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On some classification results of real singularities up to the arc-analytic equivalence

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Abstract

This note is an expanded version of a talk given during the conference *Singularity theory of differential maps and its applications* at the RIMS, Kyoto (December 6–9, 2016).

We first state the definition and some properties of the arc-analytic equivalence which is an equivalence relation with no continuous moduli on Nash (i.e. real analytic and semialgebraic) function germs. It is a semialgebraic version of the blow-analytic equivalence of T.-C. Kuo.

Then, we present an invariant of the arc-analytic equivalence which is constructed following the motivic zeta function of Denef–Loeser.

Finally, we explain how to derive from it some classification results for Brieskorn polynomials and more generally for some weighted homogeneous polynomials.

Contents

1 The arc-analytic equivalence

H. Whitney [25, Example 13.1] noticed that the cross-ratio is a continuous modulus of the family $f_t : (\mathbb{R}^2,0) \to (\mathbb{R},0)$, $t \in (0,1)$, defined by $f_t(x,y) = xy(y-x)(y-tx)$. Particularly, two distinct function germs of this family are never C^1-equivalent, i.e. if there exists a C^1-diffeomorphism $\varphi : (\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ such that $f_{t'} = f_t \circ \varphi$ then $t = t'$.

T.-C. Kuo [15] suggested the blow-analytic equivalence as a candidate to obtain a classification of real singularities without continuous moduli. He proved that this notion is an equivalence relation on real analytic function germs and that it admits no continuous moduli for isolated singularities. Indeed, a family of real analytic function germs with isolated singularities defines locally finitely many blow-analytic equivalence classes.

Up to now, the known invariants of the blow-analytic equivalence are the Fukui invariants [12] and the Koike–Parusiński zeta functions [14]. In order to construct richer invariants, G. Fichou [8, 9] introduced a semialgebraic version of the blow-analytic equivalence called the blow-Nash equivalence. It is a relation on Nash * function germs with no continuous moduli for isolated singularities. The notion of blow-Nash equivalence evolved and stabilized to the following: two Nash function germs $f,g:(\mathbb{R}^d,0)\to(\mathbb{R},0)$ are blow-Nash equivalent if, after being composed with Nash modifications †, they are Nash-equivalent via a Nash-diffeomorphism which preserves the multiplicities of the Jacobian determinants of the modifications. Initially, it was expected, but not known yet, whether this relation is an equivalence relation on Nash function germs.

The goal of this section is to introduce the arc-analytic equivalence defined in [6]. It is a characterization of the blow-Nash equivalence in terms of arc-analytic maps. It avoids to involve Nash modifications and it is an equivalence relation. Moreover, A. Parusiński and L. Păunescu [21] recently proved it admits no continuous moduli, even for families of non-isolated singularities.

Definition 1.1 ([6, Definition 7.5]). Two Nash function germs $f,g:(\mathbb{R}^d,0)\to(\mathbb{R},0)$ are arc-analytically equivalent if there exists a semialgebraic homeomorphism $\varphi:(\mathbb{R}^d,0)\to(\mathbb{R}^d,0)$ such that

(i) $g=f\circ\varphi$,
(ii) φ is arc-analytic, i.e. for $\gamma:(\mathbb{R},0)\to(\mathbb{R}^d,0)$ real analytic, the composition $\varphi\circ\gamma$ is also real analytic,
(iii) There exists $c>0$ such that $|\det d\varphi|>c$ where $d\varphi$ is defined ‡.

Remark 1.2. By [4, Corollary 3.6], for φ as in the previous definition, the converse φ^{-1} is also arc-analytic and there exists $\tilde{c}>0$ such that $|\det d\varphi^{-1}|>\tilde{c}$ where $d\varphi^{-1}$ is defined. Particularly, we get the following proposition.

Proposition 1.3 ([6, Proposition 7.7]). *The arc-analytic equivalence is an equivalence relation on Nash function germs* $(\mathbb{R}^d,0)\to(\mathbb{R},0)$.

