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Mass transportation methods in functional inequalities and a new family of sharp constrained Sobolev inequalities

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Abstract

In recent decades, developments in the theory of mass transportation have led to proofs of many sharp functional inequalities. We present some of these results, including ones due to F. Maggi and the author, and discuss related open problems.

1 Sobolev inequalities and mass transportation methods

Sobolev inequalities are among the most fundamental tools in analysis and geometry. Determining the value of the corresponding sharp constants and characterizing the associated extremal functions in these inequalities often provides interesting geometric information. For instance, for the Sobolev inequality on $\mathbb{R}^n$ for $n \geq 2$, which says that

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \geq S\|u\|_{L^{p^\star}(\mathbb{R}^n)}, \qquad p^\star = \frac{np}{n-p}, \tag{1.1}$$

for any $1 \leq p < n$ and $u \in \dot{W}^{1,p}(\mathbb{R}^n)$, the sharp constant and extremals played a crucial role in the solution of the Yamabe problem in Riemannian geometry (see [Yam60, Tru68, Aub76a, Sch84] and the survey [LP87]). It was with this motivation that Aubin showed in [Aub76b] (see also [Tal76]) that all equality cases in (1.1) are given by translations, dilations, and constant multiples of the function

$$U_S(x) = \frac{1}{(1+|x|^{p'})^{(n-p)/p}}. \tag{1.2}$$

In the Yamabe problem on manifolds with boundary, the role of (1.1) is played by the Sobolev trace inequality on the half-space $H = \{x_1 > 0\} \subset \mathbb{R}^n$ for $n \geq 2$, which states that

$$\|\nabla u\|_{L^p(H)} \geq E\|u\|_{L^{p^\sharp}(\partial H)}, \qquad p^\sharp = \frac{(n-1)p}{n-p}, \tag{1.3}$$

for all $u \in \dot{W}^{1,p}(H)$ and $1 \le p < n$. In order to address this problem in [Esc92], Escobar showed in [Esc88] that when $p = 2$, all extremal functions in (1.3) are given by the function $U_E(x) = |x + \mathrm{e}_1|^{2-n}$ and its invariant scalings, that is, the functions $c\,U_E(\lambda(x+y))$ for $c \in \mathbb{R}$, $\lambda \in \mathbb{R}_+$, and $y \in H$. Beckner gave another proof in [Bec93]. Both proofs, though different in nature, crucially exploit the conformal invariance that is specific to the case $p = 2$.

For the case $1 < p < n$, it was not until [Naz06] that the function

$$U_E(x) = \frac{1}{|x + \mathrm{e}_1|^{(n-p)/(p-1)}} \,. \tag{1.4}$$

and its invariant scalings $c\,U_E(\lambda(x+y))$ for $c \in \mathbb{R}$, $\lambda \in \mathbb{R}_+$, and $y \in H$ were shown to be extremals of (1.3), confirming a conjecture of Escobar in [Esc88].[1] Nazaret's proof was based on a new iteration of a mass transportation argument introduced in [CENV04], where Cordero-Erausquin, Nazaret, and Villani gave another proof of (1.1). This argument of [CENV04] is also the starting point of the proof of the results presented in the Section 2, so we briefly sketch it here.

Suppose that $f, g \in W^{1,p}(\mathbb{R}^n) \cap C_c^\infty(\mathbb{R}^n)$ are nonnegative and are normalized so that

$$\|f\|_{L^{p^\star}(\mathbb{R}^n)} = \|g\|_{L^{p^\star}(\mathbb{R}^n)} = 1 \,.$$

The Brenier-McCann theorem (see [Bre91, McC97] or [Vil03, Cor. 2.30]) from the theory of optimal transportation provides a map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that

- T is the gradient of a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$; and
- for every Borel measurable function $\xi : \mathbb{R}^n \to [0, \infty]$, we have

$$\int_{\mathbb{R}^n} \xi \, g^{p^\star} \, dx = \int_{\mathbb{R}^n} \xi \circ T \, f^{p^\star} \, dx \,. \tag{1.5}$$

