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ON RELATIVE SUFFICIENCY OF RELATIVE JETS

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1. Introduction

One of the most important local aspect of singularity theory is the analysis of the conditions under which a map-germ is finitely determined or sufficient and the degree of it's determinacy. More recently there has been much interest in studying mappings or varieties with non-isolated singularities. Any mapping realisation of a C^0-sufficient jet has an isolated singularity, and the zero-set of any mapping realisation of a V-sufficient jet also has an isolated singularity. Therefore the above works on sufficiency of jets only deal with the isolated singularity case. Implicit Function Theorem and Morse Lemma may be regarded as results on sufficiency. The notion of sufficiency of jets also has applications to the bifurcation problems in Differential Equation. Hence this notion has been explored by many researchers in the 1970s and 1980s (see C. T. C. Wall [36] for the survey of this field).

On the other hand, the works on characterisations of sufficiency of jets relative to a given closed set have been also started, e.g. Siersma [25], L. Wilson and Sun [26], S. Izumiya and S. Matsuoka [10], L. Kushner and B. Terra Leme [19], V. Grandjean [9], V. Thilliez [28], X. Xu [38] and so on. This relative case includes the non-isolated case. The goal of this paper is to carry on the study of germs of differentiable mappings $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, or more generally families of such mappings with non-isolated singularities. We consider the following situation, for given closed set germ Σ in $(\mathbb{R}^n, 0)$, we define the notion of map jets relative to Σ and using the group of homeomorphisms or diffeomorphisms which preserve or which fixes Σ, to define the Σ-C^0-sufficiency, Σ-V-sufficiency and Σ-SV-sufficiency for mappings jets. Finite (respectively infinite) determinacy is a way to express the stability of smooth map-germs under polynomial (resp. flat) perturbations. Let $\mathcal{E}_n$ denote the ring of C^∞ function-germs at the origin in $\mathbb{R}^n$ and $\mathfrak{m}$ its maximal ideal. Consider the ideal of flat germs $\mathfrak{m}^\infty = \bigcap_{k\geq 0} \mathfrak{m}^k$. A map germ f of $(\mathcal{E}_n)^p$ is said to be *finitely determined* if there is some integer k, such that for any h in $\mathfrak{m}^k$, $f+h$ is equivalent to f i.e. there exists a germ of C^∞-diffeomorphism at the origin φ such that $f+h = f\circ\varphi$. (resp f is *infinitely determined* if, for any element h of $\mathfrak{m}^\infty$, there exists a germ φ of C^∞-diffeomorphism at the origin such that $f+h = f\circ\varphi$.) This can be indicated by

$$f + \mathfrak{m}^k \subset f \circ \mathcal{R}^\infty, \quad (\text{resp. } f + \mathfrak{m}^\infty \subset f \circ \mathcal{R}^\infty)$$

where $\mathcal{R}^\infty$ denotes the group of germs at the origin of C^∞-diffeomorphisms of $\mathbb{R}^n$.

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In the relative case (which includes the isolated singularities), one fixes a closed set germ Σ and considers the equivalent relation on the map germs defined by homeomorphisms (or diffeomorphisms) which fix (or preserve) Σ (for example, one considers germs with a prescribed singular locus Σ.) We can generalise the action of the groupe $\mathcal{R}^r$, of germs of C^r diffeomorphisms by taking the subset $\mathcal{R}^r_\Sigma$ (respectively $\mathcal{R}^{r,\text{fix}}_\Sigma$) of diffeomorphisms which preserve Σ the set of elements φ of $\mathcal{R}^r$ which preserve Σ, that is $\varphi(\Sigma) \subset \Sigma$ (respectively which preserve Σ pointwise, that is $\varphi(x) = x$ for any $x \in \Sigma$). Remark that $\mathcal{R}^{r,\text{fix}}_\Sigma$ is a subgroup of $\mathcal{R}^r$, but it's not always the case for $\mathcal{R}^r_\Sigma$. In what follows we will consider only the equivalence according to the subgroup $\mathcal{R}^{r,\text{fix}}_\Sigma$. For example, Let $\Sigma = \mathbb{R}^- = \{x \in \mathbb{R},\ x \leq 0\}$. If $f_1 : \mathbb{R} \to \mathbb{R}$ is defined by $f_1(x) = x^5$ then any function g which coincides with f_1 on Σ, is $\mathcal{R}^{\infty,\text{fix}}_\Sigma$-equivalent to f. Now if f_2 is the identically zero function, then there are functions which are identically zero on Σ and are not even topologically equivalent to f_2. In these examples f_1 is finitely determined but f_2 is not.

An example of non finitely determined function germs is given by $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x,y) = (x^2+y^2)^2$, but any function which coincides with f on $\Sigma = \{(x,y) \in \mathbb{R}^2,\ y \leq 0\}$, is $\mathcal{R}^{\infty,\text{fix}}_\Sigma$-equivalent to f.

Before we describe the main results, we recall the conditions characterising the aforementioned sufficiencies and their related results in the non-relative case, in order to expose the difference between the two cases. For simplicity, we mention them for r-jets sufficient in C^r functions. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a C^r function germ. The r-jet of f at $0 \in \mathbb{R}^n$, $j^r f(0)$, has a unique polynomial representative z of degree not exceeding r. We do not distinguish the r-jet $j^r f(0)$ and the polynomial representative z here.

Kuiper-Kuo condition. There is a strictly positive number C such that

$$\|\operatorname{grad} z(x)\| \geq C\|x\|^{r-1}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

The Kuiper-Kuo condition is equivalent to the C^0-sufficiency of z in C^r functions (N. Kuiper [14], T.-C. Kuo [15], J. Bochnak and S. Lojasiewicz [4]).

Kuo condition. There are strictly positive numbers C, α and $\bar{w}$ such that

$$\|\operatorname{grad} z(x)\| \geq C\|x\|^{r-1}$$

in $\mathcal{H}^\Sigma_r(f;\bar{w}) \cap \{\|x\| < \alpha\}$.

Here $\mathcal{H}^\Sigma_r(f;\bar{w})$ denotes the horn-neighbourhood of $f^{-1}(0)$ of degree r and width $\bar{w}$ (see §2.2 for the definition). The Kuo condition is equivalent to the V-sufficiency of z in C^r functions (T.-C. Kuo [17]).

Condition $(\widetilde{K})$. There is a strictly positive number C such that

$$\|x\|\|\operatorname{grad} z(x)\| + |f(x)| \geq C\|x\|^r$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

This condition is the Kuo condition in a different way.

Thom type inequality. There are strictly positive numbers K and β such that

$$\sum_{i<j} \left| x_i \frac{\partial z}{\partial x_j} - x_j \frac{\partial z}{\partial x_i} \right|^2 + |f(x)|^2 \geq K\|x\|^{2r}$$

for $\|x\| < \beta$.

At almost the same time as Kuiper and Kuo, R. Thom proved in [29] that the Thom type inequality implies the C^0-sufficiency of z in C^r-functions. Using the curve selection lemma, we can show the equivalence between the Thom type inequality and condition $(\widetilde{K}_\Sigma)$ ([1]). Now, we recall the Bochnak-Lojasiewicz inequality ([4]).

Bochnak-Lojasiewicz inequality. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a C^ω function germ, and let $0 < \theta < 1$. Then $\|x\| \| \operatorname{grad} f(x) \| \geq \theta |f(x)|$ holds in some neighbourhood of $0 \in \mathbb{R}^n$.

This inequality plays a very important role in proving that the Kuo condition, or in fact condition $(\widetilde{K})$ is equivalent to the Kuiper-Kuo condition in the analytic case. It follows that V-sufficiency in C^r functions is equivalent to C^0-sufficiency in C^r functions, and we can see that Thom had proved an equivalent result to the Kuiper-Kuo theorem.

