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Existence of nontrivial solutions for scalar field equations with fractional operators

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1 Introduction

This note is a survey of [19] and also provides the nonexistence result of nontrivial solutions which is not contained in [19].

Throughout this note, we shall discuss the existence and nonexistence of nontrivial solutions of

$$\begin{cases} (1-\Delta)^{\alpha}u = f(x,u) & \text{in } \mathbf{R}^N, \\ u \in H^{\alpha}(\mathbf{R}^N). \end{cases} \tag{1}$$

Here, $N \geq 2$, $0 < \alpha < 1$ and $f(x,s) : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is a given function. Using the Fourier transform, we define the fractional operator $(1-\Delta)^{\alpha}u$ as follows:

$$(1-\Delta)^{\alpha}u := \mathscr{F}^{-1}\left((1+4\pi^2|\xi|^2)^{\alpha}\widehat{u}(\xi)\right), \quad \hat{u}(\xi) := (\mathscr{F}u)(\xi) = \int_{\mathbf{R}^N} e^{-2\pi i x\cdot\xi}u(x)\mathrm{d}x.$$

Finally, $H^{\alpha}(\mathbf{R}^N)$ denotes a fractional Sobolev space. We remark that in this note we only treat solutions of (1) which are real valued. Therefore, let $H^{\alpha}(\mathbf{R}^N)$ be consisted by real valued functions, namely,

$$H^{\alpha}(\mathbf{R}^N) := \left\{ u \in L^2(\mathbf{R}^N,\mathbf{R}) \;\middle|\; \|u\|_{\alpha}^2 := \int_{\mathbf{R}^N}(4\pi^2|\xi|^2+1)^{\alpha}|\hat{u}|^2\mathrm{d}\xi < \infty \right\}.$$

Another expression of $H^{\alpha}(\mathbf{R}^N)$ is

$$H^{\alpha}(\mathbf{R}^N) = \left\{ u \in L^2(\mathbf{R}^N,\mathbf{R}) \;\middle|\; [u]_{W^{\alpha,2}}^2 := \int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\mathrm{d}x\mathrm{d}y < \infty \right\}.$$

This can be checked by the arguments in [15].

Next, we explain the notion of solutions of (1). In this note, we only deal with weak solutions. A function $u \in H^{\alpha}(\mathbf{R}^N)$ is said to be a weak solution of (1) provided u satisfies

$$\int_{\mathbf{R}^N}(4\pi^2|\xi|^2+1)^{\alpha}\widehat{u}(\xi)\overline{\widehat{\varphi}(\xi)}\mathrm{d}\xi - \int_{\mathbf{R}^N} f(x,u(x))\varphi(x)\mathrm{d}x = 0 \quad \text{for all } \varphi \in H^{\alpha}(\mathbf{R}^N)$$

where $\overline{a}$ denotes the complex conjugate of a. Hereafter, solutions mean weak solutions.

Recently, a lot of attentions are paid for fractional operators. When $\alpha = 1/2$, the operator $(1-\Delta)^{\alpha}$ is related to pseudo-relativistic Schrödinger operator $(m^2-\Delta)^{1/2} - m$

$(m > 0)$. Many researchers study the equations involving these operators and show the existence of nontrivial solutions and infinitely many solutions. For instance, we refer to [1–3, 9–14, 16, 17, 24–26, 28] and references therein for the details.

Among them, the paper [19] is especially motivated by two papers [17] and [25]. The aim of [19] generalizes some results in [17, 25]. In [17, 19, 25], the following two cases are considered:

(i) $f(x, s) = f(s)$.
(ii) $f(x, s)$ depends on x.

In case (i), the aim of [19] is to treat general nonlinearities. When $\alpha = 1$, Berestycki and Lions [5, 6] introduce the conditions on $f(s)$ which are almost necessary and sufficient conditions for the existence of nontrivial solutions. For the case $0 < \alpha < 1$, we can consider similar conditions on $f(s)$. See (f1)–(f4) below. Under these conditions, we shall show the existence of infinitely many solutions as well as the characterization of the least energy value c_{LES} by the mountain pass value. For more precise statements, see Theorem 1.1.

On the other hand, in case (ii), the aim of [19] is to show the existence of positive solution of (1). Here the characterization of the least energy value by the mountain pass value obtained in case (i) is useful to get a positive solution. See Theorem 1.2 and section 2.

In addition to these results, we also prove the nonexistence result of nontrivial solutions of (1) if $f(x, s)$ is monotone in some direction. This is Theorem 1.3.

We first begin with case (i), namely, we consider

$$\begin{cases} (1 - \Delta)^\alpha u = f(u) & \text{in } \mathbf{R}^N, \\ u \in H^\alpha(\mathbf{R}^N). \end{cases} \tag{2}$$

For (2), we assume that the nonlinearity f is a Berestycki–Lions type ([5, 6]):

(f1) $f \in C(\mathbf{R}, \mathbf{R})$ and $f(s)$ is odd.

(f2) $-\infty < \liminf_{s \to 0} \dfrac{f(s)}{s} \le \limsup_{s \to 0} \dfrac{f(s)}{s} < 1$.

