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On minimizers of Euler's elastica energy with an adhesion effect

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1 Introduction

In general, it is complicated to comprehend the figurations of elastic bodies, in particular, if external factors and constraints are taken into consideration. The purpose of this paper is to briefly survey recent progress on an "adhesion" problem of elastic curves.

1.1 Classical elastic curve problems

We first quickly review classical variational problems of elastic curves (without adhesion). Let us consider the shape of a thin inextensible rod of clamped endpoints. Assume that the rod lies in a plane. For this problem, one of the most classical formulations is the minimizing problem for the total squared curvature energy, so-called bending energy,

$$\int_\gamma \kappa^2 ds,$$

among planar curves γ of fixed length satisfying some boundary conditions. Here κ denotes the curvature and s denotes the arc length parameter of γ. One typical example of boundary conditions is the clamped boundary condition, i.e., the positions and tangential directions of endpoints are fixed. This general variational formulation is due to D. Bernoulli in 1742, and the family of solution curves are obtained by L. Euler in 1744 (see e.g. [12, 15, 22] for the precise history). Solution curves are so-called elasticae.

Any elastica γ satisfies the equation

$$2\kappa_{ss} + \kappa^3 - \lambda\kappa = 0 \tag{1.1}$$

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for some $\lambda \in \mathbb{R}$ (depending on γ). In fact, by the method of Lagrange multipliers, for any elastica γ there is $\lambda \in \mathbb{R}$ such that γ is a critical point of the modified total squared curvature energy

$$\int_\gamma \kappa^2 ds + \lambda \int_\gamma ds,$$

among curves satisfying the same boundary conditions as γ. Calculating the first variation, we obtain the equation (1.1) (see e.g. [4, 24] for the precise derivation).

The modified total squared curvature energy still has a physical meaning since the constant λ may be interpreted as tension (normalized by a bending rigidity). Its minimizing problem has also been studied (see e.g. [2, 4, 13, 14]).

The minimizing problem for the modified total squared curvature may be regarded as a modification of the length constraint in the classical problem of Bernoulli and Euler. This modification is a "welcome relief" since the length constraint, which often makes the problem complicated, is removed. When we minimize the modified total squared curvature we often impose the natural assumption that λ is positive; for non-positive λ the modified total squared curvature is not bounded below unless we impose other constraints[2].

The adhesion problem of elastic curves discussed in this survey is, roughly speaking, the minimizing problem for the modified total squared curvature containing an effect of adhesion.

1.2 Adhesion problem for elastic curves

Our adhesion problem is motivated by materials science. When soft objects as membranes or filaments are sheeted on patterned (non-flat) solid substrates in small scale, it is observed that complex adhesion patterns can occur [8, 19, 25]. To understand the principle of pattern formation for such adhesion problems, in [21], Pierre-Louis proposed a model formulated as an energy minimizing problem in a one-dimensional setting, i.e., it is assumed that elastic bodies and substrates vary in one direction and are invariant in the other direction. In this model, it is considered that the elastic bodies are shaped by the competition between their elastic (bending) energy and attractive (adhesive) interaction with substrates. The adhesion effect is taken into account as a difference of (surface) tension as wetting problems (cf. [7]). There are many similar formulations in literature (e.g. [3, 6, 11, 20, 23]) even for higher dimensional cases.

Let us recall the formulation by Pierre-Louis [21] more precisely. Let $\Omega = \{y > \psi(x)\} \subset \mathbb{R}^2$, where $\psi \in C(\mathbb{R})$ is a given continuous function

[2]However, as mentioned in [13], we should note that the relevance between the aforementioned two minimizing problems is not trivial.

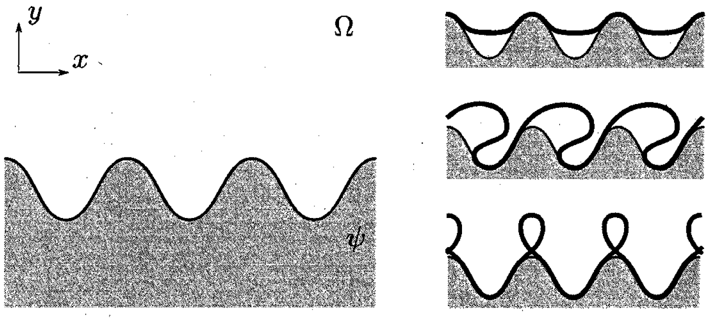

Figure 1: Curves on a substrate.

