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レプリカ解析を用いた予算制約・集中投資度制約がある場合の最小投資リスクの理論解析

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1 Introduction

Portfolio optimization problems first appeared in the theory of diversification investment introduced by Markowitz in 1952. Analytical procedures for solving portfolio optimization problems to accurately implement asset management so as to disperse the risk by diversifying investment into several assets have become well known. Over the next few decades, several issues related to portfolio optimization problems have been addressed, and recently several models and the behavior of the minimal investment risk in portfolio optimization problems have been thoroughly examined using analytical approaches which have been developed and improved through multidisciplinary collaboration [1–7]. For instance, Ciliberti *et al.* analyzed the minimal investment risk under an absolute deviation model and an expected shortfall model using replica analysis in the absolute zero temperature limit [1, 2]. Kondor *et al.* examined quantitatively the noise sensitivity of the optimal portfolio for several risk functions [3]. Pafka *et al.* discussed the relation between predicted risk, realized risk, and true risk in detail via a scenario ratio (between the number of scenarios and the number of assets) [4]. Shinzato derived the statistics minimal investment risk and investment concentration and showed that the minimal investment risk is attained and the investment concentration constraint is satisfied (e.g., by a portfolio) and that these two statistics have the self-averaging property which is frequently used in statistical mechanical informatics analysis [5]. Shinzato *et al.* developed a faster algorithm for solving portfolio optimization problems using a belief propagation method [6]. Shinzato examined the portfolio optimization problem in the case that each asset return rate is distributed independently and not identically using replica analysis [7]. These previous works have analyzed

1. Investment risk (as cost function)
2. Investment concentration (like Herfindahl-Hirschman Index)

of the optimal portfolio which can minimize the investment risk (not the expected investment risk) with budget constraint [1–7]. As one of the natural extentions, we need to examine the portfolio optimization with several constraints. As the first step of analyzing portfolio optimization problems with several constraints, noting the risk minimization problem, we develop a novel approach for solving a portfolio optimization problem with two representative constraints, a budget constraint and a constraint on investment concentration, using replica analysis. We will discuss the portfolio optimization problem with the constraints of budget and investment concentration.

2 Preliminary

- We will revisit stochastic optimization so as to clarify our target.
- We will introduce the Boltzmann distribution approach with respect to the optimization problem.
- We will explain replica analysis in order to discuss the portfolio optimization problem with two constraints

We consider the minimization problem from the viewpoint of stochastic optimization. First, we assume that with respect to controal parameter w and random variable X, the real-valued function $f(w,X)$ is prepared, where $w \in W$, W is the feasible solution subset. Moreover $f(w,X)$ is bounded below on $w \in W$ (not always convex) and here the probability of random variable X is known. Then, w.r.t. (w,X), $f(w,X) \ge \min_{w \in W} f(w,X)$ is held. Or the optimal is defined as follows; $w^*(X) = \arg\min_{w \in W} f(w,X)$. From this, we can use the following identity, $f(w^*(X),X) = \min_{w \in W} f(w,X)$. From $f(w,X) \ge \min\limits_{w \in W} f(w,X)$ and $f(w^*(X),X) = \min\limits_{w \in W} f(w,X)$, $f(w,X) \ge f(w^*(X),X)$ is obtained. Then, we can average the both sides with random variable as follows, $E[f(w,X)] \ge E[f(w^*(X),X)]$, where the notation $E[g(X)]$ means the expectation of $g(X)$. Since $E[f(w,X)] \ge E[f(w^*(X),X)]$ is held when any $w \in W$,

$$\min_{w \in W} E[f(w,X)] \ge E[f(w^*(X),X)], \tag{1}$$

is obtained. Further, from $f(w^*(X),X) = \min\limits_{w \in W} f(w,X)$,

$$\min_{w \in W} E[f(w,X)] \ge E\left[\min_{w \in W} f(w,X)\right], \tag{2}$$

is also obtained. That is, The minimum of average is not smaller than the average of minimum. From the argument in the previous work [5], for any X,

