyuki Yorioka for organizing the RIMS workshop in set theory 2016. He is also grateful to JSPS for support through the grant with JSPS KAKENHI Grant Number 15K17586.

[1]We are going to amalgamate a sequence of such measures to produce a product measure on ${}^\omega X$ as we construct the Lebesgue measure on ${}^\omega\omega$. For this purpose, the condition of having a countable support is essential.

write $[x_0, ..., x_n]$ for the basic open set $[(x_0, ..., x_n)]$. Similarly, if $x \in X$ and $\mu \in \text{Prob}(X)$, we write $\mu(x)$ for $\mu(\{x\})$. Now, for each finite sequence $s$ of elements in $X$, define

$$\mu_{\sigma,\tau}([s]) = \prod_{i=0}^{\text{lh}(s)-1} \nu(\sigma,\tau)(s\restriction i)(s(i)).$$

By using $\mathsf{AC}_\omega(\mathbb{R} \times {}^\omega X)$ (which follows from $\mathsf{AC}_\omega({}^\omega X)$), we can uniquely extend $\mu_{\sigma,\tau}$ to a Borel probability measure on ${}^\omega X$, i.e., the probability measure whose domain is the set of all Borel sets in ${}^\omega X$. Let us also use $\mu_{\sigma,\tau}$ for denoting this Borel probability measure.

Let $A$ be a subset of ${}^\omega X$. A mixed strategy $\sigma$ for player I is *optimal in $A$* if for any mixed strategy $\tau$ for player II, $A$ is $\mu_{\sigma,\tau}$-measurable and $\mu_{\sigma,\tau}(A) = 1$. Similarly, a mixed strategy $\tau$ for player II is *optimal in $A$* if for any mixed strategy $\sigma$ for player I, $A$ is $\mu_{\sigma,\tau}$-measurable and $\mu_{\sigma,\tau}(A) = 0$. We say that $A$ is *Blackwell determined* if either player I or II has an optimal strategy in $A$. Finally, $\mathsf{Bl\text{-}AD}_X$ is the statement "for any subset $A$ of ${}^\omega X$, $A$ is Blackwell determined."[2]

Note that we only need $\mathrm{AC}_\omega(\mathbb{R})$ to define $\mathrm{Bl\text{-}AD}_{\omega_1}$ because $\mathrm{AC}_\omega({}^\omega\omega_1)$ follows from $\mathrm{AC}_\omega(\mathbb{R})$.

## 2. Proof of Theorem

*Proof of Theorem 1.1.* To derive a contradiction, let us assume $\mathrm{Bl\text{-}AD}_{\omega_1}$.

We will prove the following two claims, which will be inconsistent to each other:

**Claim 1.** There is no injection from $\omega_1$ to $\mathcal{P}(\omega)$.

**Claim 2.** There is an injection from $\omega_1$ to $\mathcal{P}(\omega)$.

*Proof of Claim 1.* We will use the following fact:

**Fact 2.1.** Assume Bl-AD. Then $\omega_1$ is measurable.

*Proof.* See e.g., [2, Corollary 4.19]. □

Let $\mu$ be a normal measure on $\omega_1$ and towards a contradiction, suppose there is an injection $f\colon \omega_1 \to \mathcal{P}(\omega)$.

We will derive a contradiction by the following arguments: for each natural number $n$, let $X_n = \{\alpha < \omega_1 \mid n \in f(\alpha)\}$. Since $\mu$ is a measure on $\omega_1$, for each $n$, either $X_n \in \mu$ or $\omega_1 \setminus X_n \in \mu$. Let $X = \bigcap\{X_n \mid X_n \in \mu\} \cap \bigcap\{\omega_1 \setminus X_n \mid X_n \notin \mu\}$. Then since $\mu$ is $\sigma$-complete, $X$ is also in $\mu$. Hence one can pick $\alpha, \beta \in X$ with $\alpha \neq \beta$.

