数理解析研究所講究録
第2052巻 2017年 48-55

The double covering method for twisted knots

Naoko Kamada
Nagoya City University

1 Introduction

In [10] virtual knot theory was introduced by L. Kauffman. It is a generalization of knot theory based on Gauss diagrams and link diagrams in closed oriented surfaces. Virtual links correspond to stable equivalence classes of links in thickened orientable surfaces [2, 8]. Twisted knot theory was introduced by M. Bourgoin [1]. It is an extension of virtual knot theory. Twisted links correspond to stable equivalence classes of links in oriented 3-manifolds which are line bundles over (possibly non-orientable) closed surfaces. We construct a double covering diagram of a twisted link diagram by taking the orientation double covering of the surface on which the diagram is realized.

A virtual link diagram is called *normal* if the associated abstract link diagram is checkerboard colorable (§ 3). A virtual link is called *normal* if it has a normal diagram as a representative. Every classical link diagram is normal, and hence the set of classical link diagrams is a subset of that of normal virtual link diagrams. The set of normal virtulal link diagrams is a subset of that of virtual link diagrams. The f-polynomial (Jones polynomial) is an invariant of a virtual link [10]. It is shown in [4] that the f-polynomial of a normal virtual link has a property that the f-polynomial of a classical link has. This property may make it easier to define Khovanov homology of virtual links as stated in O. Viro [13].

In this paper, we discuss a double covering diagram of a twisted link diagram [9]. We introduce a method of converting a virtual link diagram to a normal virtual link diagram by use of the double covering technique [7]. We show some applications of our method.

Acknowledgement:
This work was supported by JSPS KAKENHI Grant Number 15K04879.

2 Double covering diagram of a twisted link diagram

A *virtual link diagram* is a generically immersed, closed and oriented 1-manifold in $\mathbb{R}^2$ with information of positive, negative or virtual crossing, on its double points. A *virtual crossing* is an encircled double point without over-under information. A *twisted link diagram* is a virtual link diagram, possibly with *bars* on arcs. A *virtual link* (or *twisted link*) is an equivalence class of virtual (or twisted) link diagrams under Reidemeister moves and virtual Reidemeister moves (or Reidemeister moves, virtual Reidemeister moves and twisted Reidemeister moves) depicted in Figures 1. We call Reidemeister moves and virtual Reidemeister moves *generalized Reidemeister moves.*

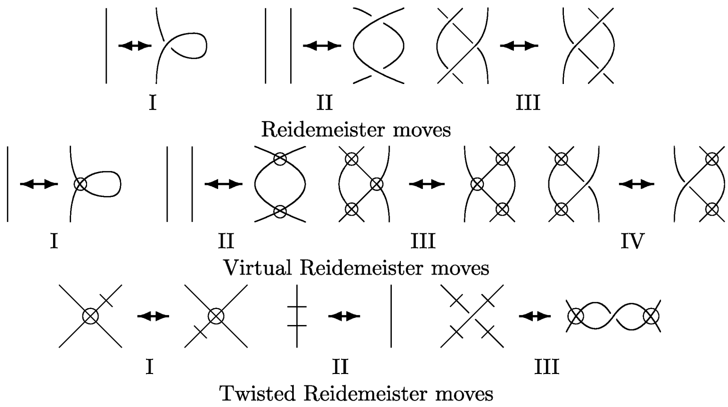

Figure 1: Generalized Reidemeister moves and twisted Reidemeister moves

An *abstract link diagram* (*ALD*) is a pair of a compact surface Σ and a link diagram D on Σ such that the underlying 4-valent graph $|D|$ is a deformation retract of Σ, denoted by (Σ, D_Σ). Two ALDs (Σ_1, D_1) and (Σ_2, D_2) are equivalent if there exist a closed surface F and embeddings $g_i : \Sigma_i \to F$ $(i = 1, 2)$ such that $g_1(D_1)$ is equivalent to $g_2(D_2)$ under Reidemeister moves I, II, and III on F. An *abstract link* is an equivalence class of abstract link diagrams. Refer to [8].

We obtain an ALD from a twisted link diagram D as in Figure 2. Such an ALD is called the *ALD associated with* D. Figure 3 shows twisted link diagrams and the ALDs associated with them.