(substrate function). For a planar curve γ as in Figure 1, i.e., constrained in the closure $\overline{\Omega}$, the total energy is defined as

$$E[\gamma] = \varepsilon^2 \int_\gamma \kappa^2 ds + \int_\gamma \Theta(\gamma) ds. \tag{1.2}$$

Here $\varepsilon > 0$ is a given constant, and the function $\Theta : \overline{\Omega} \to \mathbb{R}$ is a contact potential function defined as $\Theta \equiv 1$ in Ω and $\Theta \equiv \alpha$ on $\partial\Omega$, where $\alpha \in (0,1)$ is a given constant[3]. Then our problem is formulated as

$$\min_{\gamma \in \mathcal{A}} E[\gamma], \tag{1.3}$$

where $\mathcal{A}$ is a suitable space of admissible curves. The constant $\varepsilon > 0$ corresponds to (normalized) bending rigidity of curves. Intuitively, the larger ε is, the more gently minimizing curves bend. The constant α corresponds to adhesivity. The smaller α is, the easier minimizing curves become to adhere.

There are many mathematical studies on variational problems including a contact potential (cf. [1, 5, 16]), however they deal with first order energies. Our above energy contains the curvature which is second order.

Concerning our problem (1.3), besides [21], there are at least three mathematical papers [10, 17, 18]. In this survey we overview the results of these four papers.

One immediately notices that the above formulation is not sufficient since $\mathcal{A}$ is not defined precisely. The paper of Pierre-Louis [21] is a physical paper and does not define $\mathcal{A}$ mathematically. In the mathematical papers [10, 17, 18], the space $\mathcal{A}$ is defined, but the definition depends on the papers.

[3]The original energy in [21] is more general; in fact, the potential may take zero or a negative value. In our formulation, non-positive cases are omitted to avoid some mathematical difficulties.

2 Results on adhesion problem

In this section, we overview the papers [10, 17, 18, 21] chronologically.

2.1 Formulation and boundary conditions

As mentioned in Introduction, the problem (1.3) is formulated by Pierre-Louis [21]. In this paper, the precise definition of admissible curves (as regularity or boundary condition) is not taken care, but it is assumed that any curve $\gamma \in \mathcal{A}$ is the graph $\{y = h(x)\}$ of a height function h (as the upper right of Figure 1).

The paper [21] first states that the minimization (1.3) invokes a free boundary problem concerning elasticae. (This part does not use the fact that a curve γ is a graph.) In fact, any minimizing curve can be locally perturbed in the free part (i.e., the part that the curve is in Ω) and the total energy E is nothing but the modified total squared curvature there, thus the minimizing curve satisfies the equation (1.1) with $\lambda = 1/\varepsilon^2$. Of course, a curve follows the graph of ψ in the bounded part (i.e., the part that the curve is on $\partial\Omega$). At the contact points (free boundary), a minimizing curve satisfies boundary conditions. The boundary conditions depend on the regularity of the substrate ψ near contact points. For example, if the case that ψ is sufficiently smooth at least of class C^2 at a contact point, then the minimizing curve has the same position and tangential direction as the substrate, and the curvature has a jump there. More precisely, $\kappa_F - \kappa_B = \sqrt{1-\alpha}/\varepsilon$ holds, where κ_F is the limit of the upward curvature from the free part and κ_B from the bounded part. This kind of jump condition for curvature has also appeared in e.g. [3, 6, 11, 20, 23]. If ψ is not smooth, the conditions become more involved. See [21] for details.