The following proposition states that the arc-analytic equivalence is a characterization of the blow-Nash equivalence. Particularly, the blow-Nash equivalence is an equivalence relation as expected.

Proposition 1.4 ([6, Proposition 7.9]). *Two Nash function germs are arc-analytically equivalent if and only if they are blow-Nash equivalent.*

The following result ensures that the arc-analytic equivalence has no continuous moduli, even for families of non-isolated singularities. It is a consequence of [21, Theorem 8.5] together with the proof of [21, Theorem 3.3] and formula [21, (3.9)].

Theorem 1.5 (Parusiński–Păunescu). *Let* $F:(\mathbb{R}^d\times I,\{0\}\times I)\to(\mathbb{R},0)$ *be a Nash germ. Then the germs* $f_t(x)=F(t,x):(\mathbb{R}^d,0)\to(\mathbb{R},0)$, $t\in I$, *define locally finitely many arc-analytic classes.*

2 A motivic invariant of the arc-analytic equivalence

This section is devoted to the invariant of the arc-analytic equivalence introduced in [6]. This invariant is constructed following the motivic zeta function of Denef–Loeser [7] but with coefficients in a real analogue of the Grothendieck ring introduced by Guibert–Loeser–Merle [13]. It generalizes the motivic zeta functions of Koike–Parusiński [14] and of G. Fichou [8, 9].

* A Nash function is a real analytic function with semialgebraic graph

† A Nash modification is a proper surjective Nash map whose complexification is proper and bimeromorphic.

‡ K. Kurdyka [16, Théorème 5.2] proved that a semialgebraic arc-analytic map is real analytic outside a set of codimension 2.

Definition 2.1 ([20, §4.2]). An $\mathcal{AS}$-set is a semialgebraic subset $A \subset \mathbb{P}^n_{\mathbb{R}}$ such that given a real analytic arc $\gamma : (-1,1) \to \mathbb{P}^n_{\mathbb{R}}$ satisfying $\gamma(-1,0) \subset A$ there exists $\varepsilon > 0$ such that $\gamma(0,\varepsilon) \subset A$.

Remark 2.2 ([20, §4.2]). The $\mathcal{AS}$-subsets of $\mathbb{P}^n_{\mathbb{R}}$ form the boolean algebra spanned by semialgebraic arc-symmetric (in the sense of K. Kurdyka [16]) subsets of $\mathbb{P}^n_{\mathbb{R}}$. Particularly, $\mathcal{AS}$ is stable by $\cup, \cap, \setminus$.

Definition 2.3. We denote by $K_0(\mathcal{AS})$ the free abelian group spanned by symbols $[A]$, $A \in \mathcal{AS}$ modulo:
(i) If there is a bijection $A \to B$ with $\mathcal{AS}$-graph then $[A] = [B]$.
(ii) If B is a closed $\mathcal{AS}$-subset of A then $[A] = [A \setminus B] + [B]$.
Moreover, $K_0(\mathcal{AS})$ has a ring structure induced by the cartesian product:
(iii) $[A \times B] = [A][B]$.
We denote by $0 = [\varnothing]$ the class of the empty set which is the unit of the addition, by $1 = [\{*\}]$ the class of the point which is the unit of the product and by $\mathbb{L}_{\mathcal{AS}} = [\mathbb{R}]$ the class of the affine line.

Notation 2.4. We denote by $\mathcal{M}_{\mathcal{AS}} = K_0(\mathcal{AS})\left[\mathbb{L}_{\mathcal{AS}}^{-1}\right]$ the localization of $K_0(\mathcal{AS})$ with respect to $\left\{\mathbb{L}^i_{\mathcal{AS}}, i \in \mathbb{N}\right\}$.

The interest of working with $\mathcal{AS}$-sets here is the existence of the virtual Poincaré polynomial.