(f3)
$$\lim_{|s| \to \infty} \frac{|f(s)|}{|s|^{2_\alpha^* - 1}} = 0 \quad \text{where } 2_\alpha^* := \frac{2N}{N - 2\alpha}.$$

(f4) There exists an $s_0 > 0$ such that
$$F(s_0) - \frac{1}{2} s_0^2 > 0 \quad \text{where } F(s) := \int_0^s f(t) \mathrm{d}t.$$

Using (f1)–(f3), it is not difficult to see that a solution of (2) is characterized as a critical point of

$$I(u) := \frac{1}{2} \|u\|_\alpha^2 - \int_{\mathbf{R}^N} F(u) \mathrm{d}x \in C^1(H^\alpha(\mathbf{R}^N), \mathbf{R}). \tag{3}$$

For (2), we have

Theorem 1.1. *Assume* $N \geq 2$, $0 < \alpha < 1$ *and* (f1)–(f4).

(i) *There exist infinitely many solutions* $(u_n)_{n=1}^{\infty}$ *of* (2) *satisfying* $I(u_n) \to \infty$ *and the Pohozaev identity* $P(u_n) = 0$ *where*

$$P(u) := \frac{N-2\alpha}{2} \int_{\mathbf{R}^N} \left(1+4\pi^2|\xi|^2\right)^{\alpha} |\widehat{u}|^2 \mathrm{d}\xi - N \int_{\mathbf{R}^N} F(u) \mathrm{d}x + \alpha \int_{\mathbf{R}^N} \left(1+4\pi^2|\xi|^2\right)^{\alpha-1} |\widehat{u}|^2 \mathrm{d}\xi. \tag{4}$$

Moreover, $u_1(x) > 0$ *for all* $x \in \mathbf{R}^N$.

(ii) *Assume either* $\alpha > 1/2$ *or* $f(s)$ *is locally Lipschitz continuous. Then every solution of* (2) *satisfies the Pohozaev identity* $P(u) = 0$.

(iii) *Define* c_{MP}, c_{LES} *and* S_{LES} *by*

$$c_{\mathrm{MP}} := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \quad \Gamma := \{\gamma \in C([0,1], H^{\alpha}(\mathbf{R}^N)) \mid \gamma(0) = 0,\ I(\gamma(1)) < 0\},$$
$$c_{\mathrm{LES}} := \inf \{I(u) \mid u \not\equiv 0,\ I'(u) = 0,\ P(u) = 0\},$$
$$S_{\mathrm{LES}} := \left\{u \in H^{\alpha}(\mathbf{R}^N) \mid u \not\equiv 0,\ I'(u) = 0,\ P(u) = 0,\ I(u) = c_{\mathrm{LES}}\right\}.$$

Then, $S_{\mathrm{LES}} \neq \emptyset$ *and* $c_{\mathrm{MP}} = c_{\mathrm{LES}} > 0$ *hold. Furthermore, if* $I'(v) = 0$ *and* $P(v) = 0$, *then the path defined by* $\gamma_v(0) = 0$ *and* $\gamma_v(t) := v(x/(Tt))$ *satisfies* $\gamma_v \in \Gamma$ *for sufficiently large* $T > 0$ *and*

$$\max_{0 \leq t \leq 1} I(\gamma_v(t)) = I(v).$$

When $0 < \alpha \leq 1/2$, it seems not known whether or not every (weak) solution satisfies the Pohozaev identity. At this moment, we know that the Pohozaev identity is satisfied when a (weak) solution is of class C^1 with bounded derivatives. See [19, Proposition 3.6] (cf. [17,26]).

Next, we consider case (ii). Here, we assume that $f(x,s)$ in (1) satisfies the following:

(F1) $f(x,s) = -V(x)s + g(x,s)$ where $V \in C(\mathbf{R}^N, \mathbf{R})$, $g \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ and $g(x,-s) = -g(x,s)$ for every $(x,s) \in \mathbf{R}^N \times \mathbf{R}$.

(F2)
$$-1 < \inf_{x \in \mathbf{R}^N} V(x) \quad \text{and} \quad \lim_{s \to 0} \sup_{x \in \mathbf{R}^N} \left| \frac{g(x,s)}{s} \right| = 0.$$

(F3)
$$\lim_{|s| \to \infty} \sup_{x \in \mathbf{R}^N} \frac{|g(x,s)|}{|s|^{2_\alpha^*-1}} = 0.$$

(F4) There exist $V_\infty > -1$ and $g_\infty(s) \in C(\mathbf{R}, \mathbf{R})$ such that as $|x| \to \infty$, $V(x) \to V_\infty$ and $g(x,s) \to g_\infty(s)$ in $L^\infty_{\mathrm{loc}}(\mathbf{R}^N)$ where $g_\infty(s)$ is locally Lipschitz continuous provided $0 < \alpha \leq 1/2$. Moreover, $0 \leq F(x,s) - F_\infty(s)$ holds for all $x \in \mathbf{R}^N$ and $s \in \mathbf{R}$ where $F(x,s) := \int_0^s f(x,t)\mathrm{d}t$, $f_\infty(s) := -V_\infty s + g_\infty(s)$ and $F_\infty(s) := \int_0^s f_\infty(t)\mathrm{d}t$.