$$\min_{w \in W} f(w,X) = E\left[\min_{w \in W} f(w,X)\right], \tag{3}$$

is guaranteed mathematically (we call self-averaging). Thus, from $\min_{w\in W} E[f(w,X)] \geq E\left[\min_{w\in W} f(w,X)\right]$,

$$\min_{w\in W} E[f(w,X)] \geq \min_{w\in W} f(w,X), \tag{4}$$

is rewritten. The solution of ordinary research (OR) is determined as w^{OR}=arg $\min_{w\in W} E[f(w,X)]$. From $\min\limits_{w\in W} E[f(w,X)] \geq \min\limits_{w\in W} f(w,X)$ and $w^{\mathrm{OR}} = \arg\min\limits_{w\in W} E[f(w,X)]$,

$$E[f(w^{\mathrm{OR}},X)] \geq \min_{w\in W} f(w,X), \tag{5}$$

is also obtained. In the case of cost management, $E[f(w^{\mathrm{OR}},X)]$ in the left hand side is the cost of the solution which can not always minimize the cost. $\min\limits_{w\in W} f(w,X)$ in the right hand side is the cost of the optimal solution which the investor wants to know. In the previous work [5],

$$\textit{opportunity loss} = \frac{E[f(w^{\mathrm{OR}},X)]}{\min_{w\in W} f(w,X)} = \frac{\alpha}{\alpha-1}, \tag{6}$$

3 Portfolio optimization problem

This talk considers optimally diversified investment in N assets in an investment market with no restrictions on short selling. w_i is the amount of asset $i(=1,\cdots,N)$ in the portfolio and the full portfolio of N assets is denoted $\vec{w}=\{w_1,\cdots,w_N\}^{\mathrm{T}} \in \mathbf{R}^N$, where T indicates the transpose of a vector or matrix. $x_{i\mu}$ is the return rate of asset i under period (or scenario) $\mu(=1,\cdots,p)$. For simplicity of our discussion, similar to in the previous work, it is assumed that each return rate $x_{i\mu}$ is independently and identically normally distributed with mean 0 and variance 1 [5]. $X=\left\{\frac{x_{i\mu}}{\sqrt{N}}\right\} \in \mathbf{R}^{N\times p}$ is the return rate matrix where under this assumption, given the p return rate vectors $\vec{x}_1,\cdots,\vec{x}_p \in \mathbf{R}^N$, $\vec{x}_\mu=\{x_{1\mu},\cdots,x_{N\mu}\}^{\mathrm{T}} \in \mathbf{R}^N$.

In the mean-variance model, the investment risk $\mathcal{H}(\vec{w}|X)$ of portfolio $\vec{w}$ is defined as follows:

$$\begin{aligned}\mathcal{H}(\vec{w}|X) &= \frac{1}{2N}\sum_{\mu=1}^{p}\left(\sum_{i=1}^{N} w_i(x_{i\mu}-0)\right)^2, \\ &= \frac{1}{2N}\sum_{\mu=1}^{p}\left(\sum_{i=1}^{N} w_i x_{i\mu}\right)^2,\end{aligned} \tag{7}$$

where $E[x_{i\mu}]=0$ and $E[x_{i\mu}^2]=1$. Note that the necessary and sufficient condition for the optimal portfolio for portfolio optimization problem to be uniquely determined given in [5] is that $J=XX^{\mathrm{T}} \in \mathbf{R}^{N\times N}$ be a non-singular matrix, that is, the rank of matrix J be N or simply $p>N$. However, since J does not always need to be a regular matrix to guarantee a unique optimal portfolio for a portfolio optimization problem with several constraints, as in the present work, we do not adopt the regular matrix assumption here. The investment risk $\mathcal{H}(\vec{w}|X)$ is rewritten as follows;

