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On Three Topics of the Secretary Problem

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1 Introduction

In this note, we review three topics of the secretary problem that attracted attention of the researchers. These include

1. Robbins' problem
2. Ferguson secretary problem
3. PPS (Petruccelli-Porosinski-Samuels) paradox

As preparation for describing these topics, we briefly review the basic framework of the secretary problem.

1.1 Two informational models

In the no-information model, a known number n of objects appear one at a time in random order with all $n!$ permutations equally likely, and one of them must be chosen. If we could observe them all, we could rank them absolutely with no ties from best (rank 1) to worst (rank n) according to our own preference order. However, when an object appears, we can only observe the rank relative to its predecessors. Thus the decision to accept (select) or reject the object must be based on the relative ranks of all the objects observed so far. In contrast to the no-information model, the full-information model is the problem in which the observations are the true values of n objects $X_1, X_2, \ldots, X_n$, assumed to be i.i.d. random variables from a known continuous distribution F. We can assume without loss of generality that $X_1, X_2, \ldots, X_n$ are uniformly distributed on the interval (0, 1) for our study.

1.2 Two typical criteria of optimality

Two typical problems are the *best-choice problem* and the *rank minimization problem.* The objective of the best-choice problem is to find a stopping rule that maximizes the probability of selecting the best of all n objects and the corresponding probability of success. The objective of the rank minimization problem is to find a stopping rule that minimizes the expected (absolute) rank of the chosen object and the corresponding expected rank.

1.3 Best-choice problem

We call an object *candidate*, if it is relatively best. In the no-information model, there exists a positive integer r_n, defined as

$$r_n = \min\left\{r \geq 1 : \frac{1}{r} + \frac{1}{r+1} + \cdots + \frac{1}{n-1} \leq 1\right\},$$

for $n \geq 2$, such that the optimal rule is to reject the first $r_n - 1$ objects and then select the first candidate if any. The optimal probability is given by

$$v_n = \left(\frac{r_n - 1}{n}\right) \sum_{j=r_n}^{n} \frac{1}{j-1} \tag{1}$$

for $r_n \geq 1$. Asymptotically we have

$$\lim_{n\to\infty} \frac{r_n}{n} = \lim_{n\to\infty} v_n = e^{-1},$$

implying that, when n is large, a good approximation for r_n is $e^{-1}n \approx 0.368n$, so the optimal rule lets the fraction e^{-1} of all objects go by and then selects the first candidate. The optimal probability is then approximately $e^{-1} \approx 0.368$. In the full-model, Let $L_k = \max\left(X_1, \ldots, X_k\right), 1 \leq k \leq n$, and call the *kth* object (or X_k) *candidate* if it is a relative maximum, i.e. $X_k = L_k$. Consider a class of stopping rules of the form

$$\tau_n = \tau_n(\mathbf{a}) = \min\left\{k \geq 1 : X_k \geq a_k, X_k = L_k\right\},$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a given sequence of thresholds satisfying the monotone condition $1 \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. This rule is sometimes referred to as a *monotone rule* (with thresholds $\mathbf{a}$). Gilbert and Mosteller (1966) showed that the probability of choosing the best under a monotone rule $\tau_n(\mathbf{a})$ is calculated as

$$v_n(\mathbf{a}) = \frac{1 - a_1^n}{n} + \sum_{j=1}^{n-1} \left[\sum_{k=1}^{j} \frac{a_k^j}{j(n-j)} - \sum_{k=1}^{j} \frac{a_k^n}{n(n-j)} - \frac{a_{j+1}^n}{n}\right]$$

and that the optimal stopping rule is within the class of monotone rules and the particular thresholds $\mathbf{a}^* = (a_1^*, a_2^*, \ldots, a_n^*)$ specifys the optimal stopping rule if $a_n^* = 0$ and a_k^*, $k < n$, is a unique root $x \in (0,1)$ of the equation

$$\sum_{j=1}^{n-k} \frac{1}{j}\left(x^{-j} - 1\right) = 1.$$

If we introduce the exponential-integral functions

$$I(c) = \int_1^{\infty} \frac{e^{-cx}}{x} dx, \quad J(c) = \int_0^1 \frac{e^{cx} - 1}{x} dx = \sum_{j=1}^{\infty} \frac{c^j}{j!j}$$

and define $c^*(\approx 0.80435)$ as a solution c to the equation $J(c) = 1$, the limiting optimal probability was given by Samuels (1982) as

$$v^* = \lim_{n\to\infty} v_n^* = e^{-c^*} + (e^{c^*} - c^* - 1)I(c^*) \approx 0.580164. \tag{2}$$

