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Semigroup rings over semiprime ring semigroups

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1 Introduction

The talk presented was based on an on-going joint work with Y.Hirano (Naruto) and B.Solie (ERAU). More detailed and completed version of this manuscript will be submited elsewhere.

Let R be a ring, and let S be a semigroup. The semigroup ring $R[S]$ simultaneously encodes the semigroup structure of S and the ring structure resulting in an object of great utility in various areas of both ring theory and group theory. The interplay between the structures of R, S, and $R[S]$ is a topic with a long mathematical history full of fascinating results. For instance, much is already known about the structure of $R[G]$, where G is not just a semigroup but a group. Here, we find that the structure of $R[G]$ is exactly characterized by the structure of R and various finiteness properties of G. Maschke's famous theorem states that if G is a finite group and K is a field whose characteristic does not divide $|G|$, then $K[G]$ is semisimple [3]. Further variations include the result that $R[G]$ is prime if and only if R is a prime ring and G has no nontrivial finite normal subgroup [1]. More generally, $R[G]$ is semiprime if and only if R is semiprime and the order of every normal finite subgroup of G is a non-zero-divisor in R [5].

Consequently, one may ask whether the primeness or semiprimeness of $R[S]$ can be characterized in the case where S is an arbitrary semigroup. This is a much more difficult question on which some progress has been made in recent years. For highly structured classes of semigroups, it is again possible to determine the structure of $R[S]$. For example, when S is a cancellative semigroup, then $R[S]$ is semiprime whenever R is semiprime [4]. When F is a field and S is finite, a version of Maschke's theorem shows that the semisimplicity of $R[S]$ is characterized by the nonsingularity of the structure matrix for S when viewed as a matrix over F [2].

We consider semigroup rings over a particular class of semigroups: those semigroups which arise as the multiplicative semigroup of a ring.

2 Main Results and Examples

Let K be a field and let S be a semigroup. Recall that the *semigroup ring* $K[S]$ consists of the set of all sums $\sum_{s\in S} k_s\hat{s}$ with $k_s = 0$ for all but finitely many $s \in S$. We equip $K[S]$ with the usual addition and multiplication, where $\hat{s}\hat{t} = \widehat{st}$ for all $s,t \in S$.

Given a ring $(R,+,\cdot)$, we may forget addition and thereby obtain a semigroup $(R,\cdot)$ having both 0 and 1. We denote by $K[R]$ the semigroup ring $K[(R,\cdot)]$. Note that as $K[R] = K[0] \oplus Ann(K[0])$, $K[R]$ *is* not a prime ring. It is easy to observe that $K[R]$ is a direct sum of two simple rings if and only if every proper nonzero ideal of $K[R]$ is prime. Every prime ideal of $K[R]$ except $Ann(K[0])$ contains $K[0]$, and every ideal of $K[R]$ contained in $Ann(K[0])$ does not contain $K[0]$.

Theorem 1 *Let R be a ring and K be a field of characteristic zero. If $K[R]$ is semiprime, then R is semiprime. The converse holds if R is a commutative ring or a domain.*

Theorem 2 *Let R be a ring and let K be a field of characteristic zero. Then $K[R]$ is a semisimple Artinian ring if and only if R is a finite semisimple ring.*

For a ring R, we shall denote the Jacobson radical of R by $J(R)$.

Theorem 3 *Let R be a local ring with finitely many units, let R^* denote the group of units in R and let K be a field of characteristic zero. Then $J(K[R]) = \sum_{r\in J(R)} K(\hat{r}-\hat{0})$ and $K[R]/J(K[R]) \cong K[0] \oplus K[R^*]$.*

Let K be an algebraically closed field of characteristic zero, and let $\mathbb{Z}_n$ denote the ring of integers modulo n. It is immediate that $\mathbb{Z}_n$ is semiprime if and only if n is squarefree, and thus we have the following proposition as a corollary of Theorem 1.

Proposition *Let K be an algebraically closed field of characteristic zero, and let $\mathbb{Z}_n$ denote the ring of integers modulo n. Then $K[\mathbb{Z}_n]$ is semiprime if and only if n is squarefree.*

We now present a few examples of the structure of $K[R]$ for some finite ring R.

Example 1 *It is clear that* $K[\mathbb{Z}_2] \cong K \oplus K$.

Let V be a vector space over K. We define a multiplication on the K-linear space $K \oplus V$ by the formula $(a, v)\cdot(b, w) = (ab, aw + bv)$ for any $a, b \in K$, $v, w \in V$. Then $K \oplus V$ becomes a K-algebra, which we denote by $K \ltimes V$.