Theorem 2.5 ([17][8][18]). *There exists a unique ring morphism $\beta : K_0(\mathcal{AS}) \to \mathbb{Z}[u]$, called the* virtual Poincaré polynomial, *such that, if $A \in \mathcal{AS}$ is compact and non-singular then $\beta([A]) = \sum_i \dim H_i(A, \mathbb{Z}_2) u^i$.*

Moreover, the virtual Poincaré polynomial encodes the dimension since, if $A \in \mathcal{AS}$ is nonempty, $\deg \beta([A]) = \dim A$ (and the leading coefficient is positive).

Remark 2.6 ([22]). Notice that if we omit the arc-symmetric condition to work with all semialgebraic sets then we may deduce from the cell decomposition that every additive invariant of the semialgebraic sets up to semialgebraic homeomorphism factorises through the Euler characteristic with compact support. In this situation, it is impossible to recover the dimension, since, for example, $\chi_c(S^1) = 0$ (whereas S^1 is nonempty). Notice also that for an $\mathcal{AS}$-set A, $\beta([A])(u = -1) = \chi_c(A)$.

Definition 2.7. We denote by $K_0(\mathcal{AS}_{\mathbb{R}^*})$ the free abelian group spanned by symbols $[\varphi_X : X \to \mathbb{R}^*]$, where X and the graph Γ_{φ_X} are in $\mathcal{AS}$, modulo the relations:
(i) If there is a bijection $h : X \to Y$ with $\mathcal{AS}$-graph such that $\varphi_X = \varphi_Y \circ h$ then

$$[\varphi_X : X \to \mathbb{R}^*] = [\varphi_Y : Y \to \mathbb{R}^*]$$

(ii) If $Y \subset X$ is a closed $\mathcal{AS}$-subset then

$$[\varphi_X : X \to \mathbb{R}^*] = [\varphi_{X|X\setminus Y} : X \setminus Y \to \mathbb{R}^*] + [\varphi_{X|Y} : Y \to \mathbb{R}^*]$$

The fiber product induces a ring structure by adding the relation:
(iii) $[X \times_{\mathbb{R}^*} Y \to \mathbb{R}^*] = [\varphi_X : X \to \mathbb{R}^*][\varphi_Y : Y \to \mathbb{R}^*]$
The cartesian product induces a $K_0(\mathcal{AS})$-algebra structure by adding the relation:
(iv) $[A][\varphi_X : X \to \mathbb{R}^*] = [\varphi_X \circ \mathrm{pr}_2 : A \times X \to \mathbb{R}^*]$
We denote by $0 = [\varnothing]$ the class of the empty set which is the unit of the addition, by

$$\mathbb{1} = [\mathrm{id} : \mathbb{R}^* \to \mathbb{R}^*]$$

the class of the identity which is the unit of the product and by

$$\mathbb{L} = \mathbb{L}_{\mathcal{AS}} \mathbb{1} = [\mathrm{pr}_2 : \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}^*]$$

the class of the affine line.

Remark 2.8. The group considered in [6] is equivariant since it is assumed that X is equipped with an action of $\mathbb{R}^*$ compatible with φ_X in some sense. We also work with equivariant isomorphism classes and thus it is necessary to add technical relations in order to identify some classes.

This equivariant aspect is omitted in this note to simplify the presentation. However it is necessary to prove that the convolution formula of [6] is compatible with the one of [14]. We also believe that it is needed for a better comprehension of the so-called *real motivic Milnor fiber*.

Notation 2.9. We set $\mathcal{M} = K_0(\mathcal{AS}_{\mathbb{R}^*})[\mathbb{L}^{-1}]$. Notice that $\mathcal{M}$ has a natural structure of $\mathcal{M}_{\mathcal{AS}}$-algebra.

Proposition 2.10 ([6, §3]). *There exists a unique morphism* $\overline{\cdot} : \mathcal{M} \to \mathcal{M}_{\mathcal{AS}}$ *of* $\mathcal{M}_{\mathcal{AS}}$*-modules induced on symbols by*

$$[\varphi_X : X \to \mathbb{R}^*] \mapsto [X]$$

It is called the forgetful morphism.