Define, as a function of $c(> 0)$,

$$v_{GM}(c) = P\left(A_1 < c < A_2\right) \tag{3}$$

with

$$A_1 = E_1(1 - U_1), \quad A_2 = \left(E_1 + \frac{E_2}{U_1}\right)(1 - U_1U_2),$$

where E_1, E_2, U_1 and U_2 are independent random variables with E_1 and E_2 each exponentially distributed with parameter one and U_1 and U_2 each uniformly distributed on the interval $(0, 1)$ (see Section 4.5 for this argument). Then it can be shown that, after straightforward calculation,

$$v_{GM}(c) = \left(e^c - 1 - cJ(c)\right)I(c) + e^{-c}J(c) \tag{4}$$

and

$$v^* = v_{GM}(c^*). \tag{5}$$

1.4 Rank minimization problem

In the no-information model, there exists a non-decreasing sequence of thresholds $\{s_k, 1 \le k \le n\}$ such that the optimal rule is described as

$$\tau_n = \min\left\{k \ge 1 : R_k \le s_k\right\},$$

where R_k denotes the relative rank of the kth object. Equivalently, τ_n can be characterized by the non-decreasing cut-off points $\{t_k, 1 \le k \le n\}$ where $t_k = \min\{r \ge 1 : s_r \le k\}$, because τ_n stops with the rth object if $R_r \le k$ and $r \ge t_k$, provided that τ_n have not stopped before t_k. Such a rule is sometimes referred to as a *cut-off point* rule. The optimal rule asymptotically gives surprisingly good performance

$$\prod_{j=1}^{\infty}\left(\frac{j+2}{j}\right)^{\frac{1}{j+1}} \approx 3.8695.$$

2 Robbins' problem

The full-information version of the rank minimization problem is called Robbins' problem. Let $A_k, 1 \le k \le n$, be the (absolute) rank of the kth observation X_k. Then we have

$$A_k = \sum_{j=1}^{n} I(X_j \le X_k), \tag{6}$$

where we denote the indicator function of the event E by $I(E)$, and define the smaller observations to be the better ones for convenience (no generality is lost by doing so). The objective is to find a stopping rule τ^* that minimizes the expected rank

$$E[A_{\tau^*}] = \inf_{\tau} E[A_\tau] = V_n.$$

To examine V_n and $V = \lim_{n\to\infty} V_n$ is of interest. Robbins' problem is considered to be hard and not solved completely. What makes it hard is that the optimal stopping rule depends on the whole history of observations in the sense that the decision of either to stop or not at stage k depends on $X_1, \ldots, X_k$. Bruss and Ferguson(1993) considered the class of memoryless stopping rules

$$N = N(\mathbf{a}) = \min\{k \geq 1 : X_k \leq a_k\},$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a given sequence of thresholds with $0 \leq a_k \leq 1$ for $k < n$ and $a_n = 1$, and gave the corresponding expected rank

$$E[A_{N(\mathbf{a})}] = 1 + \frac{1}{2}\sum_{k=1}^{n}\left(\prod_{i=1}^{k-1}(1-a_i)\right)\left[(n-k)a_k^2 + \sum_{j=1}^{k-1}\frac{((a_k-a_j)^+)^2}{1-a_j}\right].$$

The sequence of thresholds that deserves special attention, suggested by the similarity to Moser's problem of minimizing the expectation of X_N, is $a_k = 2/(n-k+2)$ for which Bruss and Ferguson showed $V_n \leq E[A_{N(\mathbf{a})}] \leq 1 + 4(n-1)/3(n+1) \leq 7/3$ for all n and hence $V \leq 7/3$. To obtain a better upper bound for V, Assaf and Samuel-Cahn(1996) extended the above sequence to the class of sequences $a_k = a_k(c) = c/(n-k+c)$ for $c > 1$ and found that $c = 1.9469\ldots$ attains the minimal expected rank $2.3318\ldots$ among this class, i.e. $V \leq 2.3318$.

