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# An application of two-edge coloured graphs to group algebras of non-noetherian groups

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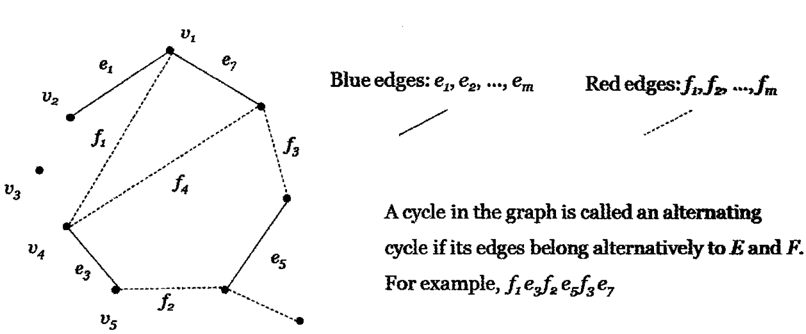
In this note, we introruce an SR-graph and an SR-cycle; we show that certain SR-graphs have SR-cycles. The class of SR-graphs is a subclass of the class of two-edge coloured graphs in which an SR-cycle is called an alternating cycle. We also consider an application of SR-graphs to group algebras; how to prove primitivity of group algebras of non-noetherian groups.

## 1 Two-edge coloured graphs

Let $\mathcal{G} = (V, E)$ be a simple graph (i.e., an undirected graph without loops or multi-edges) with vertex set $V$ and edge set $E$. $\mathcal{G}$ is a two-edge coloured graph if each of the edges is coloured either red or blue. We call a path alternating if the successive edges in $\mathcal{G}$ alternate in colour. For any $W \subseteq V$, we let $\mathcal{G}[W]$ denote the subgraph of $\mathcal{G}$ induced by $W$, i.e., $\mathcal{G}[W] := (W, \{vw \in E \mid v, w \in W\})$; let $\mathcal{G}_v := \mathcal{G}[V \setminus \{v\}]$.

A two-edge coloured graph

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We let $X(\mathcal{G})$ denote the set of all cut-vertices of $\mathcal{G}$, i.e., the set of all $v \in V$ so that $c(\mathcal{G}_v) > c(\mathcal{G})$. For any terminology and notation which we do not define, we follow [1] (which can also serve as an introductory text if needed).

The following result is due to Grossman and Häggkvist [3]:

**Theorem 1.1.** ([3, Theorem]) *Let $\mathcal{G}$ be a two-edge coloured graph so that every vertex is incident with at least one edge of each colour. Then either $\mathcal{G}$ has a cut vertex separating colours, or $\mathcal{G}$ has an alternating cycle.*