As a main contribution of the paper [21], critical points of the energy (1.2) are precisely analyzed by using a small slope approximation, i.e., it is assumed that the derivative of a height function $|h'|$ is sufficiently small and the equation (1.1) is linearized by this assumption. In this part, only special substrates, as sinusoidal and "fakir-carpet" substrates, are mainly considered. This part essentially depends on the small slope approximation (and also the graph representation of curves). Moreover, relevance to existing experimental results is also mentioned.

2.2 Singular perturbation

In the author's paper [17], a singular perturbation problem $\varepsilon \to 0$ is considered. This paper would be the first mathematical paper on our adhesion problem of elastic curves. In [17], following Pierre-Louis [21], it is assumed that admissible curves are the graphs of functions. More precisely, the set of admissible curves $\mathcal{A}$ consists of the graphs of functions $u \in H^2(I)$ with a Dirichlet boundary condition[4], where I is a bounded open interval. In addition, the substrate function ψ is assumed to be smooth at least of class C^2. In [17] the value α in the potential Θ may not be a constant (i.e., the value of $\Theta|_{\partial\Omega}$ may depend on the position) but in this survey we assume that it is a constant.

The limit $\varepsilon \to 0$ means that the bending rigidity becomes small. When $\varepsilon = 0$, the higher order energy vanishes thus the problem "degenerates" in a sense. In fact, as $\varepsilon \to 0$, the jump condition for curvature (given in the previous section) formally yields that the curvature diverges at the contact points as in Figure 2. This formal observation is valid in the sense that when $\varepsilon = 0$ any minimizer of (1.2) (in a suitable space of admissible curves) has "edge" singularities at contact points (see [17] for details). The angle θ between a minimizing curve and a substrate is determined by the adhesion coefficient α; Young's equation $\cos\theta = \alpha$ holds. In this view the limit $\varepsilon \to 0$ is a singular limit.

The limit $\varepsilon \to 0$ is a reduction in a sense since the case $\varepsilon = 0$ is rather easy. If $\varepsilon = 0$ then the boundary conditions are up to first order as Young's equation. Moreover, by (1.1), a minimizing curve satisfies $\kappa = 0$ in the free part, thus each connected component of the curve in the free part is just a segment. These conditions restrict candidates of minimizers. Note that in this case admissible curves are no longer H^2.

However, just by taking $\varepsilon = 0$, the effect of the bending energy completely vanishes. When ε is small but non-zero, what is the main effect of the bending energy? To answer this question is the main purpose of the paper [17]. The answer is natural; when $\varepsilon = 0$ the curvature of a minimizing curve is singular only at the contact points, hence when $\varepsilon \ll 1$ the main effect of the perturbation by bending depends only on the free boundary. Roughly speaking, the main theorem of [17] states that, for fixed α and smooth ψ, the total energy $E =: E_\varepsilon$ is expanded as

$$E_\varepsilon \approx E_0 + \varepsilon F + o(\varepsilon)$$

in a sense of Γ-expansion with respect to the $W^{1,1}$-topology[5], where $F[\gamma] = 4(\sqrt{2} - \sqrt{1+\alpha})N$ and N denotes the number of contact points. Since $\alpha = \cos\theta$, the energy F depends only on the number and their angles of contact points. If α depends on the position, then F is defined in terms of the zero-dimensional Hausdorff measure.

The above expansion roughly means that in the case of small ε a minimizing curve mainly minimizes E_0 and, as the next order effect, also min-

[4]The fact is that the paper [17] considers $W^{2,1}$ functions but, for our energy minimizing problem, the H^2 and $W^{2,1}$ settings are equivalent.

[5]To be more precise, in order to state Γ-expansion, all the energies E_ε, E_0, F have to be defined for any $W^{1,1}$-function (with a boundary condition). This is justified by a simple penalty method. See [17] for details.

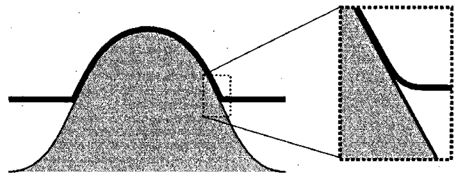

Figure 2: Minimizer on a smooth substrate for small ε.

imizes F; in our case ($\Theta|_{\partial\Omega} \equiv$ const.) the number of contact points (edge singularities) should be minimized.