(F5) There exist $\mu > 2$ and $s_1 > 0$ such that

$$0 < \mu G(x,s) \le g(x,s)s \quad \text{for each } (x,s) \in \mathbf{R}^N \times \mathbf{R} \setminus \{0\}, \quad \inf_{x \in \mathbf{R}^N} G(x,s_1) > 0$$

where $G(x,s) := \int_0^s g(x,t)\mathrm{d}t$.

Then we have the following result:

Theorem 1.2. *Assume $N \ge 2$, $0 < \alpha < 1$ and* (F1)–(F5)*. Then* (1) *admits a positive solution.*

Finally, we state the nonexistence result of nontrivial solutions of (1) when $f(x,s)$ is monotone in some direction:

Theorem 1.3. *Let $2 \le N$, $0 < \alpha < 1$ and $f \in C^1(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ satisfy*

$$|f(x,s)| \le C(|s| + |s|^{2^*_\alpha - 1}) \quad \text{for every } (x,s) \in \mathbf{R}^N \times \mathbf{R}. \tag{5}$$

Assume that for every $t > 0$, there exists a $C_t > 0$ such that

$$\left| \frac{\partial f}{\partial s}(x,s) \right| \le C_t, \quad |\nabla_x f(x,s)| \le C_t |s| \quad \text{for all } (x,s) \in \mathbf{R}^N \times [-t,t]. \tag{6}$$

Suppose also that there exists an $e \in \mathbf{R}^N$ with $|e| = 1$ such that

$$e \cdot \nabla_x f(x,s) > 0 > e \cdot \nabla_x f(x,-s) \quad \text{for all } (x,s) \in \mathbf{R}^N \times (0,\infty). \tag{7}$$

Then (1) *has no nontrivial solution.*

Remark 1.4. A typical example satisfying (5)–(7) is $f(x,s) = -V(x)s + a(x)|s|^{p-1}s$ where $1 < p \le 2^*_\alpha$, and $V(x)$ and $a(x)$ are smooth with $-\nabla V(x) \cdot e > 0$, $\nabla a(x) \cdot e > 0$ for some $e \in \mathbf{R}^N$ with $|e| = 1$.

Here we state comparison with the previous results. The equation (2) is studied in the papers [1,9,17,28]. In these papers, the nonlinearity $f(s)$ has the form of $f(s) = |s|^{p-1}s$ or $f(s) = (1-\mu)s + |s|^{p-1}s$ where $1 < p < 2^*_\alpha - 1$ and $\mu > 0$, and the existence of least energy solution and infinitely many solutions are obtained. Here, we treat general nonlinearities including the above ones, hence, Theorem 1.1 improves these results. We remark that if we replace the operator $(1-\Delta)^\alpha$ by the fractional Laplacian $(-\Delta)^\alpha$, a similar result to Theorem 1.1 is obtained in [4,8].

Next, we turn to (1), namely, $f(x,s)$ depends on x. In [17,25,26], the existence of nontrivial solution is proved. It can be checked that Theorem 1.2 generalizes some results in [17,25]. On the other hand, in [26], the author deals with a different type of nonlinearities. In fact, the nonlinearity in [26] involves a sublinear term, namely, $|u|^{p-1}u$ where $0 < p < 1$, and the existence of nontrivial solution is proved.

This note is organized as follows. In section 2, we give ideas of proofs of Theorems 1.1 and 1.2. Section 3 is devoted to the proof of Theorem 1.3.

2 Ideas of proofs of Theorems 1.1 and 1.2

To prove Theorems 1.1 and 1.2, we try to find critical points of $I(u)$ defined in (3) and $J(u)$ below, respectively:

$$
\begin{aligned}
J(u) &:= \frac{1}{2}\|u\|_\alpha^2 + \frac{1}{2}\int_{\mathbf{R}^N} V(x)u^2 \mathrm{d}x - \int_{\mathbf{R}^N} G(x,u(x))\mathrm{d}x \\
&= \frac{1}{2}\|u\|_\alpha^2 - \int_{\mathbf{R}^N} F(x,u)\mathrm{d}x \in C^1(H^\alpha(\mathbf{R}^N),\mathbf{R}).
\end{aligned} \tag{8}
$$

A main difficulty to prove Theorem 1.1 is to find bounded Palais–Smale sequences. To overcome this difficulty, we borrow the argument from [18] and introduce the following functional:

$$\tilde{I}(\theta,u) := I(u(\cdot/e^\theta)) \in C^1(\mathbf{R}\times H^\alpha(\mathbf{R}^N),\mathbf{R}).$$

Here we remark that the norm $\|u\|_\alpha^2$ is not homogeneous with respect to the scaling as already observed in [1,25]. More precisely, one can check that

$$\|u(\cdot/e^\theta)\|_\alpha^2 = e^{N\theta}\int_{\mathbf{R}^N}\left(1+4\pi^2\frac{|\xi|^2}{e^{2\theta}}\right)^\alpha |\widehat{u}(\xi)|^2\mathrm{d}\xi.$$