One can show that φ solves the Monge-Ampère equation

$$f^{p^\star}(x) = g^{p^\star}(\nabla\varphi(x)) \det \nabla^2 \varphi(x) \qquad \text{a.e. on } \operatorname{spt}(f^{p^\star}\, dx)\,; \tag{1.6}$$

see [McC97] or [Vil03, Theorem 4.8]. So, using (1.5) and (1.6), we find that

$$\int_{\mathbb{R}^n} g^{p^\sharp} \, dx = \int_{\mathbb{R}^n} (g^{p^\star} \circ T)^{-1/n} f^{p^\star} \, dx = \int_{\mathbb{R}^n} (\det \nabla^2 \varphi)^{1/n} f^{p^\sharp} \, dx \,.$$

Since φ is convex, the eigenvalues of $\nabla^2\varphi$ are nonnegative and the arithmetic-geometric mean inequality implies that

$$\int_{\mathbb{R}^n} (\det \nabla^2 \varphi)^{1/n} f^{p^\sharp} \, dx \le \frac{1}{n} \int_{\mathbb{R}^n} \Delta\varphi \, f^{p^\sharp} \, dx \,.$$

[1] The uniqueness of this family of extremals was left open in [Naz06], but was shown in [MN16].

Then, integrating by parts and applying Hölder's inequality, we find that

$$\begin{aligned}\frac{1}{n}\int_{\mathbb{R}^n}\Delta\varphi\, f^{p^\sharp}\,dx &= -\frac{p^\sharp}{n}\int f^{p^\sharp-1}\nabla f\cdot\nabla\varphi\,dx\\ &\le \frac{p^\sharp}{n}\,\|\nabla f\|_{L^p(\mathbb{R}^n)}\Big(\int_{\mathbb{R}^n}|\nabla\varphi|^{p'}f^{p^\star}\,dx\Big)^{1/p'}.\end{aligned}$$

Finally, by (1.5),

$$\int_{\mathbb{R}^n}|\nabla\varphi|^{p'}f^{p^\star}\,dx = \int_{\mathbb{R}^n}|x|^{p'}g^{p^\star}\,dx\,.$$

We have thus shown that

$$\frac{n}{p^\sharp}\int_{\mathbb{R}^n}g^{p^\sharp}\,dx\Big(\int_{\mathbb{R}^n}|x|^{p'}g^{p^\star}\,dx\Big)^{-1/p'} \le \|\nabla f\|_{L^p(\mathbb{R}^n)}\,.$$

Choosing g to be an extremal function in the Sobolev inequality, we find that the left-hand side is equal to $\|\nabla g\|_{L^p(\mathbb{R}^n)} = S$, completing the proof.

In the past two decades, the theory of optimal transport has led to the proofs of a number of other sharp functional inequalities; see, for instance, [McC97, Bar98, OV00, CEGH04, CE02, AGK04, MV05, MV08, LP09, Cas10, Ngu15] and the monograph [Vil03].

2 A new family of sharp constrained Sobolev inequalities

Recently, in [MN16], F. Maggi and the author used a variation of the mass transportation argument of [CENV04] sketched in Section 1 to prove a new family of sharp constrained Sobolev inequalities on a half-space along with a characterization of the equality cases. This family of inequalities provides a strong link between the Sobolev inequality (1.1) and Sobolev trace inequality (1.3). To explain this link, let us observe that the Sobolev inequality (1.1) is equivalent to the variational problem

$$\inf\big\{\|\nabla u\|_{L^p(\mathbb{R}^n)}\,:\,\|u\|_{L^{p^\star}(\mathbb{R}^n)}=1\big\} = S\,.$$

Considering the analogous variational problem on H, that is,

$$\inf\{\|\nabla u\|_{L^p(H)}\,:\,\|u\|_{L^{p^\star}(H)}=1,\ u\equiv 0 \text{ on } \partial H\} = \tilde S \tag{2.1}$$

we immediately see that $\tilde S \ge S$ by extending any competitor in (2.1) by zero to be defined on $\mathbb{R}^n$. On the other hand, taking a sequence of scalings of U_S concentrating in H and multiplied by cut-off functions, we find that $\tilde S \le S$. Hence $\tilde S = S$, and in this way, we see that the variational problem (2.1) is essentially equivalent to the Sobolev inequality (1.1). Similarly, the Sobolev trace inequality (1.3) can be equivalently expressed as the variational problem

$$\inf\{\|\nabla u\|_{L^p(H)} \, : \, \|u\|_{L^{p^\sharp}(H)} = 1\} = E\,. \tag{2.2}$$