The Kuiper-Kuo condition, the Kuo condition, condition $(\widetilde{K})$ and the Thom type inequality are r-compatible in the sense of [1] (see also Definition ??). Therefore we can replace z with f in those conditions.

We can consider the conditions for r-jets sufficient in C^{r+1} functions, corresponding to the Kuiper-Kuo condition, the Kuo condition and condition $(\widetilde{K})$. We call them the second Kuiper-Kuo condition, the second Kuo condition and condition $(\widetilde{K}^\delta)$, respectively. The first 4 conditions in the C^r function case and the last 3 conditions in the C^{r+1} function case can be generalised to the mapping case.

On the other hand, V. Kozyakin in [12] gave a reformulations for the Kuo condition and Thom's type inequality, from the viewpoint of the stability of the solution of a polynomial equation. His reformulations are related to $(\widetilde{K})$ and are given in term some limit conditions. They may be less geometric than the Lojasiewicz inequality conditions but simpler to check.

The usual verification of the Kuiper-Kuo condition, or the Thom condition, may be reduced to the problem on evaluation of the rate of growth of a polynomial about one of its roots, which is equivalent to calculation of the so-called local Lojasiewicz exponents of a polynomial. Recall, that according to the Lojasiewicz theorem for any polynomial $p : \mathbb{R}^n \to \mathbb{R}$ with $p(0) = 0$ there are constants $C, \alpha > 0$ such that $|p(x)| \geq C|x|^\alpha$ in a neighbourhood of the zero root. The least α for which the above inequality holds is called the *local Lojasiewicz exponent* for p and is denoted by $\mathcal{L}_0(p)$. If the zero root of p is isolated then such a least value of α exists and is rational. Moreover, in this case $\mathcal{L}_0(p) \leq (d-1)^n + 1$ where d is the degree of p. There is quite a number of publications devoted to evaluation of the Lojasiewicz exponent.

We present here some of the results obtained in a recent work with S. Koike (for the proofs and related results see [2]), and examples to illustrate. At the end we discuss some open problems.

2. Preliminaries

Throughout this paper, let us denote by $\mathbb{N}$ the set of natural numbers in the sense of positive integers. Let $\mathcal{E}_{[s]}(n,p)$ denote the set of C^s map-germs : $(\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, let $j^r f(0)$ denote the r-jet of f at $0 \in \mathbb{R}^n$ for $f \in \mathcal{E}_{[s]}(n,p)$ $(s \geq r)$, and let $J^r(n,p)$ denote the set of r-jets in $\mathcal{E}_{[s]}(n,p)$. Throughout this paper, Σ is a germ of a closed subset of $\mathbb{R}^n$ at $0 \in \mathbb{R}^n$ such that $0 \in \Sigma$. Then we denote by $\mathcal{R}^{\text{fix}}_\Sigma$ the group of germs of homeomorphisms $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ at $0 \in \mathbb{R}^n$ which fixes Σ, namely $\varphi(x) = x$ for all $x \in \Sigma$. Finally we denote by $d(x, \Sigma)$ the distance from a point $x \in \mathbb{R}^n$ to the subset Σ.

We consider on $\mathcal{E}_{[s]}(n,p)$ the following equivalence relation:

Two map-germs $f, g \in \mathcal{E}_{[s]}(n,p)$ are r-Σ-*equivalent*, denoted by $f \sim g$, if there exists a neighbourhood U of 0 in $\mathbb{R}^n$ such that the r-jet extensions of f and g satisfy $j^r f(\Sigma \cap U) = j^r g(\Sigma \cap U)$.

We denote by $j^r f(\Sigma;0)$ (or simply $j^r f(\Sigma)$) the equivalence class of f, and by $J^r_\Sigma(n,p)$ the quotient set $\mathcal{E}_{[s]}(n,p)/\sim$.

Remark 2.1. (1) In the case where $\Sigma = \{0\}$, an r-jet $j^r f(0)$ has a polynomial realisation for any $f \in \mathcal{E}_{[r]}(n,p)$. But this property does not always hold in the relative case. In fact, let $f : (\mathbb{R},0) \to (\mathbb{R},0)$ be a C^∞ function defined by

$$f(x) := \begin{cases} e^{-\frac{1}{x^2}} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Let $\Sigma = \{\frac{2}{m\pi} \mid m \in \mathbb{N}\} \cup \{0\}$. Then $f(\frac{2}{m\pi}) = 0$ for even m, but $f(\frac{2}{m\pi}) \neq 0$ for odd m. Therefore, for any $r \in \mathbb{N}$, $j^r f(\Sigma;0)$ does not have even a subanalytic C^r-realisation.

(2) Let $f \in \mathcal{E}_{[r]}(n,p)$, and let Σ be a germ of a closed subset of $\mathbb{R}^n$ at $0 \in \mathbb{R}^n$ such that $0 \in \Sigma$. Then $j^r f(\Sigma;0)$ has a C^r-realisation $\tilde{f}$ whose restriction to $\mathbb{R}^n \setminus \Sigma$ is smooth, namely of class C^∞ (Theorem 2.2, page 73 in J.-C. Tougeron [31]).

Let us introduce some equivalences for elements of $J^r_\Sigma(n,p)$.

Definition 2.2. (1) We say that $f, g \in \mathcal{E}_{[s]}(n,p)$ are Σ-C^0-*equivalent*, if there is $\varphi \in \mathcal{R}^{\text{fix}}_\Sigma$ such that $f = g \circ \varphi$.

(2) We say that $f, g \in \mathcal{E}_{[s]}(n,p)$ are Σ-V-*equivalent*, if $f^{-1}(0)$ is homeomorphic to $g^{-1}(0)$ as germs at $0 \in \mathbb{R}^n$ by a homeomorphism which fixes $f^{-1}(0) \cap \Sigma$.

(3) We say that $f, g \in \mathcal{E}_{[s]}(n,p)$ are Σ-SV-*equivalent*, if there is a local homeomorphism $h \in G_\Sigma$ such that $h(f^{-1}(0)) = g^{-1}(0)$.

Let $w \in J^r_\Sigma(n,p)$. We call the relative jet w Σ-C^0-*sufficient*, Σ-V-*sufficient*, and Σ-SV-*sufficient* in $\mathcal{E}_{[s]}(n,p)$ $(s \geq r)$, if any two realisations f, $g \in \mathcal{E}_{[s]}(n,p)$ of w, namely $j^r f(\Sigma;0) = j^r g(\Sigma;0) = w$, are Σ-C^0-equivalent, Σ-V-equivalent, and Σ-SV-equivalent, respectively.

Notation: Let $f, g : U \to \mathbb{R}$ be non-negative functions, where $U \subset \mathbb{R}^N$ is an open neighbourhood of $0 \in \mathbb{R}^N$. If there are real numbers $K > 0$, $\delta > 0$ with $B_\delta(0) \subset U$ such that $f(x) \leq Kg(x)$ for any $x \in B_\delta(0)$, where $B_\delta(0)$ is a closed ball in $\mathbb{R}^N$ of radius δ centred at $0 \in \mathbb{R}^N$. Then we write $f \precsim g$ (or $g \succsim f$). If $f \precsim g$ and $f \succsim g$, we write $f \approx g$.

2.1. **Relative Kuo condition and relative Thom's type inequality.** We suppose now on the germ Σ fixed, and introduce the relative notions to Σ of the Kuo condition and the Thom type inequality. We first give the notion of the relative Kuiper-Kuo condition. The original condition was introduced by N. Kuiper [14] and T.-C. Kuo [15] as a sufficient condition of C^0-sufficiency of jets in the function case.