On the other hand, for the operator $(-\Delta)^\alpha$ or equivalently the quantity defined by

$$[u]_\alpha^2 := \int_{\mathbf{R}^N}(4\pi^2|\xi|^2)^\alpha|\widehat{u}(\xi)|^2\mathrm{d}\xi,$$

one has

$$[u(\cdot/e^\theta)]_\alpha^2 = e^{(N-2\alpha)\theta}[u]_\alpha^2.$$

In spite of these differences, the functional $\tilde{I}(\theta,u)$ still plays a role to find bounded Palais–Smale sequences and we refer to [19, Proposition 3.1]. In addition, since $\tilde{I}$ is based on the scaling, $\tilde{I}$ is related to the Pohozaev identity: (see (4) for the definition of $P(u)$)

$$\partial_\theta\tilde{I}(0,u) = \frac{N-2\alpha}{2}\|u\|_\alpha^2 + \alpha\int_{\mathbf{R}^N}\left(1+4\pi^2|\xi|^2\right)^{\alpha-1}|\widehat{u}(\xi)|^2\mathrm{d}\xi - N\int_{\mathbf{R}^N}F(u)\mathrm{d}x = P(u).$$

Thus, if $(0,u)$ is a critical point of $\tilde{I}$, then u is a solution of (2) and satisfies the Pohozaev identity.

In order to find infinitely many critical points $((0,u_n))_{n=1}^\infty$ of $\tilde{I}$, we work on the space of radially symmetric functions

$$H_{\mathrm{r}}^\alpha(\mathbf{R}^N) := \left\{u\in H^\alpha(\mathbf{R}^N) \mid u \text{ is radially symmetric}\right\}$$

since the embedding $H_{\mathrm{r}}^\alpha(\mathbf{R}^N)\subset L^p(\mathbf{R}^N)$ $(2<p<2_\alpha^*)$ is compact (see [21]). From (f4) and the argument in [6] (see also [18]), for every $n\geq 1$, there exists a $\gamma_n\in C(D_n,H_{\mathrm{r}}^\alpha(\mathbf{R}^N))$ $(D_n := \{x\in\mathbf{R}^n \mid |x|\leq 1\})$ such that

$$\gamma_n(-\sigma) = -\gamma_n(\sigma) \quad \text{for all } \sigma\in D_n, \qquad \max_{\sigma\in\partial D_n} I(\gamma_n(\sigma)) < 0.$$

Using $(\gamma_n)_{n=1}^\infty$, we define the minimax values for I and $\tilde{I}$ by

$$\begin{aligned}
c_n &:= \inf_{\gamma\in\Gamma_n}\max_{\sigma\in D_n} I(\gamma(\sigma)), \quad \tilde{c}_n := \inf_{\tilde\gamma\in\tilde\Gamma_n}\max_{\sigma\in D_n} \tilde{I}(\tilde\gamma(\sigma)),\\
\Gamma_n &:= \left\{\gamma\in C(D_n, H^\alpha_{\mathrm{r}}(\mathbf{R}^N)) \mid \gamma(-\sigma) = -\gamma(\sigma),\ \gamma=\gamma_n \text{ on } \partial D_n\right\},\\
\tilde\Gamma_n &:= \{\tilde\gamma(\sigma) = (\theta(\sigma),\gamma(\sigma)) \in C(D_n, \mathbf{R}\times H^\alpha_{\mathrm{r}}(\mathbf{R}^N)) \mid \gamma\in\Gamma_n,\\
&\qquad\qquad \theta(-\sigma)=\theta(\sigma),\ \theta=0 \text{ on } \partial D_n\}
\end{aligned}$$

Then we can show the following:

Proposition 2.1. (i) *For each n, $c_n = \tilde{c}_n$ holds. In addition, $c_n\to\infty$ as $n\to\infty$.*

(ii) *There is a sequence $(u_n)_{n=1}^\infty \subset H^\alpha_{\mathrm{r}}(\mathbf{R}^N)$ such that $\tilde{I}(0,u_n) = c_n$ and $\partial_\theta \tilde{I}(0,u_n) = 0 = \partial_u\tilde{I}(0,u_n)$.*

(iii) *For each $u\in H^\alpha(\mathbf{R}^N)$ with $I'(u)=0$ and $P(u)=0$, the path $\gamma_u(t) := u(\cdot/t) : (0,\infty)\to H^\alpha(\mathbf{R}^N)$ satisfies*

$$\begin{aligned}
&\gamma_u(t)\to 0 \quad \text{strongly in } H^\alpha(\mathbf{R}^N) \quad \text{as } t\to 0, \quad I(\gamma_u(t))\to -\infty \quad \text{as } t\to\infty,\\
&I(\gamma_u(t)) < I(\gamma_u(1)) = I(u) \quad \text{for every } t\neq 1.
\end{aligned}$$

(iv) *For $n=1$, we have $u_1(x)>0$ in $\mathbf{R}^N$ and $c_1 = c_{\mathrm{LES}} = c_{\mathrm{MP}}$.*

For the detail, we refer to [19, section 3].