A key observation, based on a simple scaling argument, is that there is a constant $T_E > 0$ such that $\|u\|_{L^{p^\sharp}(\partial H)} = T_E \|u\|_{L^{p^*}(\partial H)}$ for all functions attaining equality in (1.3), that is, for all invariant scalings of (1.4). In particular, the infimum and minimizing functions in (2.2) are left unchanged by adding the additional constraint $\|u\|_{L^{p^*}(H)} = 1/T_E$. Or, up to multiplying by the constant T_E, the following variational problem is equivalent to (2.2):

$$\inf\{\|\nabla u\|_{L^p(H)} \, : \|u\|_{L^{p^*}(H)} = 1, \ \|u\|_{L^{p^\sharp}(\partial H)} = T_E\} = E\, T_E\,. \tag{2.3}$$

In view of (2.1) and (2.3), we see that the Sobolev inequality and the Sobolev trace inequality arise as the particular cases $T = 0$ and $T = T_E$ of the more general variational problem

$$\Phi(T) = \inf\{\|\nabla u\|_{L^p(H)} : \|u\|_{L^{p^*}(H)} = 1, \ \|u\|_{L^{p^\sharp}(\partial H)} = T\} \qquad T \geq 0\,. \tag{2.4}$$

Our main result consists of characterizing minimizers in (2.4) for every $T > 0$ and every $1 < p < n$, and then using this knowledge to give a qualitative description of the behavior of the infimum value $\Phi(T)$ as a function of T. The characterization result involves the following three families of functions:

Sobolev family: Let U_S be defined as in (1.2) and set, for every $t \in \mathbb{R}$,

$$U_{S,t}(x) = \frac{U_S(x - t\,\mathrm{e}_1)}{\|U_S(\mathrm{id} - t\,\mathrm{e}_1)\|_{L^{p^*}(H)}} \qquad x \in H\,,$$

and

$$T_S(t) = \|U_{S,t}\|_{L^{p^\sharp}(\partial H)}\,, \qquad G_S(t) = \|\nabla U_{S,t}\|_{L^p(H)}\,.$$

Thus, $U_{S,t}$ corresponds to translating the optimal function U_S in the Sobolev inequality so that its maximum point lies at signed distance t from ∂H, then multiplying the translated function by a constant factor to normalize the L^{p^*}-norm in H to be 1.

Escobar family: Letting U_E be as in (1.4), we set for every $t < 0$

$$U_{E,t}(x) = \frac{U_E(x - t\,\mathrm{e}_1)}{\|U_E(\mathrm{id} - t\,\mathrm{e}_1)\|_{L^{p^*}(H)}} \qquad x \in H\,.$$

As pointed out above, a simple computation shows that the trace and gradient norms of the $U_{E,t}$ are independent of $t < 0$, and we set

$$\|U_{E,t}\|_{L^{p^\sharp}(\partial H)} = T_E\,, \qquad \|\nabla U_{E,t}\|_{L^p(H)} = G_E$$

for these constant values. Each function $U_{E,t}$ is thus obtained by centering the fundamental solution of the p-Laplacian outside of H, and then by normalizing its $L^{p^\star}$-norm in H.