$$\begin{aligned}\mathcal{H}(\vec{w}|X) &= \frac{1}{2N}\sum_{\mu=1}^{p}\left(\sum_{i=1}^{N} w_i x_{i\mu}\right)^2 \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \left(\frac{1}{N}\sum_{\mu=1}^{p} x_{i\mu} x_{j\mu}\right) \\ &= \frac{1}{2}\vec{w}^{\mathrm{T}} J \vec{w},\end{aligned} \tag{8}$$

where i, jth component of $J = \{J_{ij}\} \in \mathbf{R}^{N\times N}$ is defined as follows,

$$J_{ij} = \frac{1}{N}\sum_{\mu=1}^{p} x_{i\mu} x_{j\mu}. \tag{9}$$

In the previous work [5], we consider the optimal portfolio which can minimize the investment risk,

$$\mathcal{H}(\vec{w}|X) = \frac{1}{2}\vec{w}^{\mathrm{T}} J \vec{w}, \tag{10}$$

with the budget constraint,

$$\sum_{i=1}^{N} w_i = N. \tag{11}$$

When $p > N$, this optimal solution is

$$\vec{w}^* = \frac{N J^{-1}\vec{e}}{\vec{e}^{\mathrm{T}} J^{-1}\vec{e}}, \tag{12}$$

where $\vec{e} = \{1, 1, \cdots, 1\}^{\mathrm{T}} \in \mathbf{R}^N$ is used. Using $\vec{w}^* = \frac{N J^{-1}\vec{e}}{\vec{e}^{\mathrm{T}} J^{-1}\vec{e}}$, the minimal investment risk $\mathcal{H}(\vec{w}^*|X)$ is calculated as follows;

$$\begin{aligned}\mathcal{H}(\vec{w}^*|X) &= \frac{1}{2}(\vec{w}^*)^{\mathrm{T}} J (\vec{w}^*) \\ &= \frac{N^2}{2\vec{e}^{\mathrm{T}} J^{-1}\vec{e}},\end{aligned} \tag{13}$$

where $J = XX^{\mathrm{T}} \in \mathbf{R}^{N\times N}$. In order to assess it accurately, we need to calculate the inverse matrix of J, however, since the computation amount $O(N^3)$ is required, it is hard to implement this approach.

In a similar way, we also consider the optimal portfolio which can minimize the investment risk,

$$\mathcal{H}(\vec{w}|X) = \frac{1}{2}\vec{w}^{\mathrm{T}} J \vec{w}, \tag{14}$$

with the constraints of budget and investment concentration as follows;

$$\sum_{i=1}^{N} w_i = N, \tag{15}$$

$$\sum_{i=1}^{N} w_i^2 = N\tau, \tag{16}$$

where Eq. (15) means budget constraint and Eq. (16) means investment concentration constraint and $\tau \geq 1$. The minimal investment risk is assessed as follows;

$$\mathcal{H}(\vec{w}^*|X) = \frac{N^2 \vec{e}^{\mathrm{T}}(J-\theta I_N)^{-1} J (J-\theta I_N)^{-1}\vec{e}}{2\left(\vec{e}^{\mathrm{T}}(J-\theta I_N)^{-1}\vec{e}\right)^2}, \tag{17}$$

where the optimal portfolio $\vec{w}^*$ is determined as follows,

$$\vec{w}^* = \frac{N(J-\theta I_N)^{-1}\vec{e}}{\vec{e}^{\mathrm{T}}(J-\theta I_N)^{-1}\vec{e}}, \tag{18}$$

and parameter θ satisfies

$$N\tau = \frac{N^2 \vec{e}^{\mathrm{T}}(J-\theta I_N)^{-2}\vec{e}}{\left(\vec{e}^{\mathrm{T}}(J-\theta I_N)^{-1}\vec{e}\right)^2}, \tag{19}$$

where $I_N \in \mathbf{R}^{N\times N}$ is the identity matrix.