Proposition 2.11 ([6, Proposition 4.16]). *For* $\varepsilon \in \{+,-\}$*, there exists a unique morphism* $F^\varepsilon : \mathcal{M} \to \mathcal{M}_{\mathcal{AS}}$ *of* $\mathcal{M}_{\mathcal{AS}}$*-algebras induced on symbols by*

$$[\varphi_X : X \to \mathbb{R}^*] \mapsto [\varphi_X^{-1}(\varepsilon 1)]$$

Remark 2.12. The forgetful morphism is not compatible with the ring structures since the one on $\mathcal{M}$ is induced by the fiber product whereas the one on $\mathcal{M}_{\mathcal{AS}}$ is induced by cartesian product. This is highlighted by computing $\beta\left(\overline{1}\right) = u + 1 \neq 1 = \beta(1)$.

However, the morphisms F^ε are compatible with the ring structures since the fiber product over one point coincides with the cartesian product.

Definition 2.13. Let $f : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ be a Nash function germ. We define the local motivic zeta function of f by

$$Z_f(T) = \sum_{n \geq 1} \left[\mathrm{ac}_f^n : \mathfrak{X}_n(f) \to \mathbb{R}^*\right] \mathbb{L}^{-nd} T^n \in \mathcal{M}[\![T]\!]$$

where $\mathfrak{X}_n(f) = \left\{\gamma = a_1 t + \ldots + a_n t^n, a_i \in \mathbb{R}^d, f(\gamma(t)) = ct^n + \cdots, c \neq 0\right\}$ and $\mathrm{ac}_f^n : \mathfrak{X}_n(f) \to \mathbb{R}^*$ is the angular component map defined by $\mathrm{ac}_f^n(\gamma) = \mathrm{ac}(f \circ \gamma) := c$.

Theorem 2.14 ([6, Theorem 7.11]). *If* $f, g : (\mathbb{R}^d, 0) \to (\mathbb{R}, 0)$ *are two arc-analytically equivalent Nash function germs then* $Z_f(T) = Z_g(T)$.

The heuristic idea of the proof is the following. First, let s be a formal variable and set $T = \mathbb{L}^{-s}$. Then, after some small changes, $Z_f(T)$ may be seen as a motivic integral with parameter s, whatever it means:

$$Z_f(T) = \int_{\mathcal{L}(\mathbb{R}^d, 0)} \mathbb{L}^{-\mathrm{ord}_t f \cdot s}$$

Now assume that f and g are arc-analytically equivalent, then there exists φ as in Definition 1.1. By a result of Bierstone–Milman [2] and A. Parusiński [19], there exists $\sigma : (M, E) \to (\mathbb{R}^d, 0)$ a finite sequence of algebraic blowings-up with non-singular centers such that $\tilde{\sigma} = \varphi \circ \sigma$ is Nash. Therefore we have the following commutative diagram

$$\begin{tikzcd} & (M,E) \arrow[dl, "\sigma"'] \arrow[dr, "\tilde{\sigma}"] & \\ (\mathbb{R}^d,0) \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & (\mathbb{R}^d,0) \arrow[dl, "g"] \\ & (\mathbb{R},0) & \end{tikzcd}$$

By the motivic change of variables formula, we get

$$Z_f(T) = \int_{\mathcal{L}(\mathbb{R}^d,0)} \mathbb{L}^{-\mathrm{ord}_t f \cdot s} = \int_{\mathcal{L}(M,E)} \mathbb{L}^{-\mathrm{ord}_t (f\circ\sigma)\cdot s - \mathrm{ord}_t \mathrm{Jac}_\sigma}$$