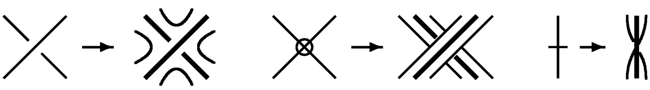

Figure 2: The correspondence from a twisted link diagrams to an ALD

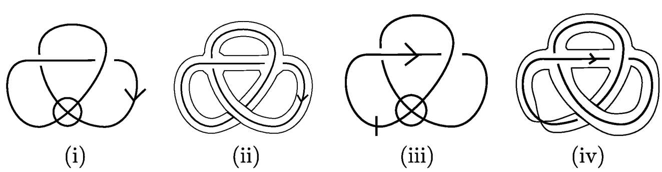

Figure 3: Twisted link diagrams and ALDs

Theorem 2.1 (cf.[1, 8, 9]) *Let D and D' be twisted link diagrams. Two ALDs which associated with D and D' are equivalent if and only if D and D' are equivalent.*

Let K be a twisted link and D be a twisted link diagram of K. Let (Σ, D_Σ) be the abstract link diagram associated with D, and let $(\widetilde{\Sigma}, \widetilde{D_\Sigma})$ be the orientation double covering of (Σ, D_Σ). Note that $\widetilde{\Sigma}$ is orientable. See Figure 4.

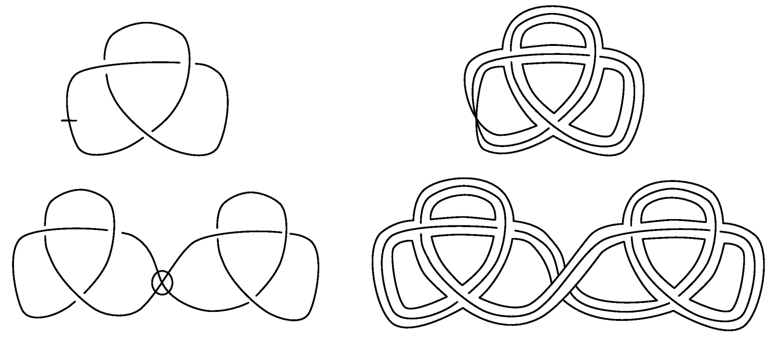

Figure 4: Double covering of an ALD

Theorem 2.2 ([9]) *The double covering of a twisted link is well defined. Namely, for two equivalent twisted link diagrams D and D', two double covering of ALDs associated with D and D' are equivalent as an abstract link.*

We show a construction of the double covering diagram of a twisted link diagram [9]. Let D be a twisted link diagram. Assume that D is on the right of the y-axis in the xy-plane and all bars are parallel to the x-axis with disjoint y-coordinates. Let D^* be the twisted link diagram obtained from D by reflection with respect to the y-axis and switching the over-under information of all classical crossings of D. Let $B = \{b_1, \dots, b_k\}$ be a set of bars of D and for $i \in \{1, \dots, k\}$, we denote by b_i^* the bar of D^* corresponding to b_i. See Figure 5 (i). For horizontal lines $l_1, \dots, l_k$ such that l_i contains b_i and the

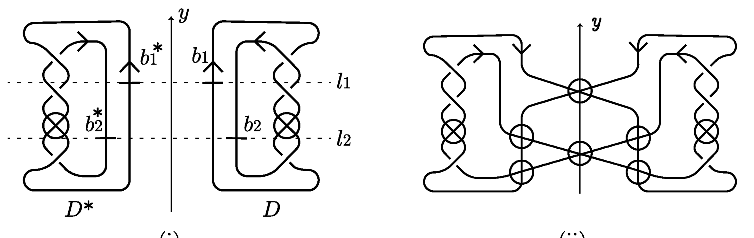

Figure 5: The double covering of a twisted link diagram

corresponding bar b_i^* of D^*, we replace each part of $D \amalg D^*$ in a neighborhood of $N(l_i)$ for each $i \in \{1, \dots, k\}$ as in Figure 6. We denote by $\psi(D)$ the virtual link diagram obtained this way.

For example, for the twisted link diagram D depicted as in Figure 5 (i), the virtual link diagram $\psi(D)$ is as in Figure 5 (ii).

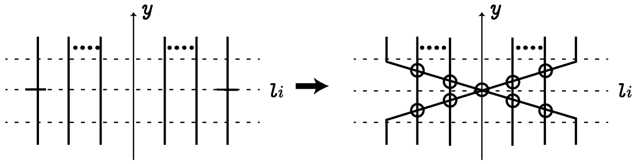

Figure 6: The replacement of diagram

We call this diagram $\psi(D)$ the *double covering diagram* of D. Then we have the followings.