Beyond-Escobar family: We consider the function

$$U_{BE}(x) = (|x|^{p'} - 1)^{(p-n)/p} \qquad |x| > 1\,, \tag{2.5}$$

and define, for every $t < -1$,

$$U_{BE,t}(x) = \frac{U_{BE}(x - t\,\mathrm{e}_1)}{\|U_{BE}(\mathrm{id} - t\,\mathrm{e}_1)\|_{L^{p^\star}(H)}} \qquad x \in H\,.$$

Correspondingly, for every $t < -1$, we set

$$T_{BE}(t) = \|U_{BE,t}\|_{L^{p^\sharp}(\partial H)}\,, \qquad G_{BE}(t) = \|\nabla U_{BE,t}\|_{L^p(H)}\,.$$

As the name of this family of functions suggests, we show that $T_{BE}(t) > T_E$ for every $t < -1$, so that $\{U_{BE}(t)\}_{t<-1}$ enters the description of $\Phi(T)$ for $T > T_E$. Notice that (2.5) defines a function on the complement of the unit ball. The function $U_{BE,t}$ is thus obtained by centering this unit ball *outside* of H, precisely at distance $|t|$ from ∂H, and then by normalizing its tail to have unit $L^{p^\star}$-norm in H.

Theorem 2.1 (Existence and Characterization of Minimizers). *Let $n \geq 2$ and $p \in (1, n)$. For every $T \in (0, +\infty)$, a minimizer exists in the variational problem* (2.4) *and is unique up to dilations and translations orthogonal to* e_1. *More precisely:*

(i) for every $T \in (0, T_E)$, there exists a unique $t \in \mathbb{R}$ such that

$$T = T_S(t)\,, \qquad \Phi(T) = G_S(t)\,,$$

and $U_{S,t}$ is the uniquely minimizer in (2.4) *up to dilations and translations orthogonal to* e_1*;*

(ii) if $T = T_E$, then, up to dilations and translations orthogonal to e_1, $\{U_{E,t} : t < 0\}$ is the unique family of minimizers of (2.4)*;*

(iii) for every $T \in (T_E, +\infty)$ there exists a unique $t < -1$ such that

$$T = T_{BE}(t)\,, \qquad \Phi(T) = G_{BE}(t)\,, \tag{2.6}$$

and $U_{BE,t}$ is the unique minimizer of (2.4) *up to dilations and translations orthogonal to* e_1.

As a consequence of Theorem 2.1, we obtain the following family of sharp constrainted Sobolev inequalities and a characterization of their extremals: for any $0 < T < \infty$,

$$\|\nabla u\|_{L^p(H)} \geq \Phi(T)\|u\|_{L^{p^\star}(H)} \tag{2.7}$$

for all $u \in \dot{W}^{1,p}(H)$ with $\|u\|_{L^{p^\sharp}(\partial H)}/\|u\|_{L^{p^\star}(H)} = T$.

Theorem 2.1 also provides an implicit description of this sharp constant Φ as a function of T on $[0,\infty)$. Starting from this characterization, we obtain a quite complete picture of its properties in Theorem 2.2 below. First, let us collect the previously known information about $\Phi(T)$. As we have seen, $\Phi(0) = S$ by (1.1), while (1.3) yields the linear lower bound

$$\Phi(T) \geq E\,T \qquad \forall T \geq 0\,, \tag{2.8}$$

with equality if $T = T_E$. Another piece of information comes from the following "gradient domain" inequality, which follows by applying (1.1) to the extension by reflection of u to $\mathbb{R}^n$:

$$\|\nabla u\|_{L^p(H)} \geq 2^{-1/n}\, S\, \|u\|_{L^{p^\star}(H)}\,,$$

with equality if and only if u is a dilation or translation orthogonal to e_1 of U_S. So, this inequality implies that

$$\Phi(T) \geq 2^{-1/n}\, S\,, \qquad \forall T \geq 0 \tag{2.9}$$

with equality if and only if $T = T_0$ where $T_0 = \|U_S\|_{L^{p^\sharp}(\partial H)}/\|U_S\|_{L^{p^\star}(H)}$. Next, a simple application of the divergence theorem and Hölder's inequality imply that for every non-negative u that is admissible in (2.4), we have

$$\int_{\partial H} u^{p^\sharp}\, d\mathcal{H}^{n-1} < p^\sharp \|\nabla u\|_{L^p(H)}\, \|u\|_{L^{p^\star}(H)}^{p^\star/p'}\,.$$