Let W be the limiting minimal expected rank among the class of memoryless rules. Assaf and Samuel-Cahn(1996) showed that $2.295 < W < 2.3267$ by elaborate analysis (note that the value $2.3267\ldots$ is obtained from the explicit fairure rate function that approximates the discrete thresholds a_k). Bruss and Ferguson(1993) gave an estimate $W = 2.32659$ by solving the system of equations $\partial E[A_{N(\mathbf{a})}]/\partial a_k = 0, 1 \leq k < n$, numerically. Though $V \leq W$, the essential question of either $V < W$ or $V = W$ is still unanswered.

To obtain a lower bound for V, Bruss and Ferguson(1993) modified the problem by changing the loss function (6) to

$$A_k^{(m)} = 1 + \min\left\{m,\ \sum_{j=1}^{k-1} I(X_j < X_k)\right\} + (n-k)X_k$$

for $m = 1, 2, \ldots$. The modified problem is in favour of the decision-maker, because the present loss (relative rank of X_k) is truncated by m i.e. rank

1 to m are counted as their value and any higher rank as $m+1$, and the future loss is replaced by its expectation $(n-k)X_k$ given $X_1, \ldots, X_k$ without loss of generality. Let $V_n^{(m)}$ be the minimal expected loss for the $A_k^{(m)}$. Then $V_n^{(m)}$, which can be shown to be non-decreasing in m and also in n, become lower bounds for V_n and hence for V. Let $V^{(m)} = \lim_{n\to\infty} V_n^{(m)}$. This truncation gives a considerable simplification for computation, yielding $V^{(1)} = 1.462, \ldots, V^{(5)} = 1.908$. Thus the best lower bound for V is known to be 1.908.

The interested reader is referred to Bruss(2005) for what is known about Robbins' problem.

3 Ferguson secretary problem

Martin Gardner's February 1960 column in *Scientific American*, described below as the game of googol, is considered to be the first appearance in print of a secretary problem.

(Googol) *Ask someone to take as many slips of paper as he pleases, and on each slip write a different positive number. The numbers may range from small fractions of one to a number the size of googol (1 followed by a hundred 0's) or even larger. These slips are turned face down and shuffled over the top of a table. One at a time you turn the slips face up. The aim is to stop turning when you come to the number that you guess to be the largest of the series. You cannot go back and pick up a previously turned slip. If you turn over all slips, then of course you must pick the last one turned.*

After reviewing various secretary problems studied so far, Ferguson (1989) returned again to googol and asked a question "Who solved the secretary problem?" His answer was "Nobody." Ferguson pointed out that googol is not the no-information best-choice problem, because the actual values of the numbers are revealed to the decision-maker (not only their relative ranks) and that googol is a two person zero sum game, because there is "someone" who may behave as an adversary, i.e. he may choose the numbers to make your decision of selecting the largest as difficult as possible.

Ferguson raised two questions. "First, can you guarantee a higher probability of selecting the largest number if you allow your decision rule to depend on the actual values of the numbers?" "Second, if you are told how this 'someone' is choosing the numbers to place on the slips, can you now guarantee a higher probability of selecting the largest number?" Ferguson also required that since the numbers are shuffled, they may as well be chosen to be an exchangeable sequence. The heart of the above problem, referred to as *Ferguson Secretary Problem*, was explicitly formulated by Samuels (1989) as follows:

Given n, either find an exchangeable sequence of continuous random variables, $X_1, X_2, \ldots, X_n$, for which, among all stopping rules, τ, based on the X's, $\sup_\tau P\{X_\tau = \max(X_1, X_2, \ldots, X_n)\}$ is achieved by a rule based only on the relative ranks of the X's-or prove that no such sequence exists.

Note that $\sup_\tau P\{X_\tau = \max(X_1, X_2, \ldots, X_n)\}$ is just v_n in (1). Continuity is assumed so that there will be no ties. Ferguson himself has come within *epsilon* of solving this problem. He has given, for each n and $\epsilon > 0$, exchangeable sequences such that the best rule based only on relative ranks has success probability within ϵ of the supremum. However, the question of whether this supremum can actually be attained still remained unsolved.

