数理解析研究所講究録
第2049巻 2017年 39-44

Wulff shapes and their duals

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1 Introduction

The Wulff construction is well-known as a geometric model of an equilibrium crystal defined as follows. Let n be a positive integer. Given a continuous function $\gamma : S^n \to \mathbb{R}_+$ where $S^n \subset \mathbb{R}^{n+1}$ is the unit sphere and $\mathbb{R}_+$ is the set consisting of positive real numbers, the *Wulff shape associated with* γ, denoted by $\mathcal{W}_\gamma$, is the following intersection (see Figure 1)

$$\mathcal{W}_\gamma = \bigcap_{\theta \in S^n} \Gamma_{\gamma,\theta}.$$

Here, $\Gamma_{\gamma,\theta}$ is the following half-space:

$$\Gamma_{\gamma,\theta} = \{x \in \mathbb{R}^{n+1} \mid x \cdot \theta \leq \gamma(\theta)\}.$$

By Wulff construction, we know that Wulff shape is a compact, convex and contains the the origin of

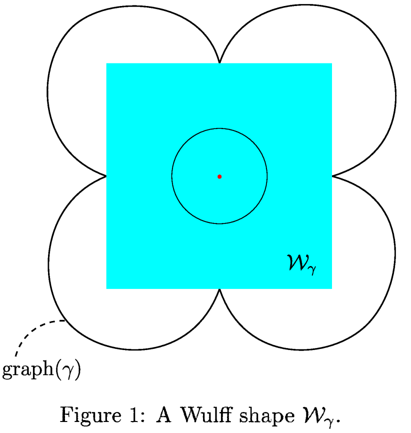

Figure 1: A Wulff shape $\mathcal{W}_\gamma$.

$\mathbb{R}^{n+1}$ as an interior point. Conversely, it is well-known that any convex body W contains the origin as

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an interior point is a Wulff shape given by appropriate support function, namely, there is a continuous function $\gamma : S^n \to \mathbb{R}_+$ such that $\mathcal{W}_\gamma = W$. For details on Wulff shapes, see for instance [1, 6, 13, 14].

For a continuous function $\gamma : S^n \to \mathbb{R}_+$, set

$$\mathrm{graph}(\gamma) = \{(\theta, \gamma(\theta)) \in \mathbb{R}^{n+1} - \{0\} \mid \theta \in S^n\},$$

where $(\theta, \gamma(\theta))$ is the polar plot expression for a point of $\mathbb{R}^{n+1} - \{0\}$. The mapping $\mathrm{inv} : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}$, defined as follows, is called the *inversion* with respect to the origin of $\mathbb{R}^{n+1}$.

$$\mathrm{inv}(\theta, r) = \left(-\theta, \frac{1}{r}\right).$$

Let Γ_γ be the boundary of the convex hull of $\mathrm{inv}(\mathrm{graph}(\gamma))$.

Definition 1 ([12, 10])
Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. If the equality $\Gamma_\gamma = \mathrm{inv}(\mathrm{graph}(\gamma))$ is satisfied, then γ is called a *convex integrand.*

The notion of convex integrand was firstly introduced by J. Taylor in [12] and it plays a key role for studying Wulff shapes (for details on convex integrands, see for instance [4, 7, 12]).

Definition 2 ([10])
Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. The convex hull of $\mathrm{inv}(\mathrm{graph}(\gamma))$ is called *dual Wulff of* $\mathcal{W}_\gamma$, denoted by $\mathcal{DW}_\gamma$.

The main topic of this paper is the relations between Wulff shapes and its duals.

2 Properties and some known results

Before proceeding further, we first introduce an equivalent definition of Wulff shape, given in [10].

(1) $Id : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \{1\}$**.**
Let $Id : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \{1\}$ be the map defined by $Id(x) = (x, 1)$.