We denote by $d(v, V)$ denote the distance from the tip of the vector $v \in \mathbb{R}^k$ to the vector subspace $V \subset \mathbb{R}^k$. Let $L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ a linear maps, $L = (L_1, \ldots, L_p)$ and let $v_j = \operatorname{grad} L_j$ (where the gradient is taken with respect to the standard inner product). Let V_j be the span of the v_j, with $j \neq i$, and let

$$\kappa(L) = \min_j d(v_j, V_j).$$

be the *Kuo distance* ([17])

Remark 2.3. Note that: $\kappa(L) = 0$ if and only if $L \in \mathcal{S}$ where $\mathcal{S}$ is the set of critical linear maps (those of rank less than p); so $\kappa(L)$ is a measure of how close L is to being critical.

For a differentiable map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, let us define:

$$\kappa(f,x) := \kappa(df(x))$$

where $df(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ denotes the derivative of f at x.
In the case where $p = 1$, $\kappa(f, x) = \| \operatorname{grad}(f)(x)\|$.

Definition 2.4 (The relative Kuiper-Kuo condition). A map germ $f \in \mathcal{E}_{[r]}(n,p)$, $n \geq p$, satisfies the *relative Kuiper-Kuo condition* (K-K_Σ) if

$$\kappa(df(x)) \gtrsim d(x, \Sigma)^{r-1}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

Definition 2.5 (The second relative Kuiper-Kuo condition). A map germ $f \in \mathcal{E}_{[r]}(n,p)$, $n \geq p$, satisfies the *second relative Kuiper-Kuo condition* (K-K_Σ^δ) if there is a strictly positive number δ such that

$$\kappa(df(x)) \gtrsim d(x, \Sigma)^{r-\delta}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

For a map germ $f \in \mathcal{E}_{[r]}(n,p)$, we denote by $\operatorname{Sing}(f)$ the singular points set of f.

Remark 2.6. For a map $f \in \mathcal{E}_{[r]}(n,p)$ satisfying the relative Kuiper-Kuo condition or the second relative Kuiper-Kuo condition, we have $\operatorname{Sing}(f) \subset \Sigma$ in a neighbourhood of $0 \in \mathbb{R}^n$. Therefore these conditions include the case where $\Sigma = \operatorname{Sing}(f)$, as a special case.

We next give the notion of the relative Kuo condition. The original condition was introduced by T.-C. Kuo [17] as a criterion of V-sufficiency of jets in the mapping case.

Definition 2.7 (The relative Kuo condition). A map germ $f \in \mathcal{E}_{[r]}(n,p)$, $n \geq p$, satisfies the *relative Kuo condition* (K_Σ) if there are strictly positive numbers C, α and $\bar{w}$ such that

$$\kappa(df(x)) \geq C d(x, \Sigma)^{r-1}$$

in $\mathcal{H}_r^\Sigma(f; \bar{w}) \cap \{\|x\| < \alpha\}$, namely

$$\kappa(df(.)) \gtrsim d(., \Sigma)^{r-1}$$

on a set of points where $\|f\| \lesssim d(., \Sigma)^r$.

In the definition 2.7, $\mathcal{H}_r^\Sigma(f; \bar{w})$ denotes the *horn-neighbourhood of* $f^{-1}(0)$ *of degree* r *and width* $\bar{w}$,

$$\mathcal{H}_r^\Sigma(f; \bar{w}) = \{x \in \mathbb{R}^n : \|f(x)\| \leq \bar{w}\ d(x, \Sigma)^r\}.$$

The notion of this horn-neighbourhood was introduced in [16].
We have also a variant of the previous condition:

Definition 2.8 (The second relative Kuo condition). A map germ $f \in \mathcal{E}_{[r+1]}(n,p)$, $n \geq p$, satisfies the *second relative Kuo condition* (K_Σ^δ) if for any map $g \in \mathcal{E}_{[r+1]}(n,p)$ satisfying $j^r g(\Sigma; 0) = j^r f(\Sigma; 0)$ there are numbers C, α, δ and $\bar{w}$ (depending on g), such that

$$\kappa(df(x)) \geq C d(x, \Sigma)^{r-\delta}$$

in $\mathcal{H}_{r+1}(g; \bar{w}) \cap \{\|x\| < \alpha\}$, namely

$$\kappa(df(.)) \gtrsim d(., \Sigma)^{r-\delta}$$

on a set of points where $\|g(.)\| \lesssim d(., \Sigma)^{r+1}$.

Remark 2.9. 1) For a map $f \in \mathcal{E}_{[r]}(n,p)$ satisfying the relative Kuo condition or the second relative Kuo condition, in a neighbourhood of $0 \in \mathbb{R}^n$, the intersection of the singular set of f, $\mathrm{Sing}(f)$, and the horn neighbourhood $\mathcal{H}_r^{\Sigma}(f;\bar{w})$ is contained in Σ, namely

$$\mathrm{Sing}(f) \cap \mathcal{H}_r^{\Sigma}(f;\bar{w}) \subset \Sigma.$$

In particular, in a neighbourhood of $0 \in \mathbb{R}^n$, $\mathrm{grad} f_1(x), \ldots, \mathrm{grad} f_p(x)$ are linearly independent on $f^{-1}(0) \setminus \Sigma$.

2) For a map $f \in \mathcal{E}_{[r]}(n,p)$ satisfying the second relative Kuo, we have for any map $g \in \mathcal{E}_{[r+1]}(n,p)$ satisfying $j^r g(\Sigma;0) = j^r f(\Sigma;0)$, in a neighbourhood of $0 \in \mathbb{R}^n$, the intersection of the singular set of f, $\mathrm{Sing}(f)$, and the horn neighbourhood $\mathcal{H}_{r+1}^{\Sigma}(g;\bar{w})$ is contained in Σ, namely $\mathrm{Sing}(f) \cap \mathcal{H}_{r+1}^{\Sigma}(g;\bar{w}) \subset \Sigma$. Since $\|(f-g)(x)\| \precsim d(x,\Sigma)^{r+1}$, we have $f^{-1}(0) \subset \mathcal{H}_{r+1}^{\Sigma}(g;\bar{w})$, then, in a neighbourhood of $0 \in \mathbb{R}^n$, $\mathrm{grad} f_1(x), \ldots, \mathrm{grad} f_p(x)$ are linearly independent on $f^{-1}(0) \setminus \Sigma$.

Definition 2.10 (Condition $(\widetilde{K}_\Sigma)$). A map germ $f \in \mathcal{E}_{[r]}(n,p)$, $n \geq p$, satisfies *condition* $(\widetilde{K}_\Sigma)$ if

$$d(x,\Sigma)\kappa(df(x)) + \|f(x)\| \succsim d(x,\Sigma)^r$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

Remark 2.11. 1) Condition $(\widetilde{K}_\Sigma)$ was introduced in [1], in the case $\Sigma = \{0\}$, in the proof of the equivalence between V-sufficiency and SV-sufficiency.

2) It is easy to see that condition $(\widetilde{K}_\Sigma)$ and the relative Kuo condition (K_Σ) are equivalent.

3) The relative Kuiper-Kuo condition $(K\text{-}K_\Sigma)$, the relative Kuo condition (K_Σ), and condition $(\widetilde{K}_\Sigma)$ are invariant under rotation.