2.3 Discretization

In the next mathematical paper [10], Kemmochi considered a discretization problem for our adhesion problem. In [10], it is still assumed that admissible curves are the graphs of H^2-functions u on the unit interval $I = (0, 1)$, and a periodic boundary condition is imposed; $u(0) = u(1)$ and $u'(0) = u'(1)$. A substrate function ψ is also assumed to be smooth.

A main purpose of the discretization is to propose a way of numerical calculation, or more simply, to simplify the minimizing problem. The problem is discretized in the sense that admissible curves are taken as (periodic) polygonal line functions of step size h. In addition, each term in the total energy (1.2) is suitably modified. In particular, the discontinuous adhesion effect and the obstacle (substrate) constraint is modified to be smooth by introducing two new parameters δ and ρ. Then the main theorem of [10] states that for fixed ε, α and ψ, under the assumption of uniformly bounded slope (i.e., for some $S > 0$ any admissible function u satisfies $\|u'\|_\infty \leq S$), the modified energy $E_{h,\delta,\rho}$ Γ-converges to the original total energy E with respect to the H^1-topology. The uniformly bounded slope assumption is essential in the proof.

Some numerical calculations are also exhibited by using the discretization. A remarkable calculation shows a "blowing-up" example for a special substrate (as a "single-needle" substrate). Here "blowing-up" means that the slope of a function diverges. This result indicates that the graph setting is not suitable for our adhesion problem. As mention in [10], the existence of such a case has been expected by the author, but Kemmochi's paper [10] is the first one to indicate it expressly.

2.4 Graph representation

All the previous works [10, 17, 21] assume that any admissible curve is represented by a graph. This yields strong topological and morphological constraints which make the problem easier to analyze, however its adequacy is nontrivial. In fact, as mentioned in the previous subsection, Kemmochi [10] shows a numerical example with blowing-up slope. The recent paper [18] by the author addresses the first rigorous study on the adequacy. This paper mathematically proves the existence of a situation such that any minimizer is not a graph.

In [18], we impose the periodic boundary condition as [10], but admissible curves are not assumed to be the graphs of functions; the space of admissible curves $\mathcal{A}$ is taken as the set of regular H^2-Sobolev curves $\gamma = (x, y) : I \to \overline{\Omega}$, where $I = (0, 1)$, with

$$x(0) = 0,\ x(1) = 1,\ y(0) = y(1),\ \dot{\gamma}(0) = \dot{\gamma}(1).$$

A substrate function ψ is assumed to have the same period as admissible curves, i.e., $\psi(x) = \psi(x + 1)$ for any $x \in \mathbb{R}$.

This non-graph setting is actually very natural in the sense that for any given ε, α and ψ there exists a minimizer for (1.3). (The uniqueness is not expected in general.) The main problem in [18] is to consider the graph representation of global minimizers. The paper [18] exhibits some sufficient conditions regarding the parameters ε, α, ψ for the graph representation of minimizers, and also examples of the parameters such that any minimizer is overhanging (non-graph).

2.4.1 Graph minimizers

We easily notice that minimizers are only straight lines in the following limiting cases; $\varepsilon = \infty$, $\alpha = 1$, and $\psi \equiv 0$. By this observation, when $\varepsilon \gg 1$, $\alpha \approx 1$ or $\psi \approx 0$, we expect that any minimizer is nearly flat and hence a graph curve, i.e., $x'(t) > 0$ for any $t \in \bar{I}$. In fact, the following two statements are proved in [18].