(2) $\alpha_N : S^{n+1}_{N,+} \to \mathbb{R}^{n+1} \times \{1\}$**.**
Denote the point $(0, \ldots, 0, 1) \in \mathbb{R}^{n+2}$ by N. The set $S^{n+1} - H(-N)$ is denoted by $S^{n+1}_{N,+}$. Let $\alpha_N : S^{n+1}_{N,+} \to \mathbb{R}^{n+1} \times \{1\}$ be the central projection relative to N, namely, α_N is defined as follows for any $P = (P_1, \ldots, P_{n+1}, P_{n+2}) \in S^{n+1}_{N,+}$:

$$\alpha_N\left(P_1, \ldots, P_{n+1}, P_{n+2}\right) = \left(\frac{P_1}{P_{n+2}}, \ldots, \frac{P_{n+1}}{P_{n+2}}, 1\right).$$

(3) $\Psi_N : S^{n+1} - \{\pm N\} \to S^{n+1}_{N,+}$**.**
Next, we consider the mapping $\Psi_N : S^{n+1} - \{\pm N\} \to S^{n+1}_{N,+}$, defined by

$$\Psi_N(\widetilde{P}) = \frac{1}{\sqrt{1 - (N \cdot \widetilde{P})^2}}(N - (N \cdot \widetilde{P})\widetilde{P}).$$

The mapping Ψ_N was introduced in [9], has the following intriguing properties:

1. For any $\widetilde{P} \in S^{n+1} - \{\pm N\}$, the equality $\widetilde{P} \cdot \Psi_N(\widetilde{P}) = 0$ holds,
2. for any $\widetilde{P} \in S^{n+1} - \{\pm N\}$, the property $\Psi_N(\widetilde{P}) \in \mathbb{R}N + \mathbb{R}\widetilde{P}$ holds,
3. for any $\widetilde{P} \in S^{n+1} - \{\pm N\}$, the property $N \cdot \Psi_N(\widetilde{P}) > 0$ holds,
4. the restriction $\Psi_N|_{S^{n+1}_{N,+} - \{N\}} : S^{n+1}_{N,+} - \{N\} \to S^{n+1}_{N,+} - \{N\}$ is a C^∞ diffeomorphism.

(4) Spherical polar transform.
For any point $\widetilde{P} \in S^{n+1}$, let $H(\widetilde{P})$ be the closed hemisphere centered at $\widetilde{P}$, namely,

$$H(\widetilde{P}) = \{\widetilde{Q} \in S^{n+1} | \widetilde{P} \cdot \widetilde{Q} \geq 0\},$$

where the dot in the center stands for the scalar product of two vectors $\widetilde{P}, \widetilde{Q} \in \mathbb{R}^{n+2}$. For any non-empty subset $\widetilde{W} \subset S^{n+1}$, the *spherical polar set of* $\widetilde{W}$, denoted by $\widetilde{W}^{\circ}$, is defined as follows:

$$\widetilde{W}^{\circ} = \bigcap_{\widetilde{P} \in \widetilde{W}} H(\widetilde{P}).$$

for details on spherical polar set, see for instance [3, 10]

Proposition 1 ([10])
Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Let graph$(\gamma) = \{(\theta, \gamma(\theta)) \in \mathbb{R}^{n+1} - \{0\} \mid \theta \in S^n\}$, where $(\theta, \gamma(\theta))$ is the polar plot expression for a point of $\mathbb{R}^{n+1} - \{0\}$. Then, $\mathcal{W}_\gamma$ is characterized as follows:

$$\mathcal{W}_\gamma = Id^{-1} \circ \alpha_N \left(\left(\Psi_N \circ \alpha_N^{-1} \circ Id\,(\mathrm{graph}(\gamma)) \right)^{\circ} \right).$$

For any Wulff shape $\mathcal{W}_\gamma$, by Proposition 1, the dual Wulff shape $\mathcal{DW}_\gamma$ can characterized as follows:

Proposition 2 ([10])
For any Wulff shape $\mathcal{W}_\gamma$, the following is holds:

$$\mathcal{DW}_\gamma = Id^{-1} \circ \alpha_N \left(\left(\alpha_N^{-1} \circ Id\,(\mathcal{W}_\gamma) \right)^{\circ} \right).$$

In general case, for given Wulff shape W, there exist uncountable many support function γ construct W(see Figure 2). Then, it is natural arise that "When does given Wulff shape has only one support fuction?". In [4], it is shown that Wulff shape W is strictly convex if and only if its convex integrand γ is of class C^1. By this result, e the following theorem is not difficult to prove.