Next, we discuss the idea of proof of Theorem 1.2. A main difficulty here is a lack of compactness of Sobolev's embedding $H^\alpha(\mathbf{R}^N)\subset L^p(\mathbf{R}^N)$ for $2\le p\le 2^*_\alpha$. To overcome this difficulty, we use the concentration compactness lemma ([20,22,23]) and the comparison of the mountain pass values. When we compare the mountain pass values, the existence of optimal paths in Theorem 1.2 plays a role.

Firstly, under (F1)–(F5), one can check that J has the mountain pass geometry and the mountain pass value is well-defined:

$$0 < d_{\mathrm{MP}} := \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} J(\gamma(t)), \quad \Gamma := \left\{\gamma\in C([0,1], H^\alpha(\mathbf{R}^N)) \mid \gamma(0)=0,\ J(\gamma(1))<0\right\}.$$

Hence, we find that there exists a Palais–Smale sequence $(u_n)_{n=1}^\infty$ at the level d_{MP}, that is, $J(u_n)\to d_{\mathrm{MP}}$ and $J'(u_n)\to 0$ strongly in $(H^\alpha(\mathbf{R}^N))^*$.

Secondly, the sequence $(u_n)_{n=1}^\infty$ is bounded in $H^\alpha(\mathbf{R}^N)$ due to (F5). In fact, set

$$\|u\|^2 := \int_{\mathbf{R}^N}\left(1+4\pi^2|\xi|^2\right)^\alpha |\hat{u}(\xi)|^2 \mathrm{d}\xi + \int_{\mathbf{R}^N} V(x)u^2\mathrm{d}x.$$

By (F2), we observe that $\|\cdot\|$ is equivalent to $\|\cdot\|_\alpha$. Moreover, by (F5), we have

$$\begin{aligned}
\mu d_{\mathrm{MP}} + o(1)\|u_n\| &\ge \mu J(u_n) - J'(u_n)u_n\\
&= \left(\frac{\mu}{2}-1\right)\|u_n\|^2 + \int_{\mathbf{R}^N} g(x,u_n)u_n - \mu G(x,u_n)\mathrm{d}x \ge \left(\frac{\mu}{2}-1\right)\|u_n\|^2,
\end{aligned}$$

which implies that $(u_n)_{n=1}^\infty$ is bounded in $H^\alpha(\mathbf{R}^N)$.

Then we describe the behavior of $(u_n)_{n=1}^\infty$:

Proposition 2.2. *There exist $k \geq 0$, $u_0 \in H^\alpha(\mathbf{R}^N)$, $\omega_j \in H^\alpha(\mathbf{R}^N)$ and $(y_{j,n})_{n=1}^\infty \subset \mathbf{R}^N$ with $j = 1, \ldots, k$ such that $u_n \rightharpoonup u_0$ weakly in $H^\alpha(\mathbf{R}^N)$ and*

(i) $|y_{j,n}| \to \infty$, $|y_{j_1,n} - y_{j_2,n}| \to \infty$ *if* $j_1 \neq j_2$.

(ii) *If $k \geq 1$, then $\omega_j \not\equiv 0$ is a critical point of*

$$J_\infty(u) := \frac{1}{2}\|u\|_\alpha^2 + \frac{1}{2}\int_{\mathbf{R}^N} V_\infty u^2 \mathrm{d}x - \int_{\mathbf{R}^N} G_\infty(u)\mathrm{d}x = \frac{1}{2}\|u\|_\alpha^2 - \int_{\mathbf{R}^N} F_\infty(u)\mathrm{d}x.$$

(iii) *If $k = 0$, then $\|u_n - u_0\|_\alpha \to 0$. On the other hand, if $k \geq 1$, then*

$$\left\| u_n - u_0 - \sum_{j=1}^k \omega_j(\cdot - y_{j,n}) \right\|_\alpha \to 0, \quad d_{\mathrm{MP}} = \lim_{n\to\infty} J(u_n) = J(u_0) + \sum_{j=1}^k J_\infty(\omega_j).$$

We remark that if $u_0 \not\equiv 0$, then u_0 is a nontrivial solution of (1) since $u_n \rightharpoonup u_0$ weakly in $H^\alpha(\mathbf{R}^N)$. Therefore, hereafter we assume $u_0 \equiv 0$. For simplicity, we strengthen (F4) slightly and assume that

(9) $$F(x,s) < F_\infty(s) \quad \text{for every } (x,s) \in \mathbf{R}^N \times (\mathbf{R} \setminus \{0\}).$$

We shall derive a contradiction in order to deduce that the case $u_0 \equiv 0$ never happens.