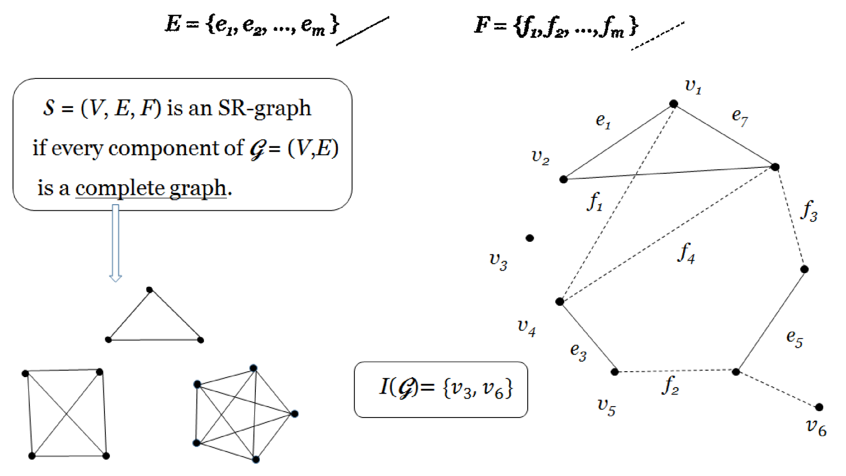
## 2 SR-graphs

In this section, we define an SR-graph and an SR-cycle; we show that certain SR-graphs have SR-cycles. We write $\mathcal{G} = (V, E)$ to denote that $\mathcal{G}$ is a simple graph (undirected and without loops or multi-edges) having vertex set $V$ and edge set $E$. We denote $\{v, w\} \in E$ by $vw$ when there is no risk of confusion. We let $I(\mathcal{G})$ denote the isolated vertices of $\mathcal{G}$, i.e., the set of all $v \in V$ for which $vw \notin E$ for all $w \in V$. We denote by $C(\mathcal{G})$ the set of components of $\mathcal{G}$, i.e., the set of subgraphs of $\mathcal{G}$ which partition $\mathcal{G}$, so that in each subgraph any two vertices are joined by a path, and so that no vertices which do not lie in the same subgraph are joined by a path in $\mathcal{G}$; we let $c(\mathcal{G}) := |C(\mathcal{G})|$. We say that $\mathcal{G}$ is connected if $c(\mathcal{G}) = 1$. We begin with two definitions:

**Definition 2.1.** Let $\mathcal{G} := (V, E)$ and $\mathcal{H} := (V, F)$. If every component of $\mathcal{G}$ is a complete graph, and if $E \cap F = \emptyset$, then we call the triple $\mathcal{S} = (V, E, F)$ a *sprint relay graph*, abbreviated SR-graph. We view $\mathcal{S}$ as the graph $(V, E \cup F)$, guaranteed simple as $E \cap F = \emptyset$, with edges partitioned into $E$ and $F$; we denote $\mathcal{S}$ by $(\mathcal{G}, \mathcal{H})$ rather than $(V, E, F)$ when convenient.

**Definition 2.2.** A cycle in an SR-graph $(V, E, F)$ is called an SR-cycle if its edges belong alternatively to $E$ and not to $E$; more formally, we call cycle $(V', E')$ an SR-cycle if there is labeling $V' = \{v_1, v_2, \ldots, v_c\}$ and $E' = \{v_1v_2, v_2v_3, \ldots, v_{c-1}v_c, v_cv_1\}$ so that $v_iv_{i+1} \in E$ if and only if $i$ is odd, for some even $c$.

An SR-graph




The class of SR-graphs is a subclass of the class of two-edge coloured graphs in which an SR-cycle is simply an alternating cycle (see the previous section).

For the remainder of this section, fix $\mathcal{S} = (V, E, F)$, $\mathcal{G} = (V, E)$, and $\mathcal{H} = (V, F)$ so that $V \neq \emptyset$, every component of $\mathcal{G}$ complete, and $\mathcal{S}$ an SR-graph. Moreover, let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ denote the components of $\mathcal{H}$ with $\mathcal{H}_i = (V_i, E_i)$ over $i \in [n]$. We first address the case in which $\mathcal{H}_i$ is a complete graph for each $i \in [n]$ as follows:

**Theorem 2.3.** ([4, Theorem 2.3]) *If $\mathcal{S}$ is connected and each component of $\mathcal{H}$ is complete, then $\mathcal{S}$ has an SR-cycle if and only if $c(\mathcal{G}) + c(\mathcal{H}) < |V| + 1$.*

Recall that $X(\mathcal{G})$ denote the set of all cut-vertices of $\mathcal{G}$. The following result follows from Theorem 1.1:

**Lemma 2.4.** *If $\mathcal{S}$ has no SR-cycle, then* $I(\mathcal{G}) \cup I(\mathcal{H}) \cup X(\mathcal{S}) \neq \emptyset$.

Before moving on, let us collect some straightforward observations:

**Remark 2.5.** Assume that $\mathcal{S}$, $\mathcal{G}$, and $\mathcal{H}$ satisfy the hypotheses of Theorem 2.3.

(I) If $v \notin X(\mathcal{S})$, then

- (i) $v \in I(\mathcal{G}) \cup I(\mathcal{H})$ implies $c(\mathcal{G}_v) + c(\mathcal{H}_v) = c(\mathcal{G}) + c(\mathcal{H}) - 1$;
- (ii) $v \notin I(\mathcal{G}) \cup I(\mathcal{H})$ implies $c(\mathcal{G}_v) = c(\mathcal{G})$ and $c(\mathcal{H}_v) = c(\mathcal{H})$.