4 Boltzmann distribution approach

We here reconsider this optimal problem using Bayesian inference. The optimization problem is formulated as follows;

$$\mathcal{H}(\vec{w}|X) \qquad s.t. \qquad \left\{ \begin{array}{l} \displaystyle\sum_{i=1}^{N} w_i = N \\ \displaystyle\sum_{i=1}^{N} w_i^2 = N\tau \end{array} \right. . \tag{20}$$

We prepare $W = \left\{\vec{w} \in \mathbf{R}^N | \sum_{i=1}^N w_i = N, \sum_{i=1}^N w_i^2 = N\tau\right\}$ as the feasible portfolio subset. In the context of Bayesian inference, constraint W is regarded as a prior $P_0(\vec{w})$ and $\mathcal{H}(\vec{w}|X)$ is also regarded as the loglikelihood.

Let us denote the Boltzmann distribution (or the posterior probability or the conditional probability) as follows;

$$P(\vec{w}|X) = \frac{P_0(\vec{w})e^{-\beta\mathcal{H}(\vec{w}|X)}}{Z(X,\beta)}, \tag{21}$$

where

$$P_0(\vec{w}) = \begin{cases} 1 & \vec{w} \in W \\ 0 & \text{otherwise} \end{cases}, \tag{22}$$

$$Z(X,\beta) = \int_{-\infty}^{\infty} d\vec{w} P_0(\vec{w}) e^{-\beta\mathcal{H}(\vec{w}|X)}, \tag{23}$$

are used. We call $Z(X,\beta)$ the partition function and $\beta(>0)$ the inverse temperature. From the property of exponential,

$$\mathcal{H}(\vec{w}_a|X) < \mathcal{H}(\vec{w}_b|X) \qquad \Longleftrightarrow \qquad P(\vec{w}_a|X) > P(\vec{w}_b|X),$$

From $\mathcal{H}(\vec{w}_a|X) < \mathcal{H}(\vec{w}_b|X) \Longleftrightarrow P(\vec{w}_a|X) > P(\vec{w}_b|X)$, in the case of sufficiently large β, $\vec{w}^* = \lim_{\beta\to\infty} \int_{-\infty}^{\infty} d\vec{w} P(\vec{w}|X)\vec{w}$ is held. Furthermore, $\mathcal{H}(\vec{w}^*|X) = \lim_{\beta\to\infty} \int_{-\infty}^{\infty} d\vec{w} P(\vec{w}|X)\mathcal{H}(\vec{w}|X)$ is also held. In addition, using the partition function $Z(X,\beta)$,

$$-\frac{\partial}{\partial\beta}\log Z(X,\beta) = \int_{-\infty}^{\infty} d\vec{w} P(\vec{w}|X)\mathcal{H}(\vec{w}|X), \tag{24}$$

is obtained. From

$$\mathcal{H}(\vec{w}^*|X) = \lim_{\beta\to\infty} \int_{-\infty}^{\infty} d\vec{w} P(\vec{w}|X)\mathcal{H}(\vec{w}|X), \tag{25}$$

$$-\frac{\partial}{\partial\beta}\log Z(X,\beta) = \int_{-\infty}^{\infty} d\vec{w} P(\vec{w}|X)\mathcal{H}(\vec{w}|X), \tag{26}$$

then

$$\mathcal{H}(\vec{w}^*|X) = \lim_{\beta\to\infty} \left\{ -\frac{\partial}{\partial\beta}\log Z(X,\beta) \right\}, \tag{27}$$

is obtained. Further, since the minimal investment risk $\mathcal{H}(\vec{w}^*|X)$ is satisfied with self-averaging property, we will examine the following;

$$\begin{aligned} \mathcal{H}(\vec{w}^*|X) &= E[\mathcal{H}(\vec{w}^*|X)] \\ &= \lim_{\beta\to\infty} \left\{ -\frac{\partial}{\partial\beta} E[\log Z(X,\beta)] \right\}. \end{aligned} \tag{28}$$