Since $d_{\mathrm{MP}} > 0$ holds, Proposition 2.2 (iii) yields $k \geq 1$. Next, we remark that Theorem 1.1 can be applied for J_∞. Moreover, by the regularity of g_∞ when $0 < \alpha \leq 1/2$, for any solution of

$$(1-\Delta)^\alpha v + V_\infty v = g_\infty(v) \quad \text{in } \mathbf{R}^N, \quad v \in H^\alpha(\mathbf{R}^N),$$

the Pohozaev identity holds. In particular, we obtain

(10) $$0 < d_{\infty,\mathrm{MP}} \leq J_\infty(\omega_j) \quad \text{for all } 1 \leq j \leq k$$

where $d_{\infty,\mathrm{MP}}$ is the mountain pass value of J_∞:

$$d_{\infty,\mathrm{MP}} := \inf_{\gamma\in\Gamma_\infty} \max_{0\leq t\leq 1} J_\infty(\gamma(t)),$$
$$\Gamma_\infty := \left\{\gamma \in C([0,1], H^\alpha(\mathbf{R}^N)) \mid \gamma(0) = 0,\ J_\infty(\gamma(1)) < 0\right\}.$$

Since $J(u) \leq J_\infty(u)$ holds for every $u \in H^\alpha(\mathbf{R}^N)$ thanks to (9) (or (F4)), one sees that

$$d_{\mathrm{MP}} \leq d_{\infty,\mathrm{MP}}.$$

Combining this inequality with (10) and Proposition 2.2 (iii), we deduce that

(11) $$k = 1, \quad d_{\mathrm{MP}} = d_{\infty,\mathrm{MP}} = J_\infty(\omega_1).$$

Now we utilize the path γ_u in Proposition 2.1 (iii). By $J(u) \leq J_\infty(u)$, we obtain $\gamma_{\omega_1} \in \Gamma$ and let $t_0 > 0$ be a maximum point of the function $t \mapsto J(\gamma_{\omega_1}(t))$. Then it follows from (9) that

$$d_{\mathrm{MP}} \leq \max_{0\leq t} J(\gamma_{\omega_1}(t)) = J(\gamma_{\omega_1}(t_0)) < J_\infty(\gamma_{\omega_1}(t_0)) \leq \max_{0\leq t} J_\infty(\gamma_{\omega_1}(t)) = J_\infty(\omega_1) = d_{\infty,\mathrm{MP}}.$$

However, this contradicts (11). Thus the case $u_0 \equiv 0$ never happens and we get a nontrivial solution of (1).

3 Proof of Theorem 1.3

In this section, we prove Theorem 1.3. We first prepare some lemmas. Denote by $\mathscr{S}(\mathbf{R}^N, \mathbf{R})$ and $(\mathscr{S}(\mathbf{R}^N, \mathbf{R}))^*$ the Schwartz class consisting of real valued functions and its dual space, respectively. Next, we introduce the function $G_{2\alpha}(x)$ by

$$G_{2\alpha}(x) := \frac{1}{(4\pi)^\alpha}\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-\pi|x|^2/t}e^{-t/(4\pi)}t^{(2\alpha-N)/2}\frac{\mathrm{d}t}{t}.$$

Using $G_{2\alpha}$, for $1 \le p < \infty$, set

$$\mathscr{L}^p_{2\alpha} := G_{2\alpha} * L^p(\mathbf{R}^N) = \left\{G_{2\alpha} * g \mid g \in L^p(\mathbf{R}^N)\right\}.$$

Lemma 3.1. (i) *Let $h \in L^p(\mathbf{R}^N)$ with $1 \le p < \infty$ and $u \in (\mathscr{S}(\mathbf{R}^N, \mathbf{R}))^*$ be a solution of*

$$(1 - \Delta)^\alpha u = h \quad \text{in } \mathbf{R}^N. \tag{12}$$

Then $u = G_{2\alpha} * h \in \mathscr{L}^p_{2\alpha}$.

(ii) *$\mathscr{L}^p_{2\alpha} \subset W^{2\alpha,p}(\mathbf{R}^N)$ where*

$$W^{\beta,p}(\mathbf{R}^N) := \left\{u \in L^p(\mathbf{R}^N) \;\middle|\; [u]^p_{W^{\beta,p}(\mathbf{R}^N)} := \int_{\mathbf{R}^N}\int_{\mathbf{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{N+\beta p}}\mathrm{d}x\mathrm{d}y < \infty\right\}$$
$$\text{if } 0 < \beta < 1,$$
$$W^{\beta,p}(\mathbf{R}^N) := \left\{u, \nabla u \in L^p(\mathbf{R}^N) \;\middle|\; [\nabla u]^p_{W^{\beta-1,p}(\mathbf{R}^N)} < \infty\right\} \quad \text{if } 1 < \beta < 2.$$

(iii) *For any $\beta \in \mathbf{R}$, the map $f \mapsto G_{2\alpha} * f : H^\beta(\mathbf{R}^N) \to H^{\beta+2\alpha}(\mathbf{R}^N)$ is isomorphism.*

Proof. It is known (see, for instance, [27]) that

$$\widehat{G_{2\alpha}}(\xi) = (4\pi^2|\xi|^2 + 1)^{-\alpha}, \quad \|G_{2\alpha}\|_{L^1} = 1.$$

Therefore, taking the Fourier transform of (12), we obtain $u = G_{2\alpha} * h$ and (i) holds. For assertions (ii) and (iii), see [27]. □

The next lemma is a variant of Brézis-Kato [7]:

Lemma 3.2. *Assume that $u \in H^\alpha(\mathbf{R}^N)$ is a solution of*

$$(1 - \Delta)^\alpha u - a(x)u = 0 \quad \text{in } \mathbf{R}^N$$

where $a(x)$ satisfies

$$|a(x)| \le C_0(1 + A(x)) \quad \text{for a.e. } x \in \mathbf{R}^N, \quad A \in L^{N/(2\alpha)}(\mathbf{R}^N).$$

Then $u \in L^p(\mathbf{R}^N)$ for all $p \in [2, \infty)$.