Definition 2.12 (Condition $(\widetilde{K}_\Sigma^\delta)$). A map germ $f \in \mathcal{E}_{[r+1]}(n,p)$, $n \geq p$, satisfies *condition* $(\widetilde{K}_\Sigma^\delta)$ if for any map $g \in \mathcal{E}_{[r+1]}(n,p)$ satisfying $j^r g(\Sigma;0) = j^r f(\Sigma;0)$ there exists $\delta > 0$ (depending on g), such that

$$d(x,\Sigma)\kappa(df(x)) + \|g(x)\| \succsim d(x,\Sigma)^{r+1-\delta}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

Remark 2.13. (1) The second relative Kuiper-Kuo condition $(K\text{-}K_\Sigma^\delta)$, the second relative Kuo condition (K_Σ^δ), and condition $(\widetilde{K}_\Sigma^\delta)$ are invariant under rotation.

(2) This condition can be equivalently written as: for any map $g \in \mathcal{E}_{[r+1]}(n,p)$ satisfying $j^r g(\Sigma;0) = j^r f(\Sigma;0)$ there exists $\delta > 0$ (depending on g), such that

$$d(x,\Sigma)\kappa(dg(x)) + \|g(x)\| \succsim d(x,\Sigma)^{r+1-\delta}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

Definition 2.14 (The relative Thom type inequality). A map germ $f \in \mathcal{E}_{[r]}(n,p)$ $(n \geq p)$, satisfies *a relative Thom type inequality* (T_Σ) if there are strictly positive numbers K, β and a such that

$$T_2(f,x) \geq K d(x,\Sigma)^a \ \ for \quad \|x\| < \beta,$$

namely

$$T_2(f,.) \succsim d(.,\Sigma)^a.$$

In [2], we give various equivalent formulations of these condition.

3. Relative C^0-sufficiency of jets

We state in this section the main results the proofs and further details one should consult [2]. Let us recall that Σ is a germ of a non-empty, closed subset at $0 \in \mathbb{R}^n$ such that $0 \in \Sigma$. In this section we give criteria for Σ-C^0-sufficiency of relative r-jets in C^r mappings and in C^{r+1} mappings, and compute some examples on relative C^0-sufficiency of jets using the criteria.

3.0.1. *Relative C^0 sufficiency of r-jets in C^r mappings.* In this subsection we give a criterion of Σ-C^0-sufficiency of r-jets in C^r mappings, using the relative Kuiper-Kuo condition.

Theorem 3.1. *Let r be a positive integer, and let $f \in \mathcal{E}_{[r]}(n,p)$ where $n \geq p$. Then the following conditions are equivalent.*

(1) f satisfies the relative Kuiper-Kuo condition $(K\text{-}K_\Sigma)$, namely

$$\kappa(df(x)) \gtrsim d(x,\Sigma)^{r-1}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

(2) The relative r-jet $j^r f(\Sigma;0)$ is Σ-C^0-sufficient in $\mathcal{E}_{[r]}(n,p)$.

3.1. **Relative C^0 sufficiency of r-jets in C^{r+1} mappings.** In this subsection we give a criterion of Σ-C^0-sufficiency of r-jets in C^{r+1} mappings, using the second relative Kuiper-Kuo condition.

Theorem 3.2. *Let r be a positive integer, and let $f \in \mathcal{E}_{[r+1]}(n,p)$ where $n \geq p$. Then the following conditions are equivalent.*

(1) f satisfies the second relative Kuiper-Kuo condition $(K\text{-}K_\Sigma^\delta)$, namely there is a strictly positive number δ such that

$$\kappa(df(x)) \gtrsim d(x,\Sigma)^{r-\delta}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$.

(2) The relative r-jet $j^r f(\Sigma;0)$ is Σ-C^0-sufficient in $\mathcal{E}_{[r+1]}(n,p)$.

3.2. **Σ-C^0-sufficiency of jets in the function case.** In this subsection we restate Theorems 3.1, 3.2 in the function case. Related to these results, we shall discuss in the next section if the Bochnak-Lojasiewicz inequality holds in the relative case, and the relationship between the relative C^0-sufficiency of jets and the relative V-sufficiency of jets through the relationship between the relative Kuiper-Kuo condition and condition $(\widetilde{K}_\Sigma)$.

Theorem 3.3. *(1) Let r be a positive integer, and let $f \in \mathcal{E}_{[r]}(n,1)$. Then the inequality*

$$\| \operatorname{grad} f(x) \| \gtrsim d(x,\Sigma)^{r-1}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$ if and only if the relative r-jet $j^r f(\Sigma;0)$ is Σ-C^0-sufficient in $\mathcal{E}_{[r]}(n,1)$.

(2) Let r be a positive integer, and let $f \in \mathcal{E}_{[r+1]}(n,1)$. Then there is a strongly positive number δ such that the inequality

$$\| \operatorname{grad} f(x) \| \gtrsim d(x,\Sigma)^{r-\delta}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$ if and only if the relative r-jet $j^r f(\Sigma;0)$ is Σ-C^0-sufficient in $\mathcal{E}_{[r+1]}(n,1)$.

Remark 3.4. X. Xu also has obtained in [38] a result that the inequality in Theorem 3.3 (1) implies Σ-C^0-sufficiency in $\mathcal{E}_{[r]}(n,1)$.

Example 3.5. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a polynomial function defined by

$$f(x,y) := x^3,$$

and let $\Sigma := \{x = 0\}$. Then we can easily see that $d((x,y),\Sigma) = |x|$, and

$$\|\mathrm{grad} f(x,y)\| \gtrsim |x|^2$$

in a neighbourhood of $(0,0) \in \mathbb{R}^2$. It follows from Theorem 3.3(1) that $j^3 f(\Sigma;0)$ is Σ-C^0-sufficient in $\mathcal{E}_{[3]}(2,1)$.

Example 3.6. Let $f_m : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $m \geq 3$, be a polynomial function ([16]) defined by

$$f_m(x,y) := x^3 - 3xy^m.$$

Then we have $\mathrm{grad} f_m(x,y) = (3(x^2 - y^m), -3mxy^{m-1})$.

(1) Let $\Sigma := \{(0,0)\}$. Then we have $d((x,y),\Sigma) = \|(x,y)\|$, and

$$\|\mathrm{grad} f_m(x,y)\| \gtrsim \|(x,y)\|^{\frac{3m}{2}-1}$$

in a neighbourhood of $(0,0) \in \mathbb{R}^2$. We can check the above inequality, dividing a neighbourhood of $0 \in \mathbb{R}^2$ into the following three regions:

$$A := \{3|x^2 - y^m| \leq |x|^2\}, \quad B := \{3|x|^2 \leq |y|^m\}, \quad \mathbb{R}^2 \setminus (A \cup B).$$

By the Kuiper-Kuo theorem [14, 15], $j^{\frac{3m-1}{2}} f(0)$ is C^0-sufficient in $\mathcal{E}_{[\frac{3m+1}{2}]}(2,1)$ if m is odd, and $j^{\frac{3m}{2}} f(0)$ is C^0-sufficient in $\mathcal{E}_{[\frac{3m}{2}]}(2,1)$ if m is even.

(2) Let $\Sigma := \{x = 0\}$. Then we can see that

$$\|\mathrm{grad} f_m(x,y)\| \gtrsim |x|^{3-\frac{2}{m}} \tag{3.1}$$

in a neighbourhood of $(0,0) \in \mathbb{R}^2$.

We can show (3.9) as follows. Let $\lambda : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be an arbitrary analytic arc on $\mathbb{R}^2$ passing through $(0,0) \in \mathbb{R}^2$, not identically zero, denoted by

$$\lambda(t) = (a_k t^k + \cdots, b_s t^s + \cdots).$$

In the case where $\lambda(t) = (0, b_s t^s + \cdots)$, $b_s \neq 0$, λ is contained in Σ, and $d((x,y),\Sigma) = |x| = 0$ on λ. Therefore we have

$$\|\mathrm{grad} f_m(x,y)\| \gtrsim |x|$$

on λ. Thus we may assume after this that $a_k \neq 0$.