As a consequence, we find that

$$\Phi(T) > \frac{T^{p^\sharp}}{p^\sharp} \qquad \forall T > 0\,. \tag{2.10}$$

Finally, given any open connected Lipschitz set $\Omega \subset \mathbb{R}^n$, let us set

$$\Phi_\Omega(T) = \inf\left\{\|\nabla u\|_{L^p(\Omega)} : \|u\|_{L^{p^\star}(\Omega)} = 1\,, \|u\|_{L^{p^\sharp}(\partial\Omega)} = T\right\} \qquad T \geq 0\,,$$

(so that $\Phi_H = \Phi$), and define $\mathrm{ISO}\,(\Omega) = P(\Omega)/|\Omega|^{(n-1)/n}$, where $P(\Omega)$ and $|\Omega|$ denote the perimeter and volume of Ω. With this notation, the Euclidean isoperimetric inequality takes the form $\mathrm{ISO}\,(\Omega) \geq \mathrm{ISO}\,(B_1)$, with equality if and only if Ω is a translation or dilation of the unit ball B_1. The following *trace-Sobolev comparison theorem* was proved in [MV05]:

$$\Phi_\Omega(T) \geq \Phi_{B_1}(T)\,, \qquad \forall T \in \left[0, \mathrm{ISO}\,(B_1)^{1/p^\sharp}\right] \tag{2.11}$$

for any open Lipschitz domain Ω. It was also shown in the same paper that Φ_{B_1} is strictly concave and decreasing on $[0, \mathrm{ISO}\,(B_1)^{1/p^\sharp}]$. Applying (2.11) with $\Omega = H$ provides an additional lower bound on $\Phi(T)$ on the interval $[0, \mathrm{ISO}\,(B_1)^{1/p^\sharp}]$.

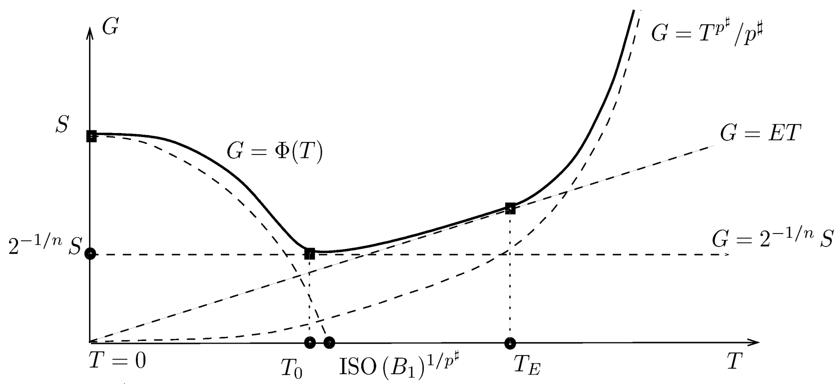

Figure 1: A qualitative picture of Theorem 2.2.

Theorem 2.2 (Properties of $\Phi(T)$). *Let $n \geq 2$ and $p \in (1, n)$. Then $\Phi(T)$ is differentiable on $(0, \infty)$, it is strictly decreasing on $(0, T_0)$ with $\Phi(0) = S$ and $\Phi(T_0) = 2^{-1/n}S$ and strictly increasing on (T_0, ∞) with*

$$\lim_{T \to +\infty} \Phi(T) = \frac{T^{p^\sharp}}{p^\sharp} + o(1) \qquad \text{as } T \to \infty\,. \tag{2.12}$$

Moreover, $\Phi(T)$ is strictly convex on $(T_0, +\infty)$, and there exists $T_* \in (0, T_0)$ such that $\Phi(T)$ is strictly concave on $(0, T_*)$.

Theorem 2.2 and the information gathered from (2.8) through (2.11) is summarized in Figure 1. In terms of the sharp constrained Sobolev inequalities (2.7), Figure 1 can be interpreted in the following way: for any function $u \in W^{1,p}(H)$, the point

$$\left(\frac{\|u\|_{L^{p^\sharp}(\partial H)}}{\|u\|_{L^{p^\star}(H)}}, \frac{\|\nabla u\|_{L^p(H)})}{\|u\|_{L^{p^\star}(H)}} \right)$$

plotted in the (T, G)-plane must lie above the curve $G = \Phi(T)$. We note that the information provided by (2.8) and (2.9) is sharp at only the single points $T = T_E$ and $T = T_0$ respectively. Furthermore, the bound given by (2.11) is only sharp at $T = 0$. Finally, we see from (2.12) that the lower bound (2.10) is saturated asymptotically as $T \to \infty$.