For a proof of Lemma 3.2, we refer to [19, Proposition 3.5].

Using Lemmas 3.1 and 3.2, we obtain the following regularity results of solutions of (1) with $f \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$.

Proposition 3.3. *Let $f(x,s) \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ satisfy* (5) *and $u \in H^\alpha(\mathbf{R}^N)$ be a solution of* (1)*. Then $u \in C_{\mathrm{b}}^\beta(\mathbf{R}^N)$ for every $\beta \in (0, 2\alpha)$ where*

$$
\begin{aligned}
C_{\mathrm{b}}^\beta(\mathbf{R}^N) &:= \left\{ u \in C(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \;\middle|\; \sup_{x,y \in \mathbf{R}^N, x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\beta} < \infty \right\} \text{ if } \beta < 1, \\
C_{\mathrm{b}}^1(\mathbf{R}^N) &:= \{ u \in C^1(\mathbf{R}^N) \mid u, \nabla u \in L^\infty(\mathbf{R}^N) \}, \\
C_{\mathrm{b}}^\beta(\mathbf{R}^N) &:= \{ u \in C_{\mathrm{b}}^1(\mathbf{R}^N) \mid \nabla u \in C_{\mathrm{b}}^{\beta-1}(\mathbf{R}^N) \} \text{ if } 1 < \beta < 2.
\end{aligned}
$$

Proof. Let $u \in H^\alpha(\mathbf{R}^N)$ be a solution of (1) and set

$$
a(x) := \begin{cases} \dfrac{f(x,u(x))}{u(x)} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0, \end{cases} \qquad A(x) := |u(x)|^{4\alpha/(N-2\alpha)}.
$$

By (5) and $u \in H^\alpha(\mathbf{R}^N) \subset L^{2_\alpha^*}(\mathbf{R}^N)$, there exists a $C_0 > 0$ such that $A \in L^{N/(2\alpha)}(\mathbf{R}^N)$ and

$$
|a(x)| \le C_0 (1 + A(x)).
$$

Moreover, u is a solution of $(1-\Delta)^\alpha u - a(x)u = 0$ in $\mathbf{R}^N$. Applying Lemma 3.2, we have $u \in L^p(\mathbf{R}^N)$ for all $2 \le p < \infty$. Hence, using (5) again, we observe that $f(x,u(x)) \in L^p(\mathbf{R}^N)$ for any $2 \le p < \infty$. Thus, by Lemma 3.1, one sees $u = G_{2\alpha} * h \in \mathscr{L}_{2\alpha}^p$ where $h(x) := f(x,u(x))$. Recalling $\mathscr{L}_{2\alpha}^p \subset W^{2\alpha,p}(\mathbf{R}^N)$, Sobolev's embedding yields $u \in C_{\mathrm{b}}^\beta(\mathbf{R}^N)$ for all $0 < \beta < 2\alpha$. Thus we complete the proof. □

Now, we prove Theorem 1.3.

Proof of Theorem 1.3. We argue indirectly and suppose that u is a nontrivial solution of (1). By Proposition 3.3, we have $u \in C_{\mathrm{b}}^\beta(\mathbf{R}^N)$ for any $\beta \in (0, 2\alpha)$.

Next we shall prove $\nabla u \in H^\alpha(\mathbf{R}^N)$. To this end, we first claim that $u \in H^\beta(\mathbf{R}^N)$ with $\beta \in (0,1)$ implies $f(x,u(x)) \in H^\beta(\mathbf{R}^N)$. In fact, let $u \in H^\beta(\mathbf{R}^N)$ and decompose $|f(x,u(x)) - f(y,u(y))|$ as follows:

$$
|f(x,u(x)) - f(y,u(y))| \le |f(x,u(x)) - f(y,u(x))| + |f(y,u(x)) - f(y,u(y))|.
$$

We estimate the first term. By $u \in L^\infty(\mathbf{R}^N)$, it follows from (5) and (6) that

$$
\begin{aligned}
|f(x,u(x)) - f(y,u(x))| &\le C|u(x)||x-y| && \text{if } |x-y| \le 1, \\
|f(x,u(x)) - f(y,u(x))| &\le C|u(x)| && \text{if } |x-y| > 1.
\end{aligned}
$$