Samuels (1989) showed that the answer is negative for $n = 2$ by using Cover's argument. Silverman and Nadas(1992) showed that the answer is positive for $n = 3$. Gnedin (1994) finally solved the Ferguson secretary problem. In fact, he gave an affirmative answer for all $n > 2$ by showing that such an exchangeable sequence exists and its probability density function is given by

$$p(x_1, \ldots, x_n) = \begin{cases} \dfrac{\epsilon}{2n}[\max(x_1, \ldots, x_n)]^{-n+\epsilon}, & 0 < \max(x_1, \ldots, x_n) < 1 \\ \dfrac{\epsilon}{2n}[\max(x_1, \ldots, x_n)]^{-n-\epsilon}, & \max(x_1, \ldots, x_n) > 1 \end{cases}$$

for sufficiently small ϵ.

Beyond the best-choice problem, Gnedin and Krengel (1996) considered more general problem of minimizing the expected loss. That is, when the loss $q(i)$ is incurred if we stop with the *ith* best of all n objects, they beged a question of whether, for given $\{q(i)\}$, there exists an exchangeable sequence of continuous random variables, $X_1, X_2, \ldots, X_n$, for which the observation of the values of the X_i's gives no advantage over the observation of just the relative ranks of the variables.

4 PPS paradox

The coincidence of the asymptptic values of the quite different discrete time optimal stopping problems is called PPS (Petruccelli-Porosinski-Samuels) paradox according to Gnedin (2004). He gave a unified approach to give a resolution to this paradox.

4.1 Petruccelli problem

Petruccelli (1980) considered a best-choice problem in which the observations are iid uniform on the unit interval $(\theta - \frac{1}{2}, \theta + \frac{1}{2})$ with θ unknown.

The search for a minimax stopping rule (i.e. a best invariant rule) is equivalently reduced to the partial-information problem in which we observe, at each stage, the *range* as well as whether or not the current observation is a relative maximum, given that the observations $X_1, \ldots, X_n$ are iid uniform on $(0,1)$. Let $R_k = \max(X_1, \ldots, X_k) - \min(X_1, \ldots, X_k)$ be range at stage k. Then the minimax rule is described as

$$\tau_n(\mathbf{a}) = \min\{k \geq 1 : R_k \geq a_k,\ X_k = \max(X_1, \ldots, X_k)\},$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a monotone sequence of thresholds with a_k for $k < n$ being a unique root $x \in [0,1]$ of the equation

$$\sum_{i=1}^{n-k} \frac{1}{i} \binom{n-k}{i} \int_x^1 v^{n-k-i}(1-v)^i dv = \int_x^1 v^{n-k} dv.$$

The limiting probability, as $n \to \infty$, of choosing the largest of $X_1, \ldots, X_n$ denoted by v_{PET} is given by

$$v_{PET} = (e^{-c_p} - c_p I(c_p))J(c_p) + (e^{c_p} - 1)I(c_p) \approx 0.4352, \tag{7}$$

where $c_p \approx 2.1198$ is a unique root c of the equation

$$1 + J(-c) = e^{-c}(1 - J(c)). \tag{8}$$

4.2 Porosinski problem

A natural extension of the basic problems is to allow the number of objects to be random. Let N denote a random variable representing the number of objects and assume N to be independent of the arrival order of the objects and all else. The objective is to select the best of all N objects. When N is uniform on $\{1, 2, \ldots, n\}$ in the full-information model, this problem is referred to as Porosinski problem. The optimal rule of Porosinski problem is described as

$$\tau_n(\mathbf{a}) = \min\{k \geq 1 : X_k \geq a_k,\ X_k = \max(X_1, \ldots, X_k)\},$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a monotone sequence of thresholds with a_k for $k < n$ being a unique root $x \in [0,1]$ of the equation

$$\sum_{i=0}^{n-k} x^i = \sum_{i=0}^{n-k-1} x^i \sum_{j=1}^{n-k-i} (1 - x^j)/j.$$

The limiting probability, as $n \to \infty$, of choosing the largest of $X_1, \ldots, X_N$ denoted by v_{POR} is given by

$$v_{POR} = (e^{-c_p} - c_p I(c_p))J(c_p) + (e^{c_p} - 1)I(c_p) \approx 0.4352, \tag{9}$$

where $c_p \approx 2.1198$ is as defined by (8). This was derived by Porosinski (2002) in a different but equivalent expression.