In the case where $2k < ms$, we have

$$\|\mathrm{grad} f_m(x,y)\| \gtrsim |t|^{2k}, \ |x| \approx |t|^k$$

on λ. Therefore we have

$$\|\mathrm{grad} f_m(x,y)\| \gtrsim |x|^2$$

on λ near $(0,0) \in \mathbb{R}^2$.

In the case where $2k \geq ms$, we have

$$\|\mathrm{grad} f_m(x,y)\| \geq |\frac{\partial f_m}{\partial y}(x,y)| \gtrsim |t|^k |t|^{(m-1)s} \gtrsim |t|^{k+\frac{2(m-1)}{m}k} = |t|^{(3-\frac{2}{m})k}, \quad |x| \approx |t|^k$$

on λ. Therefore we have $\|\mathrm{grad} f_m(x,y)\| \gtrsim |x|^{3-\frac{2}{m}}$ on λ near $(0,0) \in \mathbb{R}^2$.

Thus we have (3.9) in a neighbourhood of $(0,0) \in \mathbb{R}^2$.

Note that

$$\frac{\partial f_m}{\partial x}(t^k, t^s) \equiv 0, \quad |\frac{\partial f_m}{\partial y}(t^k, t^s)| = 3m|t|^{(3-\frac{2}{m})k}$$

in the case where $2k = ms$. Therefore it follows from Theorem 3.3(1), (2) that $j^3 f_m(\Sigma;0)$ is not Σ-C^0-sufficient in $\mathcal{E}_{[3]}(2,1)$ but Σ-C^0-sufficient in $\mathcal{E}_{[4]}(2,1)$ for any $m \geq 3$.

(3) Let $\Sigma := \{y = 0\}$. Then, using a similar computation to the above one, we can see that

$$\|\mathrm{grad} f_m(x,y)\| \gtrsim |y|^{\frac{3m}{2}-1}$$

in a neighbourhood of $(0,0) \in \mathbb{R}^2$. It follows from Theorem 3.3(1), (2) that $j^{\frac{3m-1}{2}} f(\Sigma;0)$ is Σ-C^0-sufficient in $\mathcal{E}_{[\frac{3m+1}{2}]}(2,1)$ if m is odd, and $j^{\frac{3m}{2}} f(\Sigma;0)$ is Σ-C^0-sufficient in $\mathcal{E}_{[\frac{3m}{2}]}(2,1)$ if m is even.

3.2.1. *Thom's (t)-condition and Lojasiewicz exponent.*

Definition 3.7 (see [30], [34]). Let $r \in \mathbb{N}$. Suppose X and Y are C^1-submanifolds of a C^r-manifold M and $y_0 \in Y \cap \overline{X}$; then $(X,\ Y)$ is (t^r) regula at y_0 if T a C^r-submanifold of M and $T \pitchfork_{y_0} Y$ implies $T \pitchfork X$ near y_0.

Proposition 3.8. *Suppose $\mathcal{P} = \{S_j\}$ is a finite collection of manifolds whose union is a closed set $S \subset \mathbb{R}^n \times \mathbb{R}^m$. Let M be an n-dimensional C^k-transversal to $\mathcal{P}$ (that is, $N \pitchfork S_j$, for all j). Let $d(y,\ S)$ denote the distance from y to S. If $k \geq 1$ and $\mathcal{P}$ is (t^1)-regular, then for every compact set $K \subset M$, there is a constant $C > 0$ such that*

$$d(y,\ S) \geq C \cdot d(y, N \cap S),\ \textit{for all } y \in K$$

We use this proposition to compute the Lojasiewicz exponent in certain situation:

Example 3.9. Let $f_m : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, $m \geq 3$, be a polynomial function defined by $f_m(x,y) := x^3 - 3xy^m$. Then we have $\mathrm{grad} f_m(x,y) = (3(x^2 - y^m), -3mxy^{m-1})$.
f_x has two braches $x = \pm y^{\frac{m}{2}}$, and f_y has one.
Remark that the "horn neighbourhoods" $|x - \pm y^{\frac{m}{2}}| \leq \frac{1}{2}|y|^{\frac{m}{2}}$, and $|x| \leq \frac{1}{2}|y|^{\frac{m}{2}}$ are disjoint.

(1) Let $\Sigma := \{y = 0\}$ Since any point (x,y) lies outside one of "horn neighbourhoods" you have $|f_x| \gtrsim |y|^m = d(x,\Sigma)^m$ and $|f_y| \gtrsim |y|^{\frac{3m-2}{2}} = d((x,y),\Sigma)^{\frac{3m-2}{2}}$. Thus

$$\|\mathrm{grad} f_m(x,y)\| \gtrsim d((x,y),\Sigma)^{\frac{3m}{2}-1}$$

and then $j^{\frac{3m-1}{2}} f(\Sigma;0)$ is Σ-C^0-sufficient in $\mathcal{E}_{[\frac{3m+1}{2}]}(2,1)$ if m is odd, and $j^{\frac{3m}{2}} f(\Sigma;0)$ is Σ-C^0-sufficient in $\mathcal{E}_{[\frac{3m}{2}]}(2,1)$ if m is even.

(2) Now, $\{y = 0\}$ is transverse to the branches of $\{f = 0\}$, and $\{f = 0\} \cap \{y = 0\} = \{(0,0)\}$, then by the previous proposition, we can conclude that

$$\|\mathrm{grad} f_m(x,y)\| \gtrsim \|(x,y)\|^{\frac{3m}{2}-1}$$

in a neighbourhood of $(0,0) \in \mathbb{R}^2$. By the Kuiper-Kuo theorem $j^{\frac{3m-1}{2}} f(0)$ is C^0-sufficient in $\mathcal{E}_{[\frac{3m+1}{2}]}(2,1)$ if m is odd, and $j^{\frac{3m}{2}} f(0)$ is C^0-sufficient in $\mathcal{E}_{[\frac{3m}{2}]}(2,1)$ if m is even.

(3) Let $\Sigma := \{x = 0\}$. Then we can see that

$$\|\mathrm{grad} f_m(x,y)\| \gtrsim |x|^{3-\frac{2}{m}}$$

in a neighbourhood of $(0,0) \in \mathbb{R}^2$. It follows from this that $j^3 f_m(\Sigma;0)$ is Σ-C^0-sufficient in $\mathcal{E}_{[4]}(2,1)$ for any $m \geq 3$.

4. Relative V-sufficiency of jets

4.1. Relative V-sufficiency of r-jets in C^r mappings. In this subsection we discuss the relationship between the Kuo condition and V-sufficiency of r-jets in C^r mappings which are relative to the closed set $\Sigma \subset \mathbb{R}^n$ such that $0 \in \Sigma$.

Theorem 4.1. *Let r be a positive integer, and let $f \in \mathcal{E}_{[r]}(n,p)$ where $n > p$. Then the following conditions are equivalent.*

(1) f satisfies the relative Kuo condition (K_Σ).
(2) f satisfies condition $(\widetilde{K}_\Sigma)$.
(3) The relative r-jet $j^r f(\Sigma; 0)$ is Σ-V-sufficient in $\mathcal{E}_{[r]}(n,p)$.