5 Replica analysis

So as to assess the minimal investment risk $\mathcal{H}(\vec{w}^*|X)$, we need to evaluate analytically $E[\log Z(X,\beta)]$, since it is not easy to average the logarithm function of the partition function, we apply replica trick $\log Z = \lim_{n\to 0} \frac{Z^n - 1}{n}$, then,

$$E[\log Z(X,\beta)] = \lim_{n\to 0} \frac{E[Z^n(X,\beta)] - 1}{n}, \tag{29}$$

is obtained. Or we allow to use

$$E[\log Z(X,\beta)] = \begin{cases} \lim_{n\to 0} \dfrac{\log E[Z^n(X,\beta)]}{n} \\ \lim_{n\to 0} \dfrac{\partial}{\partial n} \log E[Z^n(X,\beta)] \end{cases}, \tag{30}$$

In any case, we need to assess $E[Z^n(X,\beta)]$. For any $n \in \mathbf{R}$, it is difficult to assess $E[Z^n(X,\beta)]$, however, when $n \in \mathbf{Z}$, it is comparatively easy to calculate it as follows;

$$\begin{aligned} E[Z^n(X,\beta)] &= E\left[\left(\int_{-\infty}^{\infty} d\vec{w} P_0(\vec{w}) e^{-\beta \mathcal{H}(\vec{w}|X)}\right)^n\right] \\ &= E\left[\prod_{a=1}^{n}\left(\int_{-\infty}^{\infty} d\vec{w}_a P_0(\vec{w}_a) e^{-\beta \mathcal{H}(\vec{w}_a|X)}\right)\right] \\ &= \int_{-\infty}^{\infty} \prod_{a=1}^{n} d\vec{w}_a P_0(\vec{w}_a) E\left[e^{-\beta \sum_{a=1}^{n} \mathcal{H}(\vec{w}_a|X)}\right]. \end{aligned} \tag{31}$$

From this, the configuration average is implemented first. Thus, we use the saddle point method,

$$\begin{aligned} &\log E[Z^n(X,\beta)] \\ \simeq& \mathop{\mathrm{Extr}}_{\vec{k},\vec{\theta},Q_w,\tilde{Q}_w} \left\{ -N\vec{e}^{\mathrm{T}}\vec{k} - \frac{N\tau}{2}\mathrm{Tr}\Theta + \frac{N}{2}\mathrm{Tr}Q_w\tilde{Q}_w - \frac{p}{2}\log\det|I+\beta Q_w| \right. \\ & \left. -\frac{N}{2}\log\det\left|\tilde{Q}_w - \Theta\right| + \frac{N}{2}\vec{k}^{\mathrm{T}}\left(\tilde{Q}_w - \Theta\right)^{-1}\vec{k}\right\}, \end{aligned} \tag{32}$$

is assessed, where $\vec{k} = \{k_1, k_2, \cdots, k_n\}^{\mathrm{T}}, \vec{e} = \{1, 1, \cdots, 1\}^{\mathrm{T}} \in \mathbf{R}^n$ and $\Theta = \mathrm{diag}\{\theta_1, \theta_2, \cdots, \theta_n\}$, $Q_w = \{q_{wab}\}, \tilde{Q}_w = \{\tilde{q}_{wab}\} \in \mathbf{R}^{n\times n}$ are used and $I \in \mathbf{R}^{n\times n}$ is the identity matrix. Moreover, the notation $\mathrm{Extr}_m g(m)$ means the extremum of $g(m)$ with respect to m and k_a and θ_a auxiliary variables which are related to the budget constraint and investment concentration constraint of ath replica, respectively. In addition, $q_{wab} = \frac{1}{N}\sum_{i=1}^{N} w_{ia}w_{ib}$ is prepared. In the case that the number of assets N is sufficiently large,