(II) If $v \in X(\mathcal{S})$, then without loss of generality,

- (i) $\mathcal{S}_v$ is an SR-graph with components $(\mathcal{G}_1, \mathcal{H}_1)$ and $(\mathcal{G}_2, \mathcal{H}_2)$;
- (ii) $\sum_{i=1}^{2}(c(\mathcal{G}_i) + c(\mathcal{H}_i)) = c(\mathcal{G}) + c(\mathcal{H})$ and $|V_1| + |V_2| = |V| - 1$, where $V_1$ and $V_2$ are the vertex sets of $(\mathcal{G}_1, \mathcal{H}_1)$ and $(\mathcal{G}_2, \mathcal{H}_2)$, respectively.

We are now ready to prove Theorem 2.3.

*Proof of Theorem 2.3.* Before entering the heart of this proof, we show that

$$c(\mathcal{G}) + c(\mathcal{H}) \leq |V| + 1, \tag{1}$$

which holds trivially when $|V| = 1$. Assume, by way of induction, that $|V| > 1$ and that (1) holds for SR-graphs on fewer vertices. Fix $v \in V$. If $v \notin X(\mathcal{S})$, then $\mathcal{S}_v$ is connected and $\mathcal{H}_v$ has complete components; thus, $c(\mathcal{G}_v) + c(\mathcal{H}_v) \leq |V|$ by induction, and so (1) follows from Remark 2.5(I). If $v \in X(\mathcal{S})$, then $\mathcal{S}_v$ has components $(\mathcal{G}_1, \mathcal{H}_1)$ and $(\mathcal{G}_2, \mathcal{H}_2)$ by Remark 2.5(II)(i); by induction, $c(\mathcal{G}_i) + c(\mathcal{H}_i) \leq |V_i| + 1$ for $i \in [2]$, and thus (1) holds by Remark 2.5(II)(ii).

We are now ready for the crux of our argument. First, assume that $\mathcal{S}$ has an SR-cycle. We prove by induction on $|V|$ that $c(\mathcal{G})+c(\mathcal{H}) < |V|+1$, noting that we may assume $|V| \geq 4$. This holds trivially if $|V| = 4$, so assume $|V| > 4$ and, by way of induction, that the the result holds for SR-graphs on fewer vertices. This result holds trivially if $\mathcal{S}$ is an SR-cycle, so we may assume that there is $C \subsetneq V$ so that $\mathcal{S}[C]$ is an SR-cycle.

Consider $v \in V \setminus C$. If $v \notin X(\mathcal{S})$, then we can obtain the desired result with a similar argument to that which we used in the first paragraph when $v \notin X(\mathcal{S})$ was assumed. Assume $v \in X(\mathcal{S})$, in which case $\mathcal{S}_v$ has components $(\mathcal{G}_1, \mathcal{H}_1)$ and $(\mathcal{G}_2, \mathcal{H}_2)$ by Remark 2.5(II)(i). Since $v \in X(\mathcal{S})$ and $\mathcal{G}$ and $\mathcal{H}$ have complete components, either $C \subseteq V_1$ or $C \subseteq V_2$; say, without loss of generality, that $C \subseteq V_1$. Then, by our induction hypothesis, $c(\mathcal{G}_1)+c(\mathcal{H}_1) < |V_1|+1$. Also, by (1), $c(\mathcal{G}_2) + c(\mathcal{H}_2) \leq |V_2| + 1$. Thus, by Remark 2.5(II)(ii) that $c(\mathcal{G}) + c(\mathcal{H}) < |V| + 1$.