Theorem 4.2. *Let r be a positive integer, and let $f \in \mathcal{E}_{[r]}(n,n)$. Suppose that $j^r f(\Sigma; 0)$ has a subanalytic C^r-realisation and that Σ is a subanalytic closed subset of $\mathbb{R}^n$ such that $0 \in \Sigma$. Then the following conditions are equivalent.*

(1) f satisfies the relative Kuo condition (K_Σ).
(2) f satisfies condition $(\widetilde{K}_\Sigma)$.
(3) The relative r-jet $j^r f(\Sigma; 0)$ is Σ-V-sufficient in $\mathcal{E}_{[r]}(n,n)$.

Remark 4.3. In the non-relative case C^0-sufficiency of r-jets in $\mathcal{E}_{[r]}(n,1)$ is equivalent to V-sufficiency of r-jets in $\mathcal{E}_{[r]}(n,1)$. The Bochnak-Lojasiewicz inequality takes a very important role in the proof of the equivalence. Therefore it may be natural to ask whether the Bochnak-Lojasiewicz inequality holds also in the relative case. More precisely, if we let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ a C^ω function germ, then we ask whether the following inequality

$$d(x,\Sigma)\|\mathrm{grad} f(x)\| \gtrsim |f(x)|$$

holds in a neighbourhood of $0 \in \mathbb{R}^n$. If this Bochnak-Lojasiewicz inequality holds in the relative case, then it follows that the relative Kuiper-Kuo condition $(K\text{-}K_\Sigma)$ and condition $(\widetilde{K}_\Sigma)$ are equivalent like in the non-relative case. But we give an example below to show that conditions $(K\text{-}K_\Sigma)$ and $(\widetilde{K}_\Sigma)$ are not necessarily equivalent in the relative case. As a result, we can see that the Bochnak-Lojasiewicz inequality does not always hold in the relative case, and it follows from Theorems 3.1, 4.1 that Σ-V-sufficiency of r-jets in $\mathcal{E}_{[r]}(n,1)$ does not always imply Σ-C^0-sufficiency of r-jets in $\mathcal{E}_{[r]}(n,1)$.

Example 4.4. Let us recall the situation in Example 3.6(2). Namely, $f_m(x,y) = x^3 - 3xy^m$, $m \geq 3$, and $\Sigma = \{x = 0\}$. Let $r = 3$. In this setting, the relative Kuiper-Kuo condition is $\|\mathrm{grad} f_m(x,y)\| \gtrsim |x|^{3-1}$ in a neighbourhood of $(0,0) \in \mathbb{R}^2$. But as seen in Example 3.6(2), the above inequality does not hold along an analytic arc $\lambda(t) = (t^m, t^2)$ for $m \geq 3$. In other words, the relative Kuiper-Kuo condition $(K\text{-}K_\Sigma)$ is not satisfied. Therefore, by Theorem 3.1, $j^3 f_m(\Sigma; 0)$ is not Σ-C^0-sufficienct in $\mathcal{E}_{[3]}(2,1)$. On the other hand, condition $(\widetilde{K}_\Sigma)$ is

$$|x|\|\mathrm{grad} f_m(x,y)\| + |f_m(x,y)| \gtrsim |x|^3$$

in a neighbourhood of $(0,0) \in \mathbb{R}^2$. We show that f_m, $m \geq 3$, satisfies this condition. Let $\lambda(t) = (a_k t^k + \cdots, b_s t^s + \cdots)$ be an analytic arc passing through $(0,0) \in \mathbb{R}^2$ as in Example 3.6. Then we may assume $a_k \neq 0$, and $|x| \approx |t|^k$.

In the case where $2k < ms$, we have

$$|x|\|\mathrm{grad} f_m(x,y)\| \geq 3|x||x^2 - y^m| \geq |x||x|^2 \approx |t|^k |t|^{2k} = |t|^{3k}$$

on λ near $(0,0) \in \mathbb{R}^2$.

In the case where $2k > ms$, we have

$$|x|\|\mathrm{grad} f_m(x,y)\| \geq 3|x||x^2 - y^m| \geq |x||y|^m \gtrsim |t|^k|t|^{2k} = |t|^{3k}$$

on λ near $(0,0) \in \mathbb{R}^2$.

In the case where $2k = ms$ and $a_k \neq b_s$, we have

$$|x|\|\mathrm{grad} f_m(x,y)\| \geq 3|x||x^2 - y^m| \gtrsim |x||x|^2 \approx |t|^k|t|^{2k} = |t|^{3k}$$

on λ near $(0,0) \in \mathbb{R}^2$.

In the case where $2k = ms$ and $a_k = b_s$, we have

$$|f_m(x,y)| = |x^3 - 3xy^m| = |x||x^2 - 3y^m| \gtrsim |x||x|^2 \approx |t|^k|t|^{2k} = |t|^{3k}$$

on λ near $(0,0) \in \mathbb{R}^2$. On any analytic arc λ, condition $(\widetilde{K}_\Sigma)$ is satisfied. Therefore we can see that f_m, $m \geq 3$, satisfies condition $(\widetilde{K}_\Sigma)$. It follows that conditions $(K\text{-}K_\Sigma)$ and $(\widetilde{K}_\Sigma)$ are not necessarily equivalent in the relative case. In addition, by Theorem 4.1, we see that $j^3 f_m(\Sigma;0)$ is Σ-V-sufficient in $\mathcal{E}_{[3]}(2,1)$ for any $m \geq 3$. Incidentally, the Bochnak-Lojasiewicz inequality does not hold along an analytic arc $\lambda(t) = (t^m, t^2)$ for $m \geq 3$.

Remark 4.5. It is well-known that the Kuiper-Kuo condition and V-sufficiency of jets are equivalent for function-germs. But, by Example 4.4 and Theorem 4.1, we can see that they are not always equivalent in the relative case.

4.1.1. *Relative V-sufficiency of r-jets in C^{r+1} mappings.* In this subsection we give some characterisations for the relative r-jets to be Σ-V-sufficient in C^{r+1} mappings.

Theorem 4.6. *Let r be a positive integer, and let $f \in \mathcal{E}_{[r+1]}(n,p)$, $n \geq p$. If f satisfies condition (K^δ_Σ), then the relative r-jet, $j^r f(\Sigma;0)$ is Σ-V-sufficient in $\mathcal{E}_{[r+1]}(n,p)$.*

Example 4.7. Let $f : (\mathbb{R}^n,0) \to (\mathbb{R},0)$, $n \geq 3$, be a polynomial function defined by

$$f(x_1, x_2, \ldots, x_n) := x_1^3 - 3x_1x_2^5.$$

and $\Sigma := \{x_1 = x_2 = 0\}$. Then we have

$$\mathrm{grad}\, f_m(x,y) = (3(x_1^2 - x_2^5), -15x_1x_2^4)$$

and $d(x,\Sigma) = \|(x_1,x_2)\|$. From the computation in Example 3.6(1),

$$\|\mathrm{grad} f(x)\| \gtrsim d(x,\Sigma)^{7-\frac{1}{2}}$$

in a neighbourhood of $0 \in \mathbb{R}^n$. Therefore, by Theorem 3.3(2), $j^7 f(\Sigma;0)$ is Σ-C^0-sufficient in $\mathcal{E}_{[8]}(n,1)$.

Now, since $g(x) = x_1^3 - 3x_1x_2^5 + x_2^{\frac{15}{2}} = (x_1 - x_2^{\frac{5}{2}})^2(x_1 + 2x_2^{\frac{5}{2}})$ is a realisation of the jet $j^7 f(\Sigma;0)$ in $\mathcal{E}_{[7]}(n,1)$, which is not Σ-V-equivalent to f; therefore $j^7 f(\Sigma;0)$ is not Σ-V-sufficient in $\mathcal{E}_{[7]}(n,1)$. The proof can be carried out like in [11].