$$\begin{aligned} \Phi(n) &= \lim_{N\to\infty}\frac{1}{N}\log E[Z^n(X,\beta)] \\ &= \mathop{\mathrm{Extr}}_{\vec{k},\vec{\theta},Q_w,\tilde{Q}_w} \left\{ -\vec{e}^{\mathrm{T}}\vec{k} - \frac{\tau}{2}\mathrm{Tr}\Theta + \frac{1}{2}\mathrm{Tr}Q_w\tilde{Q}_w - \frac{\alpha}{2}\log\det|I+\beta Q_w| \right. \\ & \left. -\frac{1}{2}\log\det\left|\tilde{Q}_w - \Theta\right| + \frac{1}{2}\vec{k}^{\mathrm{T}}\left(\tilde{Q}_w - \Theta\right)^{-1}\vec{k}\right\}, \end{aligned} \tag{33}$$

is obtained where $\alpha = p/N \sim O(1)$. When $a, b(= 1, 2, \cdots, n)$, we assume the following;

$$(q_{wab}, \tilde{q}_{wab}) = \begin{cases} (\chi_w + q_w, \tilde{\chi}_w - \tilde{q}_w) & a = b \\ (q_w, -\tilde{q}_w) & \text{otherwise} \end{cases}, \tag{34}$$

$$(k_a, \theta_a) = (k, \theta), \tag{35}$$

where it is called the ansatz of replica symmetry solution. We substitute the replica symmetry solution into Eq. (33), then

$$\Phi(n) = \mathop{\mathrm{Extr}}_{k,\theta,\chi_w,q_w,\tilde{\chi}_w,\tilde{q}_w} \left\{ -nk - \frac{n\tau\theta}{2} + \frac{n(\chi_w+q_w)(\tilde{\chi}_w-\tilde{q}_w)}{2} \right.$$
$$-\frac{n(n-1)q_w\tilde{q}_w}{2} - \frac{n-1}{2}\log(\tilde{\chi}_w-\theta) - \frac{1}{2}\log(\tilde{\chi}_w-\theta-n\tilde{q}_w)$$
$$\left. +\frac{nk^2}{2(\tilde{\chi}_w-\theta-n\tilde{q}_w)} - \frac{(n-1)\alpha}{2}\log(1+\beta\chi_w) - \frac{\alpha}{2}\log(1+\beta\chi_w+n\beta q_w) \right\}, \quad (36)$$

is obtained. Note that n in Eq. (36) is integer at first, but it might assume that n at present form is real. From this,

$$\begin{aligned}
\phi &= \lim_{N\to\infty}\frac{1}{N}E[\log Z(X,\beta)] \\
&= \lim_{N\to\infty}\frac{1}{N}\lim_{n\to 0}\frac{\partial}{\partial n}\log E[Z^n(X,\beta)] \\
&= \mathop{\mathrm{Extr}}_{k,\theta,\chi_w,q_w,\tilde{\chi}_w,\tilde{q}_w} \left\{ -k - \frac{\tau\theta}{2} + \frac{(\chi_w+q_w)(\tilde{\chi}_w-\tilde{q}_w)}{2} + \frac{q_w\tilde{q}_w}{2} \right. \\
&\quad \left. -\frac{1}{2}\log(\tilde{\chi}_w-\theta) + \frac{\tilde{q}_w+k^2}{2(\tilde{\chi}_w-\theta)} - \frac{\alpha}{2}\log(1+\beta\chi_w) - \frac{\alpha\beta q_w}{2(1+\beta\chi_w)} \right\}, \qquad (37)
\end{aligned}$$

is analyzed. From $\mathcal{H}(\vec{w}^*|X) = \lim_{\beta\to\infty}\left\{-\frac{\partial}{\partial\beta}E[\log Z(X,\beta)]\right\}$ and $\phi = \lim_{N\to\infty}\frac{1}{N}E[\log Z(X,\beta)]$

$$\begin{aligned}
\varepsilon &= \lim_{N\to\infty}\frac{1}{N}\mathcal{H}(\vec{w}^*|X) \\
&= \begin{cases} \frac{\alpha\tau+\tau-1-2\sqrt{\alpha\tau(\tau-1)}}{2} & 1-\frac{1}{\tau}\le\alpha \\ 0 & \text{otherwise} \end{cases}, \qquad (38)
\end{aligned}$$

is assessed.