Since the previous diagram commutes, $f \circ \sigma = g \circ \tilde{\sigma}$ and, by 1.1.(iii), $\mathrm{ord}_t \mathrm{Jac}_\sigma = \mathrm{ord}_t \mathrm{Jac}_{\tilde{\sigma}}$. Then, again by the change of variables formula, we may conclude

$$Z_f(T) = \int_{\mathcal{L}(M,E)} \mathbb{L}^{-\mathrm{ord}_t (f\circ\sigma)\cdot s - \mathrm{ord}_t \mathrm{Jac}_\sigma} = \int_{\mathcal{L}(M,E)} \mathbb{L}^{-\mathrm{ord}_t (g\circ\tilde{\sigma})\cdot s - \mathrm{ord}_t \mathrm{Jac}_{\tilde{\sigma}}} = \int_{\mathcal{L}(\mathbb{R}^d,0)} \mathbb{L}^{-\mathrm{ord}_t g \cdot s} = Z_g(T)$$

Notice that, in [6] (and before in [14] and [8]), we avoid to introduce the motivic measure (for which we would need to work with a completion of $\mathcal{M}$) and the motivic integral. For this purpose, the change of variable formula is hidden in a computation of $Z_f(T)$ in terms of σ directly with the coefficients of $Z_f(T)$ as a power series in T, in a way similar to Denef–Loeser for their proof of the rationality of their motivic zeta functions. Then we compare these rational formulae of $Z_f(T)$ and $Z_g(T)$ to conclude.

3 A convolution formula

Proposition 3.1. *There exists a unique $K_0(\mathcal{AS})$-bilinear map $* : K_0(\mathcal{AS}_{\mathbb{R}^*}) \times K_0(\mathcal{AS}_{\mathbb{R}^*}) \to K_0(\mathcal{AS}_{\mathbb{R}^*})$ satisfying the following relation on symbols*

$$\begin{aligned} &[\varphi_X : X \to \mathbb{R}^*] * [\varphi_X : X \to \mathbb{R}^*] \\ &\quad = -\left[\varphi_X + \varphi_Y : X \times Y \setminus (\varphi_X + \varphi_Y)^{-1}(0) \to \mathbb{R}^*\right] + \left[\mathrm{pr}_2 : (\varphi_X + \varphi_Y)^{-1}(0) \times \mathbb{R}^* \to \mathbb{R}^*\right] \end{aligned}$$

It is called the convolution product.

Remark 3.2. It induces a $\mathcal{M}_{\mathcal{AS}}$-bilinear map $* : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. It is associative, commutative and it admits $\mathbb{1}$ as unit.

Definition 3.3. The modified zeta function of a Nash function germ $f : (\mathbb{R}^d,0) \to (\mathbb{R},0)$ is defined by

$$\tilde{Z}_f(T) = Z_f(T) - \frac{\mathbb{1} - Z_f^{\mathrm{naive}}(T)}{\mathbb{1} - T} + \mathbb{1}$$

where $Z_f^{\mathrm{naive}}(T)$ is defined by applying $\alpha \mapsto \overline{\alpha}\mathbb{1}$ coefficientwise to $Z_f(T)$.

Remark 3.4 ([6, Corollary 6.14]). The modified zeta function and the zeta function encode the same information since

$$Z_f(T) = \tilde{Z}_f(T) + \frac{\mathbb{1} - \mathbb{L}^{-1}\tilde{Z}_f^{\mathrm{naive}}(T)}{\mathbb{1} - \mathbb{L}^{-1}T} - \mathbb{1}$$

Theorem 3.5 (The convolution formula [6, Theorem 6.15]). *For $i = 1,2$, let $f_i : (\mathbb{R}^{d_i},0) \to (\mathbb{R},0)$ be a Nash function germ and define $f_1 \oplus f_2 : (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2},0) \to (\mathbb{R},0)$ by $f_1 \oplus f_2(x_1,x_2) = f_1(x_1)+f_2(x_2)$. Then*

$$\tilde{Z}_{f_1 \oplus f_2}(T) = -\tilde{Z}_{f_1}(T) \circledast \tilde{Z}_{f_2}(T)$$

where $\circledast$ is defined by applying the convolution product $*$ coefficientwise.