Comparing (9) to (7), Porosinski (2002) noticed an unexpected coincidence $v_{POR} = v_{PET}$ between two quite different best-choice problems. However, Samuels (2004) pointed out that (9) is in effect correct but the argument to derive it is not. Samuels also said that his mistake nevertheless reveals a new way of proving $v_{POR} = v_{PET}$ by showing that each is equal to $v_{GM}(c_p)$ (see (3) for $v_{GM}(c)$). These coincidence will be confirmed using PPP (planar Poisson process) model in Section 5.

4.3 Duration problem

As a version of the secretary problem, Ferguson et al. (1992) considered the *duration problem*, in which the objective is to maximize the time of possession of a relatively best object (i.e. candidate). We only select a candidate, receiving a payoff of 1 plus the number of future observations before a new candidate appears or until the final stage n is reached. Define T_k as the arrival time of the first candidate after time k if there is one, and as $n+1$ if there is none. Then the payoff earned by possessing a candidate selected at time k is $(T_k - k)/n$ (division by n is for normarization).
The optimal rule of thef ull-information model is described as

$$\tau_n(\mathbf{a}) = \min\{k \geq 1 : X_k \geq a_k,\ X_k = \max(X_1, \ldots, X_k)\},$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a monotone sequence of thresholds with a_k for $k < n$ being a unique root $x \in [0,1]$ of the equation

$$\sum_{i=0}^{n-k} (h_{n-k-i} - h_i - 1)\, x^i = 0$$

where $h_k = \sum_{j=1}^{k} 1/j,\ k \geq 1$ and $h_0 = 0$. The optimal payoff is

$$v_n = \frac{1}{n}\left[h_n + \sum_{k=1}^{n}\sum_{j=k}^{n} \frac{1}{j}\,(h_{n-j} - h_{j-k} - 1)\, a_k^j\right].$$

Moreover, the limiting value $v_{DUR} = \lim_{n\to\infty} v_n$ is given by

$$v_{DUR} = (e^{-c_p} - c_p I(c_p))J(c_p) + (e^{c_p} - 1)I(c_p) \approx 0.4352, \tag{10}$$

where $c_p \approx 2.1198$ is as defined by (8). See Samuels (2004), Gnedin (2004) and Mazalov and Tamaki (2006) for (10). Look (10)! We have another coincidence $v_{POR} = v_{DUR}$.

4.4 Planar Poisson process model

The planar Poisson process (PPP) model is widely known to be an appropriate setting in which we can define the infinite version of the corresponding finite problems. Gnedin (1996) used PPP to study the full-information best-choice problem posed by Gilbert and Mosteller (1966). Further applicability to other full-information problems, e.g. Petruccelli problem in Section 4.1, Porosinski problem in Section 4.2, or duration problem in Section 4.3 was demonstrated by Samuels (2004). We use the PPP developed by Samuels (see Section 9, 2004) and show how to use this PPP to solve the full-information best-choice problem of Gilbert and Mosteller (1966). The semi-infinite strip can be scanned from left to right by shifting a vertical detector and the scanning can be stopped each time a point in the PPP, referred to as atom, is detected. Let $P(t,y)$ denote the probability of success if we choose the point (t,y) in the PPP, i.e., we stop at time t with a relatively best atom having value y. Then, introducing Poisson probability $Pois(k,\lambda)=e^{-\lambda}\frac{\lambda^k}{k!}$, we have $P(t,y)=Pois(0,y(1-t))$. On the other hand, let $Q(t,y)$ denote the probability of success if we do not choose the point (t,y), but instead choose the point related to the next relatively best atom, if any, then $Q(t,y)=\sum_{j=1}^{\infty}\frac{1}{j}Pois(j,y(1-t))$. Solving for the locus of point (t,y) at which $P(t,y)=Q(t,y)$ yields $y(1-t)=c^*$ for $c^*\approx 0.80435$ given as a root c of $J(c)=1$ in Section 1.3. Moreover, since $P(t,y)\geq Q(t,y)$ implies that $P(t',y')\geq Q(t',y')$ for $t'>t$, $y'<y$, we are in the *monotone* case of optimal stopping and can conclude that the optimal rule stops with the first relatively best atom, if any, that lies below the threshold curve $y=c^*/(1-t)$.