To prove the converse, by (1), it suffices to show that if $\mathcal{S}$ has no SR-cycle, then $c(\mathcal{G}) + c(\mathcal{H}) = |V| + 1$. To that end, assume $\mathcal{S}$ has no SR-cycle. Our proof will again be by induction on $|V|$. If $X(\mathcal{S}) \neq \emptyset$ then we may consider $v \in X(\mathcal{S})$ and obtain the result with a similar argument to that which we used in the first paragraph when $v \in X(\mathcal{S})$ was assumed. Assume $X(\mathcal{S}) = \emptyset$. By Lemma 2.4, there is $v \in I(\mathcal{G}) \cup I(\mathcal{H})$. By induction, $c(\mathcal{G}_v) + c(\mathcal{H}_v) = |V|$. It follows from Remark 2.5(I)(i) that $c(\mathcal{G}) + c(\mathcal{H}) = |V| + 1$. $\square$

Let $I := I(\mathcal{G})$, $W := V \setminus I$, $W_i := V_i \setminus I$, and say $\mathcal{H}[W_i] = (W_i, F_i)$. For any $m_1, m_2, \ldots, m_k \in \mathbb{N}$, we let $K_{m_1,m_2,\ldots,m_k}$ denote the complete multipartite graph with partite sets of size $m_1, m_2, \ldots, m_k$, i.e., the graph $(V', E')$ so that $V'$ can be partitioned into sets $P_1, P_2, \ldots, P_k$ called partite sets, with $|P_i| = m_i$ and $vw \in E'$ if and only if $v$ and $w$ are in different partite sets for all $v, w \in V$. We let $\mu(K_{m_1,m_2,\ldots,m_k}) := \max_{i \in [k]}\{m_i\}$. We now handle the case in which each component of

$\mathcal{H}$ is complete multipartite. We can then get the following theorem:

**Theorem 2.6.** ([4, Theorem 2.6]) *Assume that $\mathcal{H}_i$ is a complete multipartite graph for each $i \in [n]$. If $|I| \leq n$ and $|V_i| > 2\mu(\mathcal{H}_i)$ for each $i \in [n]$, then $\mathcal{S}$ has an SR-cycle.*

In order to build to a proof of Theorem 2.6, we need two lemmas (see [4]).

**Lemma 2.7.** *Let $U \subseteq V$ with $U \cap I = \emptyset$, and let $U' := V \setminus U$. Then, $|I \cap U'| \leq |I(\mathcal{G}[U'])| \leq |I \cap U'| + |U|$.*

**Lemma 2.8.** *If $\mathcal{H}[W_i] \not\simeq K_{1,m}$ for all $m \geq 2$ and $I(\mathcal{H}[W]) = \emptyset$, then $\mathcal{S}$ has an SR-cycle.*

We are now read to prove Theorem 2.6.

*Proof of Theorem 2.6.* Our proof is by induction on $n$. Assume $n = 1$, and say $\mathcal{H}_1$ has partite sets $P_1, P_2, \ldots, P_p$. We note that if there are distinct $i, j \in [p]$, and $v_i, w_i \in P_i$ and $v_j, w_j \in P_j$ with $v_i w_i, v_j w_j \in E$, then $\mathcal{S}[\{v_i, w_i, v_j, w_j\}]$ is an SR-cycle by definition. So, we my assume, without loss of generality, that elements of $E$ join only vertices of $P_1$ (and thus, that $P_i \subseteq I$ for $i \neq 1$). However, as $|V_1| > 2|P_1|$, this implies that $|I| \geq |V_1 \setminus P_1| > 1$, so this case cannot occur, and thus the desired result holds when $n = 1$. Assume, by way of induction, that this result holds for all SR-graphs $(V', E', F')$ satisfying analogous hypotheses, if $(V', F')$ has less than $n$ components.

Suppose that there is $i \in [n]$ with $\mathcal{H}[W_i] \simeq K_{1,m}$ for some $m \geq 2$. Since $|W_i| = |V_i| - |I \cap V_i|$ by definition, and since $|W_i| = m + 1$ by assumption, it follows from our hypotheses that