Theorem 2.1. *Suppose that* $(\pi^2\varepsilon^2 + 1)\alpha \geq 1$. *Then, independently of* ψ, *any minimizer is a graph curve.*

Theorem 2.2. *Suppose that* $\psi \in W^{2,\infty}(\mathbb{R})$ *and* $\|\psi''\|_\infty^2 \leq \frac{8\pi^2}{8/\alpha + 1/\varepsilon^2}$. *Then any minimizer is a graph curve.*

Theorem 2.1 immediately implies that, if we fix ε and take $\alpha \approx 1$, or fix α and take $\varepsilon \gg 1$, any minimizer is a graph curve. Theorem 2.2 states that, for any ε and α which may be small, if the substrate ψ is sufficiently flat in the second order sense $\psi'' \approx 0$ then our problem still admits only graph curve minimizers. The proofs in [18] rely only on energy arguments; we obtain a lower bound for the energy of all non-graph curves and choose suitable graph curves so that, under the assumptions in the theorems, their energies are less than the lower bound.

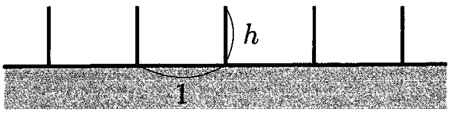

Figure 3: Fakir carpet of height h and period 1.

2.4.2 Overhanging minimizers

On the other hand, it turns out that there is a combination of $\varepsilon \ll 1$, $\alpha \ll 1$ and "almost singular" ψ of special shape such that any minimizer of (1.3) is overhanging, i.e., $x'(t) < 0$ for some $t \in \bar{I}$.

Theorem 2.3. *Let $h > 0$ and $m_h := \frac{\min\{1,h\}}{1+2h}$. Then for any $\alpha < m_h$ and $\varepsilon < \frac{(1+2h)(m_h-\alpha)}{20\pi}$ there exists $\psi \in C^\infty(\mathbb{R}; [0,h])$ such that any minimizer is overhanging.*

Theorem 2.4. *Let $h > 0$ and $m_h := \frac{\min\{1,h\}}{1+2h}$. Then for any $\alpha < m_h$ there exists $\psi \in \mathrm{Lip}(\mathbb{R}; [0,h])$ such that for any small $\varepsilon > 0$ any minimizer is overhanging.*

A difference between the above two theorems indicates that the regularity of ψ is essential for the uniformity of ε. In particular, for any fixed α and sufficiently smooth ψ (at least more than Lipschitz) if we take ε sufficiently small then there "may" not exist an overhanging minimizer.

These theorems are also proved by only energy arguments. Substrates ψ are taken as slightly modified fakir carpets (the shape of a singular fakir carpet is as in Figure 3). The key step is a classification of non-overhanging curves by using the shape of ψ to obtain a lower bound for their energies. We then construct an overhanging competitor whose energy is less than the lower bound provided that $\varepsilon, \alpha \ll 1$.

2.4.3 Self-intersections

The paper [18] also mentions some problems about self-intersections. One notices that our admissible curves may have a self-crossing (as in the lower right of Figure 1) which is not suitable for membrane problems. However, as in [18], our problem can be adapted to membrane problems by considering the minimizing problem on a subset $\mathcal{A}' \subset \mathcal{A}$. The set $\mathcal{A}'$ is taken as the H^2-weak closure of the set of curves without self-intersection. Then any curve in $\mathcal{A}'$ has no self-crossing (it may have only self-contacts). The existence of minimizers in $\mathcal{A}'$ is proved by a similar argument for $\mathcal{A}$.

3 Perspectives

The adhesion problem of elastic curves is developing, thus there are still many remaining problems. Some of them are mentioned in [18] precisely. One main remaining problem is, as mentioned in Section 2.4.2, the graph representation of minimizers for given α, smooth ψ, and sufficiently small ε. We announce that our forthcoming paper addresses this open case.

Original physical membrane problems are two-dimensional, thus it is natural to consider challenging higher-dimensional problems. For the generalization we need to return to modeling. A simple mathematical generalization is to consider the Willmore energy (the total squared mean curvature) as the bending energy. The adhesion energy can be simply generalized by a weighted surface energy (a weighted area functional). A more suitable setting for cell membranes would be the Helfrich energy (cf. [9] and references therein) with an adhesion effect. Such an energy has appeared in literature, e.g. in [3, 6, 23], with a volume term.

References

[1] H. W. Alt, L. A. Caffarelli, Existence and regularity for a minimum problem with free boundary, J. Reine Angew. Math. 325 (1981), 105–144.