3 Remarks and open problems

The minimization problem (2.4) was first considered by Carlen and Loss for the case $p = 2$ in [CL94]. There, they used the method of competing symmetries developed in [CL90b, CL90a, CL92] to give a characterization of the minimizers for all values of $T > 0$. Hence, Theorem 2.1 can be seen as a generalization of [CL94] from the case

$p = 2$ to the full range $p \in (1, n)$. This method, like the results of [Esc88] and [Bec93] characterizing extremals in (1.3), relies in an essential way on the conformal invariance that is present only in the case $p = 2$. Outside of this case, these methods cannot be applied, and the mass transportation methods employed in our proof of Theorem 2.1 provide an effective approach.

As discussed in Section 2, a related variational problem is

$$\Phi_\Omega(T) = \inf\{\|\nabla u\|_{L^p(\Omega)} : \|u\|_{L^{p^*}(\Omega)} = 1,\ \|u\|_{L^{p^\sharp}(\partial\Omega)} = T\} \qquad T \geq 0 \tag{3.1}$$

for any suitably regular open domain $\Omega \subset \mathbb{R}^n$. The main result of [MV05] is the trace-Sobolev comparison theorem (2.11). Theorem 2.1 can be seen as a complementary trace-Sobolev comparison theorem, providing an upper bound on $\Phi_\Omega(T)$ for any open Lipschitz domain Ω:

Corollary 3.1 (Half-spaces have the best Sobolev inequalities). *If Ω is a non-empty open set with Lipschitz boundary on $\mathbb{R}^n$, then*

$$\Phi_\Omega(T) \leq \Phi(T) \qquad \forall T \geq 0\,.$$

Notice that, while Corollary 3.1 provides an upper bound for any $T \geq 0$, the lower bound (2.11) cannot hold on a larger interval. Indeed, $\Phi_{B_1}(T) > 0$ for $T \neq \mathrm{ISO}\,(B_1)^{1/p^\sharp}$. Hence, if Ω is not a ball and thus $\mathrm{ISO}\,(\Omega) > \mathrm{ISO}\,(B_1)$, then

$$\Phi_{B_1}(\mathrm{ISO}\,(\Omega)^{1/p^\sharp}) > 0 = \Phi_\Omega(\mathrm{ISO}\,(\Omega)^{1/p^\sharp})\,.$$

A number of open questions remain. First of all, what is the behavior of $\Phi_{B_1}(T)$ for $T > \mathrm{ISO}\,(B_1)^{1/p^\sharp}$? Are there minimizers in the variational problem for this range of T, and if so, can one characterize these minimizers? For more general domains Ω, can one find lower bounds on $\Phi_\Omega(T)$ for $T > \mathrm{ISO}\,(\Omega)^{1/p^\sharp}$?

In [MV05], it is also shown that the trace-Sobolev comparison theorem (2.11) holds with strict inequality if the variational problem (3.1) admits a minimizer and the domain Ω is connected and not homothetic to B_1. However, the problem of existence of of minimizers in (3.1) for $T > 0$ is open, to the author's knowledge, even for the case $p = 2$. We remark that mass transportation methods may be too rigid for a general domain and it would likely be more effective to prove existence using a concentration compactness argument as in [Lio85].

That said, it would also be interesting to see if there are domains other than the half-space and the ball for which one could give an explicit description of minimizers. For instance, in [MV08], mass transportation methods were used to prove Sobolev inequalities on a certain class of cones. It is natural to ask whether one could prove an analogue of Theorem 2.1 for these domains.

References

[AGK04] M. Agueh, N. Ghoussoub, and X. Kang. Geometric inequalities via a general comparison principle for interacting gases. *Geom. Funct. Anal.*, 14(1):215–244, 2004.

[Aub76a] T. Aubin. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl. (9)*, 55(3):269–296, 1976.