$$m + 1 > 2\mu(\mathcal{H}_i) - |I \cap V_i| \geq 2m - |I \cap V_i|, \tag{2}$$

since $\mu(\mathcal{H}_i) \geq \mu(\mathcal{H}[W_i]) = m$. Let $P_1, P_2, \ldots, P_k$ be the partite sets of $\mathcal{H}_i$, and let $Q_1 = \{w_0\}$ and $Q_2 = \{w_1, w_2, \ldots, w_m\}$ be the partite sets of $\mathcal{H}[W_i]$; without loss of generality, say $Q_1 \subseteq P_1$ and $Q_2 \subseteq P_2$. Now, since $|V_i| > 2\mu(\mathcal{H}_i)$, $k \geq 3$; since $\mathcal{H}[W_i] \simeq K_{1,m}$, this implies that there is $v \in P_3 \cap I$. Let $V'$ be obtained from $V$ by replacing $V_i$ with $V_i' := \{w_0, w_1, v\}$, and consider $\mathcal{S}[V']$. Since $\mathcal{H}[V_i'] \simeq K_{1,1,1}$, we have $|V_i'| > 2\mu(\mathcal{H}[V_i'])$. Moreover, if the vertices in $Q_2 \setminus \{w_1\}$ are removed from $V$, then the number of additional isolated vertices caused by the removing of those vertices is at most $|Q_2 \setminus \{w_1\}|$ by Lemma 2.7. Moreover $|(I \cap V_i)| \geq m$ by (2), and so it holds that

$$\begin{aligned} |I(\mathcal{G}[V'])| &\leq |I| - |(I \cap V_i) \setminus \{v\}| + |Q_2 \setminus \{w_1\}| \\ &\leq n - (m-1) + (m-1) = n. \end{aligned}$$

Therefore, $\mathcal{S}[V']$ still satisfies the hypotheses of our theorem, and clearly, if $\mathcal{S}[V']$ has an SR-cycle then so must $\mathcal{S}$. Moreover, by considering corresponding $W_i' = \{w_0, w_1\}$, we see that $\mathcal{H}[W_i'] \simeq K_{1,1}$ (and, in particular, no longer isomorphic to $K_{1,m}$ for any $m \geq 2$). Thus, we may assume that $\mathcal{H}[W_i] \not\simeq K_{1,m}$ (by applying this procedure to any component of $\mathcal{H}$ if necessary).

Since $\mathcal{H}[W_i] \not\simeq K_{1,m}$ for any $m \geq 2$, if $F_i \neq \emptyset$ for all $i \in [n]$ (as this is equivalent to $I(\mathcal{H}[W]) = \emptyset$ in this case), then we obtain the desired result by Lemma 2.8. So, it remains to assume that $\mathcal{H}[W_i] \not\simeq K_{1,m}$, but that $F_i = \emptyset$ for some $i$. Let $V' := V \setminus V_i$ and say $\mathcal{S}[V'] = (V', E', F')$. Since the number of components of $(V', F')$ is $n-1$, we may apply our induction hypothesis and prove this result if $|I(\mathcal{G}[V'])| \leq n-1$; we show that this must be the case. Let $m := |W_i|$. Since $\mathcal{H}_i$ is a complete $k$-partite graph and $F_i = \emptyset$, $W_i$ is contained in a partition of $\mathcal{H}_i$, and so $|V_i| > 2m$ by assumption; thus, $|I \cap V_i| = |V_i| - m > m$. Since $I \cap V' = I \setminus (I \cap V_i)$ and $|I| \leq n$, we have $|I \cap V'| \leq n - m - 1$. On the other hand, by Lemma 2.7, $|I(\mathcal{G}[V'])| - |I \cap V'| \leq m$. Hence,

$$m \geq |I(\mathcal{G}[V'])| - |I \cap V'| \geq |I(\mathcal{G}[V'])| - (n - m - 1),$$