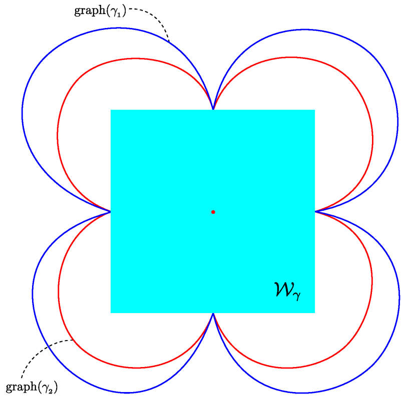

Figure 2: A Wulff shape $\mathcal{W}_\gamma$.

Theorem 1 ([6])
Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function and let $\mathcal{W}_\gamma$ be the Wulff shape associated with γ. Suppose that the boundary of $\mathcal{W}_\gamma$ is a C^1 submanifold. Then, γ must be the convex integrand of $\mathcal{W}_\gamma$.

For Wulff shapes and their dual Wulff shapes, we have the following relations.

Proposition 3 ([4])
A Wulff shape in $\mathbb{R}^{n+1}$ is strictly convex if and only if the boundary of its dual Wulff shape is C^1 diffeomorphic to S^n.

Proposition 4 ([4])
A Wulff shape in $\mathbb{R}^{n+1}$ is strictly convex and its boundary is C^1 diffeomorphic to S^n if and only if its dual Wulff shape is strictly convex and the boundary of it is C^1 diffeomorphic to S^n.

Proposition 5 ([10])
A Wulff shape in $\mathbb{R}^{n+1}$ is a polytope if and only if its dual Wulff shape is a polytope.

Definition 3 ([2])
Let γ_1, γ_2 be convex integrands. Define $\gamma_{_{max}}$ and $\gamma_{_{min}}$ as natural way.

$$\gamma_{_{max}} : S^n \to \mathbb{R}_+, \gamma_{_{max}}(\theta) = \max\{\gamma_1(\theta), \gamma_2(\theta)\}.$$

$$\gamma_{_{min}} : S^n \to \mathbb{R}_+, \gamma_{_{min}}(\theta) = \min\{\gamma_1(\theta), \gamma_2(\theta)\}.$$

Proposition 6 ([2])
Let $\mathcal{W}_{\gamma_1}, \mathcal{W}_{\gamma_2}$ be dual Wulff shapes. Then $\mathcal{W}_{\gamma_{_{max}}}$ is the dual Wulff shape of $\mathcal{W}_{\gamma_{_{min}}}$.

3 Self-dual Wulff shapes

Definition 4 ([5])
Let W be a Wulff shape. If Wulff shape W and its dual Wulff shape are same convex body, then W is said to be *self-dual* Wulff shape.

By Proposition 1, we have the following.

Corollary 1
Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Then the following are equivalent.

1. $\mathcal{W}_\gamma = \mathcal{D}\mathcal{W}_\gamma$.
2. $\mathcal{W}_\gamma = Id^{-1} \circ \alpha_N \left(\left(\alpha_N^{-1} \circ Id\,(\mathcal{W}_\gamma) \right)^\circ \right)$.
3. *$\mathcal{W}_\gamma$ is exactly the convex hull of inv(graph(γ)).*

Moreover, self-dual Wulff shape can characterized as follows.

Definition 5 ([3])
1. A subset $\widetilde{W}$ of S^{n+1} is said to be *hemispherical* if there exists a point $\widetilde{P} \in S^{n+1}$ such that $\widetilde{W} \cap H(\widetilde{P}) = \emptyset$.

