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NON-INJECTIVITY OF GENERALIZED DISTANCE-SQUARED MAPPINGS OF EQUIDIMENSIONAL CASES

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ABSTRACT. Generalized distance-squared mappings are quadratic mappings of $\mathbb{R}^m$ into $\mathbb{R}^\ell$ of a special type. In this paper, it is shown that any generalized distance-squared mapping of equidimensional cases is not injective.

1. INTRODUCTION

Throughout this paper, i, j, ℓ, m, n stand for positive integers. Let $p_i = (p_{i1}, p_{i2}, \ldots, p_{im})$ $(1 \leq i \leq \ell)$ (resp., $A = (a_{ij})_{1\leq i\leq \ell, 1\leq j\leq m}$) be a point of $\mathbb{R}^m$ (resp., an $\ell \times m$ matrix with non-zero entries). Set $p = (p_1, p_2, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell$. Let $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^\ell$ be the mapping defined by

$$G_{(p,A)}(x) = \left(\sum_{j=1}^{m} a_{1j}(x_j - p_{1j})^2, \sum_{j=1}^{m} a_{2j}(x_j - p_{2j})^2, \ldots, \sum_{j=1}^{m} a_{\ell j}(x_j - p_{\ell j})^2\right),$$

where $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. The mapping $G_{(p,A)}$ is called a *generalized distance-squared mapping*, and the ℓ-tuple of points $p = (p_1, \ldots, p_\ell) \in (\mathbb{R}^m)^\ell$ is called the *central point* of the generalized distance-squared mapping $G_{(p,A)}$. A *distance-squared mapping* D_p (resp., *Lorentzian distance-squared mapping* L_p) is the mapping $G_{(p,A)}$ satisfying that each entry of A is 1 (resp., $a_{i1} = -1$ and $a_{ij} = 1$ $(j \neq 1)$).

In [1] (resp., [2]), a classification result on distance-squared mappings D_p (resp., Lorentzian distance-squared mappings L_p) is given.

In [5], a classification result on generalized distance-squared mappings of the plane into the plane is given. If the rank of A is two, a generalized distance-squared mapping having a generic central point is a mapping of which any singular point is a fold point except one cusp point (for details on fold points and cusp points, refer to [6]). The singular set is a rectangular hyperbola. If the rank of A is one, a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to the normal form of definite fold mapping $(x_1, x_2) \to (x_1, x_2^2)$.

In [3], a classification result on generalized distance-squared mappings of $\mathbb{R}^{m+1}$ into $\mathbb{R}^{2m+1}$ is given. If the rank of A is $m+1$, a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to the mapping called the normal form of Whitney umbrella as follows:

$$(x_1, \ldots, x_{m+1}) \mapsto (x_1^2, x_1x_2, \ldots, x_1x_{m+1}, x_2, \ldots, x_{m+1}).$$

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If the rank of A is less than $m+1$, a generalized distance-squared mapping having a generic central point is $\mathcal{A}$-equivalent to the inclusion as follows:

$$(x_1, \ldots, x_{m+1}) \mapsto (x_1, \ldots, x_{m+1}, 0, \ldots, 0).$$

In [4], the properties of compositions by generalized distance-squared mappings having a generic central point are investigated. As an appendix of [4], the following lemma is proved.

Lemma 1.1. *Any generalized distance-squared mapping of equidimensional cases* $G_{(p,A)} : \mathbb{R}^m \to \mathbb{R}^m$ *is not injective.*

The main purpose of this paper is to give another proof of this lemma (for the proof of this lemma, see Section 2).

2. Proof of Lemma 1.1

If $m = 1$, then we get the mapping $G_{(p,A)} : \mathbb{R} \to \mathbb{R}$ defined by $G_{(p,A)}(x_1) = a_{11}(x_1 - p_{11})^2$. It is clearly seen that the mapping $G_{(p,A)} : \mathbb{R} \to \mathbb{R}$ is not injective. Hence, it is sufficient to consider the cases of $m \geq 2$.

Let $h : \mathbb{R}^m \to \mathbb{R}^m$ be the diffeomorphism defined by

$$h(x_1, \ldots, x_m) = (x_1 + p_{m1}, \ldots, x_m + p_{mm}).$$

The composition of $G_{(p,A)}$ and h is as follows:

$$\begin{aligned} & G_{(p,A)} \circ h(x) \\ = & \left(\sum_{j=1}^{m} a_{1j} \left(x_j + p_{mj} - p_{1j}\right)^2, \ldots, \sum_{j=1}^{m} a_{m-1,j} \left(x_j + p_{mj} - p_{m-1,j}\right)^2, \sum_{j=1}^{m} a_{mj} x_j^2 \right). \end{aligned}$$

Let $H : \mathbb{R}^m \to \mathbb{R}^m$ be the diffeomorphism of the target for deleting constant terms. The composition of H and $G_{(p,A)} \circ h$ is as follows:

$$\begin{aligned} & H \circ G_{(p,A)} \circ h(x) \\ = & \left(\sum_{j=1}^{m} a_{1j} x_j^2 + \sum_{j=1}^{m} b_{1j} x_j, \ldots, \sum_{j=1}^{m} a_{m-1,j} x_j^2 + \sum_{j=1}^{m} b_{m-1,j} x_j, \sum_{j=1}^{m} a_{mj} x_j^2 \right), \end{aligned}$$

where $b_{ij} = 2a_{ij}(p_{mj} - p_{ij})$ $(1 \leq i \leq m-1, 1 \leq j \leq m)$.

Now, consider the following:

$$\sum_{j=1}^{m} b_{1j} x_j = \cdots = \sum_{j=1}^{m} b_{m-1,j} x_j = 0. \tag{1}$$

By (1), we get $(x_1, \ldots, x_m)B = (0, \ldots, 0)$, where $B = (b_{ij})_{1 \leq i \leq m-1, 1 \leq j \leq m}$. It is clearly seen that $m = \text{rank } B + \dim \text{Ker } B$. Hence, by $\text{rank } B = m - \dim \text{Ker } B$ and $\text{rank } B \leq m-1$, we have $\dim \text{Ker } B \geq 1$. Therefore, there exists a non-zero vector $(c_1, \ldots, c_m) \in \text{Ker } B$. Set $c = (c_1, \ldots, c_m)$. Then, it follows that $H \circ G_{(p,A)} \circ h(c) = H \circ G_{(p,A)} \circ h(-c)$. Since H and h are diffeomorphisms, we see that $G_{(p,A)}$ is not injective.

□

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