[Aub76b] T. Aubin. Problèmes isopérimétriques et espaces de Sobolev. *J. Differential Geom.*, 11(4):573–598, 1976.

[Bar98] F. Barthe. On a reverse form of the Brascamp-Lieb inequality. *Invent. Math.*, 134(2):335–361, 1998.

[Bec93] W. Beckner. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. *Ann. of Math. (2)*, 138(1):213–242, 1993.

[Bre91] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4):375–417, 1991.

[Cas10] P. Castillon. Submanifolds, isoperimetric inequalities and optimal transportation. *J. Funct. Anal.*, 259(1):79–103, 2010.

[CE02] D. Cordero-Erausquin. Some applications of mass transport to Gaussian-type inequalities. *Arch. Ration. Mech. Anal.*, 161(3):257–269, 2002.

[CEGH04] D. Cordero-Erausquin, W. Gangbo, and C. Houdré. Inequalities for generalized entropy and optimal transportation. In *Recent advances in the theory and applications of mass transport*, volume 353 of *Contemp. Math.*, pages 73–94. Amer. Math. Soc., Providence, RI, 2004.

[CENV04] D. Cordero-Erausquin, B. Nazaret, and C. Villani. A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. *Adv. Math.*, 182(2):307–332, 2004.

[CL90a] E. A. Carlen and M. Loss. Competing symmetries of some functionals arising in mathematical physics. In *Stochastic processes, physics and geometry (Ascona and Locarno, 1988)*, pages 277–288. World Sci. Publ., Teaneck, NJ, 1990.

[CL90b] E. A. Carlen and M. Loss. Extremals of functionals with competing symmetries. *J. Funct. Anal.*, 88(2):437–456, 1990.

[CL92] E. A. Carlen and M. Loss. Competing symmetries, the logarithmic HLS inequality and Onofri's inequality on S^n. *Geom. Funct. Anal.*, 2(1):90–104, 1992.

[CL94] E. A. Carlen and M. Loss. On the minimization of symmetric functionals. *Rev. Math. Phys.*, 6(5A):1011–1032, 1994.

[Esc88] J. F. Escobar. Sharp constant in a Sobolev trace inequality. *Indiana Univ. Math. J.*, 37(3):687–698, 1988.

[Esc92] J. F. Escobar. The Yamabe problem on manifolds with boundary. *J. Differential Geom.*, 35(1):21–84, 1992.

[Lio85] P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.

[LP87] J. M. Lee and T. H. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)*, 17(1):37–91, 1987.

[LP09] M. Ledoux and I. Popescu. Mass transportation proofs of free functional inequalities, and free Poincaré inequalities. *J. Funct. Anal.*, 257(4):1175–1221, 2009.

[McC97] R.J. McCann. A convexity principle for interacting gases. *Adv. Math.*, 128(1):153–179, 1997.

[MN16] F. Maggi and R. Neumayer. A bridge between the Sobolev and Escobar inequalities and beyond. *Preprint, arXiV:1609.02346*, 2016.

[MV05] F. Maggi and C. Villani. Balls have the worst best Sobolev inequalities. *J. Geom. Anal.*, 15(1):83–121, 2005.

[MV08] F. Maggi and C. Villani. Balls have the worst best Sobolev inequalities. II. Variants and extensions. *Calc. Var. Partial Differential Equations*, 31(1):47–74, 2008.

[Naz06] B. Nazaret. Best constant in Sobolev trace inequalities on the half-space. *Nonlinear Anal.*, 65(10):1977–1985, 2006.

[Ngu15] V. H. Nguyen. Sharp weighted Sobolev and Gagliardo-Nirenberg inequalities on half-spaces via mass transport and consequences. *Proc. Lond. Math. Soc. (3)*, 111(1):127–148, 2015.

[OV00] F. Otto and C. Villani. Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. *J. Funct. Anal.*, 173(2):361–400, 2000.

[Sch84] R. Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.*, 20(2):479–495, 1984.

[Tal76] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372, 1976.

[Tru68] N. S. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. *Ann. Scuola Norm. Sup. Pisa (3)*, 22:265–274, 1968.

[Vil03] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[Yam60] H. Yamabe. On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.*, 12:21–37, 1960.

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