The idea of the proof is the following. Assume that we want to compute $\mathfrak{X}_n(f_1 \oplus f_2)$, i.e. we look for $\gamma_1(t)$ and $\gamma_2(t)$ such that $f_1(\gamma_1(t)) + f_2(\gamma_2(t)) = ct^n + \cdots$, $c \neq 0$. Assume that $f_1(\gamma_1(t)) = c_1 t^{n_1} + \cdots$ and $f_2(\gamma_2(t)) = c_2 t^{n_2} + \cdots$. We encounter the following cases:

1. $n_1 = n_2 = n$ and $c_1 + c_2 \neq 0$, in this case $c = c_1 + c_2$.
2. $n_1 = n_2 < n$ and $c_1 + c_2 = 0$.
3. $n_1 = n < n_2$, in this case $c = c_1$.
4. $n_1 > n_2 = n$, in this case $c = c_2$.

The two first items are naturally handled by the definition of the convolution product. The two last items are why we need to work with the modified zeta function). For technical reasons, in the current proof, we need to work with a resolution of f_i in order to do the required computations.

4 Applications: some classification results

4.1 Arc-analytic classification of Brieskorn polynomials

Definition 4.1. A polynomial $f \in \mathbb{R}[x_1,\ldots,x_d]$ is said to be a Brieskorn polynomial if it is of the following form

$$f(x) = \sum_{i=1}^{d} \varepsilon_i x_i^{k_i}, \varepsilon_i \neq 0, k_i \geq 1$$

Since we are only interested in the arc-analytic classification of Brieskorn polynomials, we first do the following simplifications.

Remark 4.2. Since we may reorder the variables without changing the arc-analytic type of a polynomial, we will always assume that

$$k_1 \leq k_2 \leq \cdots \leq k_d$$

In the same vein, we may assume that $\varepsilon_i = \pm 1$.

Remark 4.3. We may first elude the non-singular case. Indeed, a Brieskorn polynomial $f(x) = \sum_{i=1}^{d} \varepsilon_i x_i^{k_i}$ is non-singular if and only if there exists $i = 1,\ldots,d$ such that $k_i = 1$. Without loss of generality, we may assume in this case that $k_1 = 1$. Then, f is arc-analytically equivalent to $(x_1,\ldots,x_d) \mapsto x_1$ by applying the Nash inverse mapping theorem to $(x_1,\ldots,x_d) \mapsto (f(x),x_2,\ldots,x_d)$. Notice that, in this case, $\tilde{Z}_f(T) = 0$.

From now on, we assume that $k_i \geq 2$.

The following theorem is a real analogue of a result of Yoshinaga–Suzuki [26] stating that the topological type of a Brieskorn singularity determines its exponents.

Theorem 4.4 ([6, Corollary 8.4]). *Assume that the Brieskorn polynomials*

$$f(x) = \sum_{i=1}^{d} \varepsilon_i x_i^{k_i} \quad \text{and} \quad g(x) = \sum_{i=1}^{d} \eta_i x_i^{l_i}$$

with

$$2 \leq k_1 \leq \ldots \leq k_d \quad \text{and} \quad 2 \leq l_1 \leq \ldots \leq l_d$$

are arc-analytically equivalent, then

$$\forall i = 1,\dots,d,\ k_i = l_i$$

Since the modified motivic zeta function is an invariant of the arc-analytic equivalence, it is enough to show that we may recover the exponents of a Brieskorn polynomial f from $\tilde{Z}_f(T)$. This fact may be proved following the next plan divided in three steps.