Corollary 4.8. *Let r be a positive integer, and let $f \in \mathcal{E}_{[r+1]}(n,p)$, $n \geq p$. If there exists $\delta > 0$ such that*

$$d(x,\Sigma)\kappa(df(x)) + \|f(x)\| \gtrsim d(x,\Sigma)^{r+1-\delta}$$

holds in some neighbourhood of $0 \in \mathbb{R}^n$, then $j^r f(\Sigma;0)$ is Σ-V-sufficient in $\mathcal{E}_{[r+1]}(n,p)$.

Remark 4.9. In the non-relative case C^0-sufficiency of r-jets in $\mathcal{E}_{[r+1]}(n,1)$ is equivalent to V-sufficiency of r-jets in $\mathcal{E}_{[r+1]}(n,1)$, too. But this does not holds in the relative case, namely we give an example below to show that Σ-V-sufficiency of r-jets in C^{r+1} functions does not always imply Σ-C^0-sufficiency of r-jets in C^{r+1} functions, either.

Example 4.10. Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ be a polynomial function defined by

$$f(x,y) := (x - y^3)^2 + y^{10},$$

and let $\Sigma = \{x = 0\}$. Then we have $\mathrm{grad} f(x,y) = (2(x - y^3), -6y^2(x - y^3) + 10y^9)$. Let $\lambda(t) = (a_k t^k + \cdots, b_s t^s + \cdots)$ be an analytic arc passing through $(0,0) \in \mathbb{R}^2$ as in Example 3.6. Then we may assume $a_k \neq 0$, and then $|x| \approx |t|^k$.

In the case where $k < 3s$, we have $\|\mathrm{grad} f(x,y)\| \geq 2|x - y^3| \geq |x|$ on λ near $(0,0) \in \mathbb{R}^2$.

In the case where $k > 3s$, we have $\|\mathrm{grad} f(x,y)\| \geq 2|x - y^3| \geq |y|^3 \geq |x|$ on λ near $(0,0) \in \mathbb{R}^2$.

In the case where $k = 3s$, $|x| \approx |y|^3$. Therefore we have

$$|f(x)| = (x - y^3)^2 + y^{10} \geq y^{10} \approx |x|^{4 - \frac{2}{3}}$$

on λ near $(0,0) \in \mathbb{R}^2$.

On any analytic arc λ, $|x| \|\mathrm{grad} f(x,y)\| + |f(x)| \gtrsim |x|^{4-\frac{2}{3}}$ holds near $(0,0) \in \mathbb{R}^2$. Therefore the above inequality holds in a neighbourhood of $(0,0) \in \mathbb{R}^2$. It follows from Corollary 4.8 that $j^3 f(\Sigma; 0)$ is Σ-V-sufficient in $\mathcal{E}_{[4]}(2,1)$.

Let $\lambda(t) := (t^3, t)$. Then $|x| = |t|^3 = |y|$ on λ. Therefore we have

$$\|\mathrm{grad} f(x,y)\| = \|(0, 10t^9)\| = 10|t|^9 = 10|x|^3$$

on λ near $(0,0) \in \mathbb{R}^2$. By Theorem 3.3(2), $j^3 f(\Sigma; 0)$ cannot be Σ-C^0-sufficient in $\mathcal{E}_{[4]}(2,1)$.

We gave a sufficient condition for the relative r-jets to be Σ-V-sufficient in C^{r+1} mappings. We next give a necessary condition.

Definition 4.11 (Σ-Regular Horn neighbourhood). Let $f \in \mathcal{E}_{[r]}(n,p)$ and $d \in \mathbb{N}$.

We say that $\mathcal{H}_d(f)$ is Σ-regular if for some $w > 0$,

$$\kappa(df(x)) \gtrsim d(x, \Sigma)^{d-1}$$

for $x \in \mathcal{H}_d(f; w)$, x near 0.

Remark 4.12. For germ $f \in \mathcal{E}_{[r]}(n,p)$, $n \geq p$, the following conditions are equivalent:

1) $\mathcal{H}_r(f)$ is Σ-regular
2) f satisfies condition $(\widetilde{K}_\Sigma)$.

Proposition 4.13. *Let r be a positive integer, and let $f \in \mathcal{E}_{[r+1]}(n,p)$, $n \geq p$, such that the relative r-jet $j^r f(\Sigma; 0)$ is Σ-V-sufficient in $\mathcal{E}_{[r+1]}(n,p)$. Then for any realisation g of $j^r f(\Sigma; 0)$ in $\mathcal{E}_{[r+1]}(n,p)$, the horn neighbourhood $\mathcal{H}_{r+1}(g)$ is Σ-regular.*

5. Rigidity and Relative SV-determinacy

Let $\mathcal{E}(n)^p$, $n \geq p$, be the set of C^∞ map-germs : $\mathbb{R}^n \to \mathbb{R}^p$ at $0 \in \mathbb{R}^n$, and let Σ be a germ of closed subset of $\mathbb{R}^n$ such that $0 \in \Sigma$. We say that $f \in \mathcal{E}(n)^p$ is *finitely $\Sigma - SV$-determined* (resp. *finitely $\Sigma - V$-determined*) if there is a positive integer k such that for any $g \in \mathcal{E}(n)^p$ having the same k-jet as f at $0 \in \mathbb{R}^n$, g is $\Sigma - SV$-equivalent (resp. $\Sigma - V$-equivalent) to f. Concerning finite SV-determinacy or finite V-determinacy, lots of characterisations have been obtained in the case of isolated singularities (see J. Bochnak - T.-C. Kuo [3]).

Let $\varphi = (\varphi_1, \ldots, \varphi_p) : \mathbb{R}^n \to \mathbb{R}^p$, $n \geq p$, be a C^∞ map-germ at $0 \in \mathbb{R}^n$. We denote by $I_K(\varphi)$ the ideal of $\mathcal{E}(n)$ generated by $\varphi_1, \ldots, \varphi_p$ and the Jacobian determinants

$$\frac{D(\varphi_1, \ldots, \varphi_p)}{D(x_{i_1}, \ldots, x_{i_p})}(x) \qquad (1 \leq i_1 < \ldots < i_p \leq n),$$

and we let $Z(\varphi, x) := \sum_{1 \le i_1 < \ldots < i_p \le n} \left| \frac{D(\varphi_1, \ldots, \varphi_p)}{D(x_{i_1}, \ldots, x_{i_p})}(x) \right|^2 + \sum_{j=1}^{p} \varphi_j(x)^2$. In the case where $n > p$, we define also the ideal of $\mathcal{E}(n)$, denoted by $I_T(\varphi)$, generated by $\varphi_1, \ldots, \varphi_p$ and the Jacobian determinants $\frac{D(\varphi_1, \ldots, \varphi_p, \rho)}{D(x_{i_1}, \ldots, x_{i_{p+1}})}(x) \quad (1 \le i_1 < \ldots < i_{p+1} \le n)$. In the case where $n = p$, we define the ideal $I_T(\varphi)$ of $\mathcal{E}(n)$, as the ideal generated by only $\varphi_1, \ldots, \varphi_p$.

Recall that for $1 \le s \le \infty$, $\mathfrak{m}^s_\Sigma$ is the ideal of $\mathcal{E}(n)$ of germs of $(s-1)$-flat functions at Σ, namely $\mathfrak{m}^s_\Sigma = \{f \in \mathcal{E}(n) : j^{s-1}f(\Sigma; 0) = 0\}$. Therefore we have, $\mathfrak{m}^\infty_\Sigma = \cap_{s=1}^{\infty} \mathfrak{m}^s_\Sigma$.