[2] G. Bellettini, L. Mugnai, Characterization and representation of the lower semicontinuous envelope of the elastica functional, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), no. 6, 839–880.

[3] E. M. Blokhuis, W. F. C. Sager, Helfrich free energy for aggregation and adhesion, J. Chem. Phys. 110, (1999) 3148.

[4] G. Brunnett, A new characterization of plane elastica, In: Mathematical methods in computer aided geometric design II, Academic Press, Boston, MA, 1992, pp. 43–56.

[5] L. A. Caffarelli, A. Friedman, Regularity of the boundary of a capillary drop on an inhomogeneous plane and related variational problems, Rev. Mat. Iberoamericana 1 (1985), no. 1, 61–84.

[6] R. Capovilla, J. Guven, Geometry of lipid vesicle adhesion, Phys. Rev. E 66 (2002), 041604.

[7] P.-G. de Gennes, F. Brochard-Wyart, D. Quéré, Capillarity and Wetting Phenomena: Drops, Bubbles, Pearls, Waves, Springer, New York, 2004.

[8] V. Derycke, R. Martel, J. Appenzeller, P. Avouris, Carbon Nanotube Inter- and Intramolecular Logic Gates, Nano Lett. 1 (2001), 453–456.

[9] C. M. Elliott, H. Fritz, G. Hobbs, Small deformations of Helfrich energy minimising surfaces with applications to biomembranes, preprint (arXiv:1610.05240)

[10] T. Kemmochi, Numerical analysis of elastica with obstacle and adhesion effects, preprint (arXiv:1604.03400)

[11] L. Landau, E. Lifshitz, Theory of Elasticity, Elsevier, Amsterdam, 1986.

[12] R. Levien, The elastica: a mathematical history, Technical Report No. UCB/EECS-2008-10, University of California, Berkeley, 2008.

[13] A. Linnér, Explicit elastic curves, Ann. Global Anal. Geom. 16 (1998), no. 5, 445–475.

[14] A. Linnér, Curve-straightening and the Palais-Smale condition, Trans. Amer. Math. Soc. 350 (1998), no. 9, 3743–3765.

[15] A. E. H. Love, A treatise on the Mathematical Theory of Elasticity, Fourth Ed. Dover Publications, New York, 1944.

[16] A. Mellet, Some mathematical aspects of capillary surfaces, In: Singularities in mechanics: formation, propagation and microscopic description, Panor. Synthèses, 38, Soc. Math. France, Paris, 2012, pp. 91–124.

[17] T. Miura, Singular perturbation by bending for an adhesive obstacle problem, Calc. Var. Partial Differential Equations, in press.

[18] T. Miura, Overhanging of membranes adhering to periodic graph substrates, submitted (arXiv:1612.08532)

[19] J. Nicolle, D. Machon, P. Poncharal, O. Pierre-Louis, A. San-Miguel, Pressure-Mediated Doping in Graphene, Nano Lett. 11 (2011), 9, 3564–3568.

[20] X. Oyharcabal, T. Frisch, Peeling off an elastica from a smooth attractive substrate, Phys. Rev. E 71 (2005), 036611.

[21] O. Pierre-Louis, Adhesion of membranes and filaments on rippled surfaces, Phys. Rev. E 78 (2008), 021603.

[22] Y. L. Sachkov, Maxwell strata in the Euler elastic problem, J. Dyn. Control Syst. 14 (2008), no. 2, 169–234.

[23] U. Seifert, R. Lipowsky, Adhesion of vesicles, Phys. Rev. A 42 (1990), 4768.

[24] D. A. Singer, Lectures on elastic curves and rods, In: Curvature and variational modeling in physics and biophysics, 1002, Amer. Inst. Phys., Melville, NY, 2008, pp. 3–32.

[25] M. Yamamoto, O. Pierre-Louis, J. Huang, M. Fuhrer, T. Einstein, W. Cullen, "The Princess and the Pea" at the Nanoscale: Wrinkling and Delamination of Graphene on Nanoparticles, Phys. Rev. X 2 (2012), 041018.