1. First, by the convolution formula, we may deduce the modified zeta function $\tilde{Z}_f(T)$ of f from the one $\tilde{Z}_{\varepsilon x^k}$ of a pure monomial εx^k. An easy computation gives

$$\begin{aligned}\tilde{Z}_{\varepsilon x^k}(T) = &-T - \dots - T^{k-1}\\ &-\left(\mathbb{1} - \left[\varepsilon x^k : \mathbb{R}^* \to \mathbb{R}^*\right]\right)\mathbb{L}^{-1}T^k - \mathbb{L}^{-1}T^{k+1} - \dots - \mathbb{L}^{-1}T^{2k-1}\\ &-\left(\mathbb{1} - \left[\varepsilon x^k : \mathbb{R}^* \to \mathbb{R}^*\right]\right)\mathbb{L}^{-2}T^{2k} - \mathbb{L}^{-2}T^{2k+1} - \dots - \mathbb{L}^{-2}T^{3k-1}\\ &-\dots\end{aligned}$$

   Particularly, by the convolution formula, if n is not a multiple of an exponent k_i, the coefficient a_n of T^n in $\tilde{Z}_f(T)$ is $-\mathbb{L}^{-\sum_{i=1}^d \left\lfloor \frac{n}{k_i} \right\rfloor}$.

2. Next, we deduce from this an upper bound of k_d. Indeed, if p is a prime number big enough, p is not a multiple of an exponent k_i, then

$$\lim_{p \text{ prime}} \frac{1 - \deg \beta\left(\overline{a_p}\right)}{p} = \lim_{p \text{ prime}} \frac{\sum_{i=1}^d \left\lfloor \frac{p}{k_i} \right\rfloor}{p} = \sum_{i=1}^d \frac{1}{k_i}$$

   Since there are only finitely many $(k'_1,\dots,k'_d)$ such that $\sum_{i=1}^d \frac{1}{k_i} = \sum_{i=1}^d \frac{1}{k'_i}$, we may deduce from $\tilde{Z}_f(T)$ an upper bound K of $\{k_1,\dots,k_d\}$.

3. We conclude by constructing from the coefficients of $\tilde{Z}_f(T)$ a linear system in the unknowns $M_k = \#\{i, k_i = k\}$, for $k = 1,\dots,K$, and we solve it.

Since we are working with real numbers, it is natural to wonder what is the impact of the signs of the coefficients of f on its arc-analytic type. Some preliminary results we present below allow us to state the following conjecture telling that the motivic zeta function is a complete invariant of the arc-analytic type of a Brieskorn polynomial. They also give conditions on the exponents and coefficients of a Brieskorn polynomial to characterize its arc-analytic type.

Conjecture 4.5 ([3, Conjecture 1.10.1]). *Let*

$$f(x) = \sum_{i=1}^d \varepsilon_i x_i^{k_i} \qquad \text{and} \qquad g(x) = \sum_{i=1}^d \eta_i x_i^{l_i}$$

be two Brieskorn polynomials with $\varepsilon_i, \eta_i \in \{\pm 1\}$. We assume that $2 \le k_1 \le \dots \le k_d$ and $2 \le l_1 \le \dots \le l_d$, and, moreover, that if $k_i = k_{i+1} = \dots = k_{i+m}$ then $\varepsilon_i \ge \dots \ge \varepsilon_{i+m}$ (resp. if $l_i = l_{i+1} = \dots = l_{i+m}$ then $\eta_i \ge \dots \ge \eta_{i+m}$).
Then the following are equivalent:

(1) f and g are arc-analytically equivalent.
(2) $Z_f(T) = Z_g(T)$
(3) (i) $\forall i, k_i = l_i$
(ii) For j such that k_j is even and not multiple of an odd exponent k_m, we have $\varepsilon_j = \eta_j$.

First, notice that this conjecture is compatible with the classifications of Koike–Parusiński in the two variable case and of G. Fichou in the three variable case.

We have already shown that 4.5.(1)⇒4.5.(2) and that 4.5.(2)⇒4.5.(3).(i). It is already known [3, Lemme 1.10.2] that 4.5.(3)⇒4.5.(1). The idea to prove this last step is to adapt an argument of Koike–Parusiński [14, p2095] which consists in embedding f and g in a same family of Nash function germs with isolated singularities and to use the absence of continuous moduli to conclude.