Let T be the arrival time of the first (leftmost) atom that lies below the threshold curve $y=c^*/(1-t)$ and S the time when the value of the best (lowest) atom above threshold is now equal to the threshold. Then, T and S are independent and their distributions are given by

$$P\{T>t\}=Pois\left(0,\int_0^t g(r)dr\right),\quad P\{S>s\}=Pois\left(0,\int_0^s (g(s)-g(r))dr\right),$$

where $g(r)=c^*/(1-r), 0<r<1$. From the form of the optimal rule, the optimal success probability can be calculated as

$$\begin{aligned} v^* &= \int_0^1\int_0^t P\left(s,\frac{c^*}{1-s}\right)f_S(s)f_T(t)dsdt \\ &\quad +\int_0^1\int_0^s\left[\frac{1-t}{c^*}\int_0^{c^*/(1-t)}P(t,y)dy\right]f_T(t)f_S(s)dtds, \end{aligned}$$

where $f_T(t)$ and $f_S(s)$ are the densities of T and S respectively. The straightforward calculation yields (2).

Samuels (2004) called this derivation of v^* a forward-looking argument. In addition, he gave a backward-looking argument introducing a threshold curve $y = c/(1-t)$ for any $c > 0$ (c is taken as c^* in the forward-looking argument). Let U_1 be the time of the best arrival and E_1 be its value. Then the arrival time and value of the best previous arrival are of the form U_1U_2 and $E_1 + E_2/U_1$ respectively from the PPP property, where E_1, E_2, U_1 and U_2 are independent random variables with E_1 and E_2 each exponentially distributed with parameter one and U_1 and U_2 each uniformly distributed on the interval $(0,1)$. Since we can choose the best if the best arrival is below the thereshold and the best previous arrival is above, the corresponding probability is expressed as

$$P\left(E_1 < \frac{c}{1-U_1} \text{ and } E_1 + \frac{E_2}{U_1} > \frac{c}{1-U_1U_2}\right),$$

which can be written as $v_{GM}(c)$ in (3), yielding (4) and (5).

4.5 Fourfold coincidence

Corresponding to Porosinski problem in Section 4.2 and Petruccelli problem in Section 4.1, we can define two functions of $c(>0)$, analogous to (3),

$$\begin{aligned} v_{POR}(c) &= P\left(B_1 < c < B_2\right), \\ v_{PET}(c) &= P\left(C_1 < c < C_2\right) \end{aligned}$$

with

$$\begin{aligned} B_1 &= \frac{E_1}{U_1}(1-U_1U_2), \quad B_2 = \left(\frac{E_1}{U_1} + \frac{E_2}{U_1U_2}\right)(1-U_1U_2U_3), \\ C_1 &= \left(E_1 + \frac{E_2}{U_1}\right)(1-U_1), \quad C_2 = \left(E_1 + \frac{E_2}{U_1} + \frac{E_3}{U_1U_2}\right)(1-U_1U_2) \end{aligned}$$

where E_1, E_2, E_3, U_1, U_2 and U_3 are independent random variables with E_1, E_2 and E_3 each exponentially distributed with parameter one and U_1, U_2 and U_3 each uniformly distributed on the interval $(0,1)$. Samuels (2004) then showed that

$$\begin{aligned} v_{POR} &= v_{POR}(c_p), \\ v_{PET} &= v_{PET}(c_p), \end{aligned}$$

where c_p is as defined by (8).

Remember the coincidence $v_{POR} = v_{PET}$ discovered by Porosinski (2002) in Section 4.2. Why his argument was incorrect is that he erroneously took $v_{GM}(c_p)$ as v_{POR}. He should have computed $v_{POR}(c_p)$ for v_{POR} instead. Samuels (2004) correctly proved this coincidence by showing that $v_{POR}(c_p) = v_{GM}(c_p)$ and $v_{PET}(c_p) = v_{GM}(c_p)$ (i.e. $v_{GM}(c_p)$ plays a role of mediator). Including v_{DUR}, we have fourfold coincidence $v_{POR} = v_{PET} = v_{DUR} = v_{GM}$, where $v_{GM} = v_{GM}(c_p)$.

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