Example 2 *Consider the ring $K[\mathbb{Z}_4]$ and let $g_i = \hat{i} - \hat{0}$ for $i = 1,2,3$. Then $K[\mathbb{Z}_4]$ is the direct sum of two sided ideals $K[0]$ and $S = Kg_1 + Kg_2 + Kg_3$. The identity of the ring S is g_1. Let us set $e_1 = \frac{1}{2}(g_1 - g_3)$, $e_2 = \frac{1}{2}(g_1 + g_3)$. Then e_1, e_2 are orthogonal central primitive idempotents of S and $g_1 = e_1 + e_2$.* We can easily see that $K[\mathbb{Z}_4] \cong K^2 \oplus (K \ltimes K)$.

Example 3 *Consider the ring $K[\mathbb{Z}_6]$ and let $g_i = \hat{i} - \hat{0}$ for $i = 1,2,\cdots,5$. Then $K[\mathbb{Z}_6]$ is the direct sum of $K[0]$ and $S = Kg_1 + Kg_2 + \cdots + Kg_5$. Set $e_1 = \frac{1}{2}(g_1 + g_5)$ and $e_2 = \frac{1}{2}(g_1 - g_5)$. We again have orthogonal central primitive idempotents e_1 and e_2 in S and $g_1 = e_1 + e_2$, and moreover $e_1S \cong K^3$ and $e_2S \cong K^2$. Thus we have that $K[\mathbb{Z}_6] \cong K^6$.*

Example 4 *Consider the ring $K[\mathbb{Z}_8]$ and let $g_i = \hat{i} - \hat{0}$ for $i = 1,2,\cdots,7$. Then $K[\mathbb{Z}_8]$ is the direct sum of two sided ideals $K[\hat{0}]$ and $S = Kg_1 + Kg_2 + \cdots + Kg_7$. The identity of the ring S is g_1. Let us set*

$$e_1 = \frac{1}{4}(g_1 - g_3 - g_5 + g_7)$$
$$e_2 = \frac{1}{4}(g_1 + g_3 - g_5 - g_7)$$
$$e_3 = \frac{1}{4}(g_1 - g_3 + g_5 - g_7)$$
$$e_4 = \frac{1}{4}(g_1 + g_3 + g_5 + g_7).$$

Then e_1, e_2, e_3, e_4 are orthogonal central primitive idempotents of S and $g_1 = e_1 + e_2 + e_3 + e_4$. We can easily see $e_1S \cong K$, $e_2S \cong K$, $e_3S \cong K \ltimes K$, and $e_4S \cong K \ltimes (K \oplus K)$. Therefore we have that $K[\mathbb{Z}_8] \cong K^3 \oplus (K \ltimes K) \oplus (K \ltimes (K \oplus K))$.

Example 5 *Consider the ring* $R=\left\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a,b,c,d \in GF(2)\right\}$ *of order* $|R|=2^4=16$.

Then $R^*=\left\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mid a\neq 0, b,c,d \in GF(2)\right\}$. We can easily see that R^* is the dihedral group D_8 of order 8 and so $K[R^*]\cong K^4 \oplus M_2(K)$. Therefore, by Theorem 3, we have that $K[R]/J(K[R])\cong K^5 \oplus M_2(K)$.

Example 6 *Let* $M_2(GF(2))$ *denote the ring of* 2×2 *matrices over the field* $GF(2)$. *Then we can prove that* $Z[M_2(GF(2))]$ *is a semiprime ring. Let us set* $H=M_2(GF(2))-GL_2(GF(2))$. *Then we can see that* $Q[H]\cong Q\oplus M_3(Q)$. *In fact, let*

$$O=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},\quad e_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\quad e_2=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\quad e_3=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$e_4=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\quad e_5=\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},\quad e_6=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},\quad e_7=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

$$e_8=\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},\quad e_9=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then the elements $E=\hat{O},\ F_1=\hat{e}_1-\hat{O},\ F_2=\hat{e}_2-\hat{O},$

$$F_3=(\hat{e}_1-\hat{O})+(\hat{e}_2-\hat{O})+(\hat{e}_3-\hat{O})+(\hat{e}_4-\hat{O})-(\hat{e}_5-\hat{O})$$
$$-(\hat{e}_6-\hat{O})-(\hat{e}_7-\hat{O})-(\hat{e}_8-\hat{O})+(\hat{e}_9-\hat{O})$$

are primitive orthogonal idempotents, and $Q[H]=QE\oplus Q[H](F_1+F_2+F_3)\cong Q\oplus M_3(Q)$.
Since $GL_2(GF(2))\cong S_3$, we have $Q[M_2(GF(2))]/Q[H]\cong Q[S_3]$.
It is easily see that $Q[S_3]\cong Q\oplus Q\oplus M_2(Q)$. Hence $Q[M_2(GF(2))]$ is isomorphic to the semisimple Artinian ring $Q\oplus Q\oplus Q\oplus M_2(Q)\oplus M_3(Q)$.

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