6 Discussion

When $1-\frac{1}{\tau}\le\alpha$, we calculate

$$\frac{\alpha\tau+\tau-1-2\sqrt{\alpha\tau(\tau-1)}}{2} = \frac{\alpha-1}{2} + \frac{(\sqrt{\alpha(\tau-1)}-\sqrt{\tau})^2}{2}. \qquad (39)$$

From this finding and the minimal investment risk per asset of the mean-variace model with budget constraint only, $\frac{\alpha-1}{2}$, discussed in [5], it is interpreted that $\frac{(\sqrt{\alpha(\tau-1)}-\sqrt{\tau})^2}{2}$ is the risk term induced from the investment concentration constraint. Moreover, when $\sqrt{\alpha(\tau-1)} - \sqrt{\tau} = 0$, then $\tau = \frac{\alpha}{\alpha-1}$ is obtained.

$$\tau = \chi_w + q_w = \frac{1}{N}\sum_{i=1}^{N} w_i^2, \qquad (40)$$

where $\chi_w + q_w$ is the diagonal of Q_w, that is, q_{waa}. In similar way, we assess the minimal expected investment risk per asset $\varepsilon^{\rm OR} = \lim_{N\to\infty}\frac{1}{N}\min_{\vec{w}\in W} E[\mathcal{H}(\vec{w}|X)]$ using the approach of ordinary reseach (OR),

$$\varepsilon = \begin{cases} \frac{\alpha\tau+\tau-1-2\sqrt{\alpha\tau(\tau-1)}}{2} & 1-\frac{1}{\tau}\le\alpha \\ 0 & \text{otherwise} \end{cases}, \tag{41}$$

$$\varepsilon^{\rm OR} = \frac{\alpha\tau}{2}, \tag{42}$$

Thus, the opportunity loss $\frac{\varepsilon^{\rm OR}}{\varepsilon}$,

$$\frac{\varepsilon^{\rm OR}}{\varepsilon} = \begin{cases} \frac{\alpha\tau}{\alpha\tau+\tau-1-2\sqrt{\alpha\tau(\tau-1)}} & 1-\frac{1}{\tau}\le\alpha \\ +\infty & \text{otherwise} \end{cases}, \tag{43}$$

where $\tau-1-2\sqrt{\alpha\tau(\tau-1)}\le 0$ is held when $1-\frac{1}{\tau}\le\alpha$ and $\tau\ge 1$. In order to verify our proposed approach based on the assumption of a replica symmetry solution (or simply replica analysis), we compare the results derived using the proposed method with those from numerical simulation and those obtained using the standard approach in ordinary research (OR). From Eq. (17) to Eq. (19), given $X=\left\{\frac{x_{i\mu}}{\sqrt{N}}\right\}\in\mathbf{R}^{N\times p}$ and $J=XX^{\rm T}$, the minimal investment risk per asset ε is,

$$\varepsilon = \frac{1}{N}\mathcal{H}(\vec{w}^*|X) = \frac{N\vec{e}^{\rm T}(J-\theta I_N)^{-1}J(J-\theta I_N)^{-1}\vec{e}}{2\left(\vec{e}^{\rm T}(J-\theta I_N)^{-1}\vec{e}\right)^2}, \tag{44}$$

where $\tau=\frac{N\vec{e}^{\rm T}(J-\theta I_N)^{-2}\vec{e}}{(\vec{e}^{\rm T}(J-\theta I_N)^{-1}\vec{e})^2}$.