and thus $|I(\mathcal{G}[V'])| \leq n-1$. $\square$

# 3 How to apply SR-graph theory to algebras

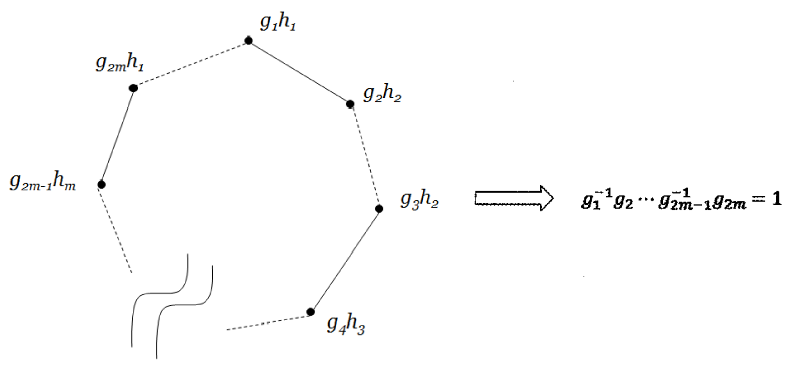
In order to prove the group algebra $R = KG$ of a group $G$ over a field $K$ to be primitive, according to the method of Formanek [2], it suffices to show that for each non-zero $a \in R$, there exists an element $\varepsilon(a)$ in the ideal $RaR$ generated by $a$ such that the right ideal $\rho = \sum_{a \in R\setminus\{0\}}(\varepsilon(a)+1)R$ is proper. The main difficulty here is how to choose elements $\varepsilon(a)$'s so as to make $\rho$ be proper. Now, $\rho$ is proper if and only if $r \neq 1$ for all $r \in \rho$. Since $\rho$ is generated by the elements of form $(\varepsilon(a)+1)$ with $a \neq 0$, $r$ has the presentation, $r = \sum_{(a,b)\in\Pi}(\varepsilon(a)+1)b$, where $\Pi$ is a subset of $R \times R$ consisting of a finite number of elements both of whose components are non-zero. Moreover, since $\varepsilon(a)$ and $b$ are linear combinations of elements of $G$, $r$ is presented as follows:

$$r = \sum_{(a,b)\in\Pi} \sum_{g \in S_a, h \in T_b} (\alpha_g \beta_h gh + \beta_h h), \tag{3}$$

where $S_a$ and $T_b$ are the support of $\varepsilon(a)$ and $b$ respectively and both $\alpha_g$ and $\beta_h$ are elements in $K$. In the above presentation (3), if there exists $gh$ such that $gh \neq 1$ and does not coincide with the other $gh$'s and $h$'s, then $r \neq 1$ holds.

On the contrary, if $r = 1$, then for each $gh$ in (3) with $gh \neq 1$, there exists another $g'h'$ or $h'$ in (3) such that either $gh = g'h'$ or $gh = h'$ holds. Suppose here that there exist $(g_{2i-1}, h_i)$ and $(g_{2i}, h_{i+1})$ $(i = 1, \cdots, m)$ in $V = \bigcup_{(a,b)\in\Pi} S_a \times T_b$ such that the following equations hold:

$$
\begin{aligned}
g_1h_1 = \;& g_2h_2, \\
& g_3h_2 = \; g_4h_3, \\
& \qquad \ddots \\
& \qquad\qquad g_{2m-1}h_m = g_{2m}h_{m+1} \quad \text{and} \quad h_{m+1} = h_1.
\end{aligned}
\tag{4}
$$




Eliminating $h_i$'s in the above, we can see that (4) above implies the equation $g_1^{-1}g_2\cdots g_{2m-1}^{-1}g_{2m}=1$. If we can choose $\varepsilon(a)$'s so that their supports $g_i$'s never satisfy such an equation, then we can prove that $r\neq 1$ holds by contradiction. We need therefore only to see when supports $g$'s of $\varepsilon(a)$'s satisfy equations as described in (4) provided $r=1$ holds.

In order to see this, we consider a graph which has two distinct edge sets $E$ and $F$ on the same vertex set $V$; an SR-graph $\mathcal{S}=(V,E,F)$. Roughly speaking, we regard $V=\bigcup_{(a,b)\in\Pi}S_a\times T_b$ above as the set of vertices and for $v=(g,h)$ and $w=(g',h')$ in $V$, we take an element $vw$ as an edge in $E$ provided $gh=g'h'$ in $G$, and take $vw$ as an edge in $F$ provided $g\neq g'$ and $h=h'$ in $G$. In this situation, if there exists an SR-cycle $v_1w_1v_2w_2,\cdots,v_pw_pv_1$ in the SR-graph $(V,E,F)$, then there exist $(g_i,h_j)$'s in $V$ satisfying the desired equations as described in (4). Thus the problem can be reduced to find an SR-cycle in a given SR-graph.

In fact, by making use of the method described above, we can show primitivity of group algebras of groups which belong to many classes of non-noetherian groups, including free groups, locally free groups, free products, amalgamated free products, HNN-extensions and one relator groups.

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