2. A hemispherical subset $\widetilde{W} \subset S^{n+1}$ is said to be *spherical convex* if for any $\widetilde{P}, \widetilde{Q} \in \widetilde{W}$ the following arc $\widetilde{P}\widetilde{Q}$ is contained in $\widetilde{W}$:

$$\widetilde{P}\widetilde{Q} = \left\{ \frac{(1-t)\widetilde{P} + t\widetilde{Q}}{||(1-t)\widetilde{P} + t\widetilde{Q}||} \;\middle|\; t \in [0,1] \right\}.$$

3. A hemispherical subset $\widetilde{W}$ is called a *spherical convex body* if it is closed, spherical convex and has an interior point. A hemisphere $H(\widetilde{P})$ is said to *support a spherical convex body* $\widetilde{W}$ if both $\widetilde{W} \subset H(\widetilde{P})$ and $\partial\widetilde{W} \cap \partial H(\widetilde{P}) \neq \emptyset$ hold.

Definition 6 ([8])

1. For any two $\widetilde{P}, \widetilde{Q} \in S^{n+1}$ ($\widetilde{P} \neq \pm\widetilde{Q}$), the intersection $H(\widetilde{P}) \cap H(\widetilde{Q})$ is called a *lune* of S^{n+1}.

2. The *thickness of the lune* $H(\widetilde{P}) \cap H(\widetilde{Q})$, denoted by $\triangle(H(\widetilde{P}) \cap H(\widetilde{Q}))$, is the real number $\pi - |\widetilde{P}\widetilde{Q}|$, where $|\widetilde{P}\widetilde{Q}|$ stands for the length of the arc $\widetilde{P}\widetilde{Q}$.

3. For a spherical convex body $\widetilde{W}$ and a hemisphere $H(\widetilde{P})$ supporting $\widetilde{W}$, the *width of* $\widetilde{W}$ *determined by* $H(\widetilde{P})$, denoted by $\mathrm{width}_{H(\widetilde{P})}\widetilde{W}$, is the minimum of the following set:

$$\left\{\triangle(H(\widetilde{P}) \cap H(\widetilde{Q})) \;\middle|\; \widetilde{W} \subset H(\widetilde{P}) \cap H(\widetilde{Q}), H(\widetilde{Q}) \text{ supports } \widetilde{W}\right\}.$$

4. For any $\rho \in \mathbb{R}_+$ less than π, a spherical convex body $\widetilde{W} \subset S^{n+1}$ is said to be *of constant width* ρ if $\mathrm{width}_{H(\widetilde{P})}\widetilde{W} = \rho$ for any $H(\widetilde{P})$ supporting $\widetilde{W}$.

Theorem 2 ([5])
Let $\gamma : S^n \to \mathbb{R}_+$ be a continuous function. Then, the Wulff shape W_γ is self-dual if and only if the spherical convex body $\widetilde{W}_\gamma = \alpha_N^{-1} \circ Id(W_\gamma)$ is of constant width $\pi/2$.

Definition 7 ([8])
Let $\widetilde{W}$ be a spherical convex body of S^{n+1}.

1. Thickness $\Delta(\widetilde{W})$ of $\widetilde{W} \subset S^{n+1}$ defined as follows:

$$\Delta(\widetilde{W}) = inf\{width_K(\widetilde{W}); K \text{ is a supporting hemisphere of } \widetilde{W}\}.$$

2. $\widetilde{W} \subset S^{n+1}$ is said to be *reduced* if $\Delta(\widetilde{Y}) < \Delta(\widetilde{W})$ for every convex body $\widetilde{Y} \subset \widetilde{W}$ different from $\widetilde{W}$.

Theorem 3 ([8])
Every smooth reduced body on S^n is of constant width.

In the case of Wulff shapes, the following seems to be open.

Definition 8 ([8])
Let $\widetilde{W} \subset S^{n+1}$ be a spherical convex body. Then, the following number is called the *diameter* of $\widetilde{W}$ and is denoted by $\mathrm{diam}\left(\widetilde{W}\right)$:

$$\max\left\{|\widetilde{P}\widetilde{Q}| \;\middle|\; \widetilde{P}, \widetilde{Q} \in \widetilde{W}\right\}.$$

Question: Let W be a Wulff shape. Are the following equivalent?

1. Wulff shape W is self-dual.

2. Spherical convex body $\widetilde{W}_\gamma = \alpha_N^{-1} \circ Id(W_\gamma)$ is reduced and $\mathrm{diam}(\widetilde{W}) = \pi/2$.

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