As the numerical setting; The number of assets $N=500$ and the number of periods $p=1000$, that is, $\alpha=p/N=2$; As the sample sets, $C=100$ return rate matrices, $X^1,X^2,\cdots,X^{100}$, are prepared where $X^c=\left\{\frac{x^c_{i\mu}}{\sqrt{N}}\right\}\in\mathbf{R}^{N\times p}$; $x^c_{i\mu}$ is independently and identically assigned with the probability with mean 0 and variace 1, respectively; $\varepsilon^c=\frac{1}{N}\mathcal{H}(\vec{w}^*|X^c)$ is evaluated with respect to each return rate matrix; The minimal investment risk per asset is estimated by $\varepsilon=\frac{1}{C}\sum_{c=1}^C\varepsilon^c$.

Our analytical procedure is:

1. Stochastic optimization: $E[f(w^{\rm OR},X)]\ge\min_{w\in W}f(w,X)$.
2. Boltzmann distribution: $P(\vec{w}|X)=\frac{P_0(\vec{w})e^{-\beta\mathcal{H}(\vec{w}|X)}}{Z(X,\beta)}$ and $-\frac{\partial}{\partial\beta}\log Z(X,\beta)=\int_{-\infty}^{\infty}d\vec{w}P(\vec{w}|X)\mathcal{H}(\vec{w}|X)$.
3. Self-averaging: $\mathcal{H}(\vec{w}^*|X)=E[\mathcal{H}(\vec{w}^*|X)]=\lim_{\beta\to\infty}\left\{-\frac{\partial}{\partial\beta}E[\log Z(X,\beta)]\right\}$.
4. Replica trick: We estimate $E[Z^n(X,\beta)]$ of $n\in\mathbf{R}$ using $E[Z^n(X,\beta)]$ of $n\in\mathbf{Z}$.
5. Replica symmetry ansatz: $q_{wab}=\chi_w+q_w$ if $a=b$, otherwise $q_{wab}=q_w$.
6. Numerical simulations: Our proposed approach is supported.

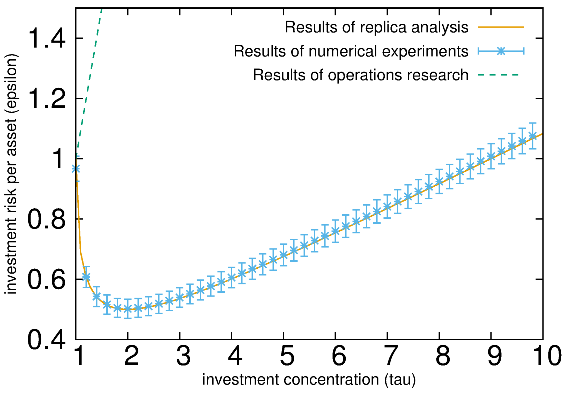

Figure 1 Minimal investment risk per asset ε at $\alpha = p/N = 2$ results from replica analysis (orange line), numerical simulation (sky-blue asterisks with error bars), and operations research approach (green dashed line) versus investment concentration τ. Results of replica analysis are consistent with the averages obtained from a numerical experiment with 100 samples and $N = 500$ assets.

7 Conclusion and the future works

In this talk, we have discussed the minimal investment risk per asset for a portfolio optimization problem with a budget constraint and an investment concentration constraint, using replica analysis, which was developed for and improved during interdisciplinary research. Unlike the minimal investment risk per asset and portfolio optimization problem with a budget constraint which has been discussed in previous work, we assessed quantitatively the deviation of the minimal investment risk per asset from the budget constraint only case caused by the inclusion of an investment concentration constraint. In contrast, the standard operations research approach failed to identify accurately the minimal investment risk of the portfolio optimization problem, since the obtained optimal portfolio only minimizes the expected investment risk, not the investment risk, making it clear that this approach cannot provide investors information about the optimal investment strategy.

As the future work, we need to improve and develop the model in order to be able to treat a more realistic depiction of the investment market. For instance, we need to analyze the portfolio optimization problem in an investment market comprising a risk-free asset and assets of varying risk levels. As alternative constraints to a budget constraint or an investment concentration constraint, we need to consider, for instance, an expected return constraint for the case that the return rate is not normalized and linear inequality constraints.

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