Let r be a positive integer, and let $\varphi = (\varphi_1, \ldots, \varphi_p) : \mathbb{R}^n \to \mathbb{R}^p$, $n \ge p$, be a C^r map-germ at $0 \in \mathbb{R}^n$. We denote by $\mathcal{E}_{[r]}(n)$ be the ring of C^r function-germs : $\mathbb{R}^n \to \mathbb{R}$ at $0 \in \mathbb{R}^n$, by $\mathcal{E}_{[r]}(n)^p$ the set of C^r map-germs : $\mathbb{R}^n \to \mathbb{R}^p$ at $0 \in \mathbb{R}^n$, and by $\mathcal{E}_{[r]}(n)(\varphi)$ the ideal of $\mathcal{E}_r(n)$ generated by $\varphi_1, \ldots, \varphi_p$.

Definition 5.1. We call $\varphi \in \mathcal{E}(n)^p$, *$\Sigma$-$C^r$-rigid* if there is a positive integer k for which the following holds:
for any $\psi \in \mathcal{E}(n)^p$ such that $j^k\varphi = j^k\psi$ on Σ, there exists $\tau \in \mathcal{R}^{\mathrm{fix}}_\Sigma$ such that

$$\mathcal{E}_{[r]}(n)(\varphi \circ \tau) = \mathcal{E}_{[r]}(n)(\psi).$$

Definition 5.2. Let I be an ideal of $\mathcal{E}(n)$. We say that I is *Σ-elliptic* if there is $f \in I$ such that

$$|f(x)| \ge Cd(x, \Sigma)^\alpha$$

in a neighbourhood of 0, where C and α are positive constants. We call such f an *elliptic element* of I.

Remark 5.3. If the ideal I is Σ-elliptic and generated by $f_1, \ldots, f_k$, then $f_1^2 + \ldots + f_k^2$ is an elliptic element of I.

Proposition 5.4. *For $\varphi \in \mathcal{E}(n)^p$, the following conditions are equivalent:*

(1) There exist $C, \alpha, \beta > 0$ such that $Z(\varphi, x) \ge Cd(x, \Sigma)^\alpha$ for $|x| < \beta$.
(2) $\mathfrak{m}^\infty_\Sigma \subset I_K(\varphi)$.
If moreover Σ is subanalytic and φ is analytic, they are also equivalent to:
(3) $\mathfrak{m}^\infty_\Sigma \subset I_T(\varphi)$.
(4) The set germ at 0, $\mathrm{Sing}(\varphi) \cap \varphi^{-1}(0)$, is contained in Σ.

Definition 5.5. A germ of closed subset Σ of $\mathbb{R}^n$ is called *coherent* if $\mathfrak{m}_\Sigma$ is a finitely generated ideal of $\mathcal{E}(n)$.

This definition is inspired by the following result of W. Kucharz proved in [13]: an analytic and semi-algebraic subset X in an open subset U of $\mathbb{R}^n$ is coherent if and only if $\mathfrak{m}_X$ is a finitely generated ideal of $\mathcal{E}(n)$. In particular, $\Sigma = \{0\}$ is coherent.
Let us give a generalisation of the Bochnak-Kuo theorem in [3] as follows.

Theorem 5.6. *Let Σ be a coherent germ of closed subset of $\mathbb{R}^n$ such that $0 \in \Sigma$. Then the following conditions are equivalent for $\varphi \in \mathcal{E}(n)^p$ where $n > p$:*

(1) For each $r \in \mathbb{N}$, φ is Σ-C^r-rigid.
(2) φ is finitely Σ-SV-determined.
(3) φ is finitely Σ-V-determined.
(4) $I_K(\varphi)$ is Σ-elliptic.
(5) $\mathfrak{m}^\infty_\Sigma \subset I_K(\varphi)$.
If moreover φ is analytic, they are also equivalent to:
(6) $\mathfrak{m}^\infty_\Sigma \subset I_T(\varphi)$.

6. Some comments and open problems

Definition 6.1. 1) A germ $f \in \mathcal{E}(n)$ is say to be Σ-elliptic on a subset X of $\mathbb{R}^n$ if there exists a real number $\alpha > 0$ and a neighbourhood U of 0 in $\mathbb{R}^n$ such that for all $x \in U \cap X$ we have

$$|f(x)| \geq d(x, \Sigma)^\alpha$$

2) A germ $f \in \mathcal{E}(n)$ is say to be Σ-flat on a subset X of $\mathbb{R}^n$ if for every real number $\alpha \geq 0$, $f(x) = o(d(x, \Sigma)^\alpha)$, for $x \in X \setminus \Sigma$.
3) If $\Sigma = \{0\}$, we say f is elliptic (resp. flat) on X.

Remark 6.2. a) $f = 0 \iff f$ is flat on $X = \mathbb{R}^n$.
b) If f is not Σ-elliptic on X, then there exists $Y \subset X$, such that f is Σ-flat on Y.
c) If $f(0) = 0$ and f is not Σ-flat on X, then there exists $Y \subset X$, such that f is Σ-elliptic on Y.

Theorem 6.3. *Let Σ be a nonempty set germ of a closed set at 0 of $\mathbb{R}^n$. Let $f_1, \dots, f_l$ be elements of $\mathcal{E}(n)$ and $k \leq l$. The following conditions are equivalents:*
1) There exists a subset X of $\mathbb{R}^n \setminus \Sigma$, such the f_i are flat on X for $i \in \{1, \dots, k\}$ and Σ-elliptic for $i \in \{k+1, \dots, l\}$.
2) There exists a smooth curve $g : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ whose germ at 0 is not flat, and such that $f_i \circ g$ is flat at 0 for $i \in \{1, \dots, k\}$ and Σ-elliptic for $i \in \{k+1, \dots, l\}$ and positive on $\mathbb{R}_+ \setminus \{0\}$.
3) When $k = l$, the ideal generated by the f_i is not Σ-elliptic.

This results may be seen is a generalisation of a result of Merrien-Lassalle [23], [20], in the absolute case i.e. $\Sigma = \{0\}$, they use the Sturm theory for for equations on ordered fields, in instance the field for fractions on Puiseux series with coefficients in $\mathbb{R}$.

Corollary 6.4 (Relative curve selection lemma). *For $f \in \mathcal{E}(n)^p$ such that $I_K(f)$ is Σ-elliptic and $f^{-1}(0)$ is not contained in Σ (as germs at 0).*
Then there exists a smooth germ curve $\gamma : (\mathbb{R}, 0) \to (\mathbb{R}^n, 0)$ such that $\gamma((0, +\infty)) \subset f^{-1}(0) \setminus \Sigma$.

As an application me may improve the results of Theorems 4.2, where we get ride, in the case $n = p$, of the subanalytic restriction.

Open problems: The followings results are known to holds true in the case $\Sigma = \{0\}$. We suppose Σ a subanalytic set or definable set in some polynomially bounded 0-minimal structure (see [7] for definition of 0-minimal structures), or more generally a germ of closed set at 0 :

1) Is there a converse to theorem 4.6 i.e. Let r be a positive integer, and let $f \in \mathcal{E}_{[r+1]}(n, p)$, $n \geq p$. If f the relative r-jet, $j^r f(\Sigma; 0)$ is Σ-V-sufficient in $\mathcal{E}_{[r+1]}(n, p)$ then satisfies condition (K^δ_Σ).
2) The same question for Proposition 4.13
3) For $f \in \mathcal{E}_{[r]}(n, p)$, is V-sufficiency equivalent to SV-sufficiency?
4) Find for $f \in \mathcal{E}_{[r]}(n, p)$ an analytic characterisation of Σ-V-sufficiency in $\mathcal{E}_{[k]}(n, p)$, $k \geq r + 2$.

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