Hence, by $\beta \in (0,1)$, we see

$$\begin{aligned}
&\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{|f(x,u(x)) - f(y,u(x))|^2}{|x-y|^{N+2\beta}} \mathrm{d}y\mathrm{d}x \\
=& \int_{\mathbf{R}^N} \mathrm{d}x \left(\int_{|y-x|\le 1} + \int_{|y-x|>1} \right) \frac{|f(x,u(x)) - f(y,u(x))|^2}{|x-y|^{N+2\beta}} \mathrm{d}y \\
\le& C \int_{\mathbf{R}^N} \mathrm{d}x \left(\int_{|y-x|\le 1} \frac{|u(x)|^2}{|x-y|^{N-2+2\beta}} \mathrm{d}y + \int_{|y-x|>1} \frac{|u(x)|^2}{|x-y|^{N+2\beta}} \mathrm{d}y \right) \\
\le& C\|u\|_{L^2}^2 < \infty.
\end{aligned}$$

Similarly, for the second term, since the inequality

$$|f(y,u(x)) - f(y,u(y))| \le C|u(x) - u(y)| \quad \text{for each } x,y \in \mathbf{R}^N$$

holds due to $u \in L^\infty(\mathbf{R}^N)$ and (6), the fact $u \in H^\beta(\mathbf{R}^N)$ implies

$$\begin{aligned}
\int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{|f(y,u(x)) - f(y,u(y))|^2}{|x-y|^{N+2\beta}} \mathrm{d}x\mathrm{d}y &\le C \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\beta}} \mathrm{d}x\mathrm{d}y \\
&= [u]^2_{W^{\beta,2}(\mathbf{R}^N)} < \infty.
\end{aligned}$$

Thus, we have $f(x,u(x)) \in H^\beta(\mathbf{R}^N)$ provided $u \in H^\beta(\mathbf{R}^N)$.

Since $u \in H^\alpha(\mathbf{R}^N)$ and $u = G_{2\alpha} * f(x,u(x))$, we observe from Lemma 3.1 that $u \in H^{3\alpha}(\mathbf{R}^N)$. This yields $f(x,u(x)) \in H^{3\alpha}(\mathbf{R}^N)$ and $u = G_{2\alpha} * f(x,u(x)) \in H^{5\alpha}(\mathbf{R}^N)$. Iterating this argument, we get $u \in H^\beta(\mathbf{R}^N)$ for all $0 < \beta < 1$, hence, $u \in H^{\beta+2\alpha}(\mathbf{R}^N)$ for each $0 < \beta < 1$. Therefore, we have $\nabla u \in H^\alpha(\mathbf{R}^N)$.

Now, we derive a contradiction. Since $I'(u) = 0$ and $e \cdot \nabla u \in H^\alpha(\mathbf{R}^N)$, we have
(13)

$$\begin{aligned}
0 &= I'(u)[e \cdot \nabla u(x)] \\
&= \int_{\mathbf{R}^N} (1 + 4\pi|\xi|^2)^\alpha \hat{u}(\xi) e \cdot (-2\pi i \xi) \overline{\hat{u}(\xi)} \mathrm{d}\xi - \int_{\mathbf{R}^N} f(x,u) e \cdot \nabla u(x) \mathrm{d}x \\
&= -2\pi i \int_{\mathbf{R}^N} (1 + 4\pi|\xi|^2)^\alpha |\hat{u}(\xi)|^2 \xi \cdot e \mathrm{d}\xi - \int_{\mathbf{R}^N} e \cdot \nabla_x (F(x,u)) - e \cdot (\nabla_x F)(x,u) \mathrm{d}x.
\end{aligned}$$

Since it follows from (6) that

$$|F(x,s)| + |\nabla_x F(x,s)| \le C|s|^2 \quad \text{for } s \in [-\|u\|_{L^\infty}, \|u\|_{L^\infty}]$$

by $\nabla u \in L^2(\mathbf{R}^N)$, we have $F(x,u(x)), \nabla_x(F(x,u(x))) \in L^1(\mathbf{R}^N)$. From

$$\int_0^\infty \int_{\partial B_r(0)} |F(r\sigma, u(r\sigma))| \mathrm{d}\sigma \mathrm{d}r < \infty,$$

we may find a sequence $(R_n)_{n=1}^\infty$ such that

$$R_n \to \infty, \quad \int_{\partial B_{R_n}(0)} |F(R_n\sigma, u(R_n\sigma))| \mathrm{d}\sigma \to 0.$$

Using the divergence theorem, we infer that

$$\int_{\mathbf{R}^N} \nabla_x(F(x,u))\mathrm{d}x = \lim_{n\to\infty}\int_{B_{R_n}(0)} \nabla_x(F(x,u))\mathrm{d}x = -\lim_{n\to\infty}\int_{\partial B_{R_n}(0)} F(x,u(x))\frac{x}{|x|}\mathrm{d}\sigma = 0.$$

Thus, taking the real part in (13), we get

$$0 = \int_{\mathbf{R}^N} e\cdot(\nabla_x F)(x,u)\mathrm{d}x.$$

However, from (7) and the fact that u is nontrivial, it follows that $\int_{\mathbf{R}^N} e\cdot(\nabla_x F)(x,u)\mathrm{d}x > 0$ and this is a contradiction. Thus (1) has no nontrivial solution and we complete the proof. □

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