We end this section by giving, which we believe to be, a promising way to prove the previous conjecture. Our goal is to prove 4.5.(2)⇒4.5.(3).(ii). Again, let f be a Brieskorn polynomial and define a_n by $\tilde{Z}_f(T) = \sum_{n\geq 1} a_n T^n$. Assume that $n \geq 1$ is a multiple of an even exponent of f but is not a multiple of an odd exponent. Then, by a closer look at the convolution formula, we get that

$$\beta(\overline{a_n})u^{\sum_{i=1}^{d}\left\lfloor\frac{n}{k_i}\right\rfloor} = \beta\left(\sum_{i,k_i|n} \varepsilon_i x_i^{k_i} \neq 0\right) - (u-1)\beta\left(\sum_{i,k_i|n} \varepsilon_i x_i^{k_i} = 0\right) = u^{\#\{i,k_i|n\}} - u\beta\left(\sum_{i,k_i|n} \varepsilon_i x_i^{k_i} = 0\right)$$

$$\beta\left(F^{\varepsilon}(a_n)\right)u^{\sum_{i=1}^{d}\left\lfloor\frac{n}{k_i}\right\rfloor} = \beta\left(\sum_{i,k_i|n} \varepsilon_i x_i^{k_i} = \varepsilon 1\right) - \beta\left(\sum_{i,k_i|n} \varepsilon_i x_i^{k_i} = 0\right)$$

So that, for $\varepsilon = +,-$, we may recover $\beta\left(\sum_{i,k_i|n} \varepsilon_i x_i^{k_i} = \varepsilon 1\right)$ from $\tilde{Z}_f(T)$:

$$\beta\left(\sum_{i,k_i|n} \varepsilon_i x_i^{k_i} = \varepsilon 1\right) = \left(\beta\left(F^{\varepsilon}(a_n)\right) - \beta(\overline{a_n})u^{-1}\right)u^{\sum_{i=1}^{d}\left\lfloor\frac{n}{k_i}\right\rfloor} + u^{\#\{i,k_i|n\}-1}$$

Hence, if we were able to find the number of positive (or negative) coefficients of a Brieskorn polynomial with even exponents from the virtual Poincaré polynomials of its preimages over 1 and -1, we could conclude by induction. We believe that it is possible so that the conjecture is reduced to the computation of some virtual Poincaré polynomials. Notice it already holds for a homogeneous Brieskorn polynomial of even degree by [10, Corollary 2.5 & Corollary 2.6].

4.2 Arc-analytic classification of some weighted homogeneous polynomials

In the complex case, it is known that the local analytic type of a singular weighted homogeneous polynomial with isolated singularity at the origin determines its weights [24]. It also holds by considering merely the topological type in two [27] and three [23] variables.

T. Fukui [12, Conjecture 9.2] conjectures the real counterpart for weighted homogeneous real polynomials with isolated singularity in the blow-analytic context. This conjecture has been proven by O. M. Abderrahmane [1] in two variables and by G. Fichou and T. Fukui [11] in three variables in the blow-Nash context for convenient weighted homogeneous polynomials which are non-degenerate with respect to their Newton polyhedra.

Since Brieskorn polynomials are convenient weighted homogeneous polynomials which are non-degenerate with respect to their Newton polyhedra, it is natural to ask whether the material presented in the previous section would allow one to generalize the result of G. Fichou and T. Fukui with no condition on the number of variables.

A first obstacle is that we can't use the convolution formula anymore since we can't assume that such a polynomial is a sum of pure monomials. However, it is still possible to adapt the strategy used to prove that the arc-analytic type of a Brieskorn polynomial determines its exponents. It relies on a formula to compute the modified zeta function of a polynomial non-degenerate with respect to its Newton polyhedron.

Theorem 4.6 ([5]). *Let $f, g \in \mathbb{R}[x_1, \ldots, x_d]$ be two arc-analytically equivalent weighted homogeneous polynomials which are non-degenerate with respect to their Newton polyhedra. Then*

1. *Either they are both non-singular, and in this case they are both arc-analytically equivalent to $(x_1, \ldots, x_d) \mapsto x_1$,*
2. *Or they share the same weights (up to permutation and positive common multiplicative factor).*

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