We argue that $f(\alpha) = f(\beta)$, which would contract the assumption that $f$ is injective. In fact, for each natural number $n$,

$$n \in f(\alpha) \iff \alpha \in X_n \iff \beta \in X_n \iff n \in f(\beta),$$

[2] This formulation of Blackwell determinacy axioms does not involve imperfect information games; the original formulation due to Blackwell did, but these axioms turned out to be equivalent to the version we defined here which could be described as "perfect information determinacy with mixed strategies". The imperfect information axiom would allow the players to move simultaneously, but at each move at least one of the players would have only finitely many choices. The proof of [4] adapts to show that the perfect information axiom implies the imperfect information one. Vervoort's proof in [5] shows that optimal strategies exist for the perfect information games. For more details, cf. [3, § 5].

where the second equivalence holds because both $\alpha$ and $\beta$ are in $X$. Therefore, $f(\alpha) = f(\beta)$. $\square$

*Proof of Claim 2.* We will construct an injection $g\colon \omega_1 \to \mathcal{P}(\omega)$ using determinacy of Blackwell games with choosing countable ordinals.

Let us consider the following game $\mathcal{G}$: Player I chooses a countable ordinal $\alpha$ at first and then Player II chooses either 0 or 1 $\omega$-many times and produces a real $y \subseteq \omega$. Player II wins if the real $y$ codes $\alpha$. Otherwise Player I wins.

This game can be formulated as a game in the definition of Bl-AD$_{\omega_1}$ and therefore, one of the players has an optimal strategy in the game $\mathcal{G}$.

Notice that Player I cannot have an optimal strategy in the game $\mathcal{G}$ because of the following argument: Suppose $\sigma$ is an optimal strategy in the game $\mathcal{G}$ for Player I and then by the definition of a mixed strategy, there is an ordinal $\alpha < \omega_1$ such that $\sigma(\emptyset)(\{\alpha\}) > 0$. Then let $\tau_0$ be the mixed strategy for Player II in the game $\mathcal{G}$ such that $\tau_0$ let II play a real $y$ coding $\alpha$ with measure 1. Then the probability of the payoff set in the $\mathcal{G}$ via $\mu_{\sigma,\tau_0}$ is less than 1 and hence the mixed strategy $\sigma$ is not optimal.

Hence Player II has an optimal strategy $\tau$ in the game $\mathcal{G}$ instead. We will produce an $\omega_1$-sequence $(x_\alpha \mid \alpha < \omega_1)$ of distinct reals using this strategy $\tau$ in the following way:

Given a countable ordinal $\alpha$, let $\tau_\alpha$ be the mixed strategy of Player II in the game $\mathcal{G}$ after Player I plays $\alpha$ with probability 1. Then $\tau_\alpha$ can be regarded as a Borel probability on the Baire space and hence it can be seen as a real $x_\alpha$.

We claim that the function $g(\alpha) = x_\alpha$ is injective. Let $\alpha < \beta$ be countable ordinals. Then the strategy $\tau_\alpha$ concentrates on reals coding $\alpha$ with probability 1 while $\tau_\beta$ concentrates on reals coding $\beta$ with probability 1. Therefore, $\tau_\alpha$ and $\tau_\beta$ are distinct and so are $x_\alpha$ and $x_\beta$, as desired.

$\square$

$\square$

## References

[1] Thomas Jech. *Set theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. The third millennium edition, revised and expanded.

[2] Benedikt Löwe. The simulation technique and its applications to infinitary combinatorics under the axiom of Blackwell determinacy. *Pacific J. Math.*, 214(2):335–358, 2004.

[3] Benedikt Löwe. Set theory of infinite imperfect information games. In Alessandro Andretta, editor, *Set theory: recent trends and applications*, volume 17 of *Quad. Mat.*, pages 137–181. Dept. Math., Seconda Univ. Napoli, Caserta, 2006.

[4] Donald A. Martin. The determinacy of Blackwell games. *J. Symbolic Logic*, 63(4):1565–1581, 1998.

[5] Donald A. Martin, Itay Neeman, and Marco Vervoort. The strength of Blackwell determinacy. *J. Symbolic Logic*, 68(2):615–636, 2003.

(D. Ikegami) Department of Mathematics, School of Engineering, Tokyo Denki University, 5 Asahi-cho, Senju, Adachi-ward, Tokyo 120-8551, JAPAN

*E-mail address*, D. Ikegami: ikegami@mail.dendai.ac.jp