Theorem 2.3 (**[9]**) *Let D and D' be twisted link diagrams. If D and D' are equivalent as a twiated link, then $\psi(D)$ and $\psi(D')$ are equivalent as a virtual link.*

As a consequence of Theorem 2.3 we have the following.

Corollary 2.4 *For a virtual knot invariant X, $X \circ \psi$ is an invariant of twisted links.*

Bourgoin introduced the twisted knot group in [1], which is a virtual knot group of a double covering diagram of a twisted link diagram.

Theorem 2.5 (**[9]**) *Let D be a twisted link diagram. Let $\widetilde{L}$ be a link in $\Sigma \times [-1, 1]$ presented by the abstract link of the double covering of D. Then a twisted knot group coincides with $\pi_1(\Sigma \times [-1, 1] - \widetilde{L}/\Sigma \times \{1\})$ (or $\pi_1(\Sigma \times [-1, 1] - \widetilde{L}/\Sigma \times \{-1\})$).*

In [5], the author introduced a twisted quandle. A fundamental twisted quandle of a twisted link diagram D coincides with a fundamental quandle of the double covering diagram of D. The JKSS invariant is an invariant of virtual links [12]. The doubled JKSS invariant is an invariant of twisted links [6], which coincides with the JKSS invariant [3, 12] of the double covering diagram of a twisted link diagram.

3 Normal virtual links

The diagram D is said to be *normal* or *checkerboard colorable* if the regions of $\Sigma - |D_\Sigma|$ can be colored black and white such that colors of two adjacent regions are different. In Figure 7, we show an example of a normal diagram. A classical link diagram is normal. A twisted link is said to be *normal* if it has a normal twisted link diagram. Note that normality is not necessary to be preserved under generalized Reidemeister moves. For example the virtual link diagram in the right of Figure 8 is not normal and is equivalent to the trefoil knot diagram in the left which is normal. However we have the following.

Theorem 3.1 (**[13]**) *Let D and D' two normal virtual link diagrams. If they are equivalent, there is a sequence of normal virtual link diagrams $D_0 = D, D_1, \ldots D_{n-1}, D_n = D'$ such that D_i and D_{i+1} are related with one of generalized Reidemeister moves.*

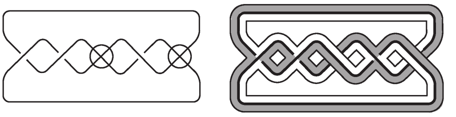

Figure 7: A normal twisted link diagram and its associated ALD with a checkerboard coloring

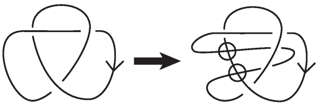

Figure 8: A diagram of a normal virtual link which is not normal

Proposition 3.2 ([7]) *For a normal twisted link diagram D, the double covering diagram of D $\psi(D)$ is normal.*

H. Dye introduced the notion of cut points to a virtual link diagram in her talk presented in the Special Session 35, "Low Dimensional Topology and Its Relationships with Physics", held in Porto, Portugal, June 10-13, 2015 as part of the 1st AMS/EMS/SPM Meeting.

Let (D, P) be a pair of a virtual link diagram D and a finite set P of points on edges of D. We call the ALD associated with the twisted link diagram which is obtained from (D, P) by replacing all points of P with bars, the *ALD associated with* (D, P). See Figure 9 (ii) and (iii). If the ALD associated with (D, P) is normal, then we call the set of points P a *cut system* of D and call each point of P a *cut point*. For the virtual link diagram in Figure 9 (i) we show an example of a cut system in Figure 9 (ii) and the ALD associated with it in Figure 9 (iii).

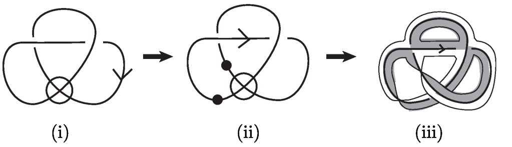

Figure 9: Example of cut points

A cut system of a virtual link diagram is not unique. Dye introduced the cut point moves depicted in Figure 10.

The author showed the following.

Theorem 3.3 ([7]) *For a virtual link diagram D, two cut systems of D are related by a sequence of cut point moves I, II and III.*

4 Conversion to a normal virtual link diagram

Let (D, P) be a virtual link diagram D with a cut system P. We replace all cut points of P with bars. Then we obtain a twisted link diagram, denoted by $t(D, P)$. We put

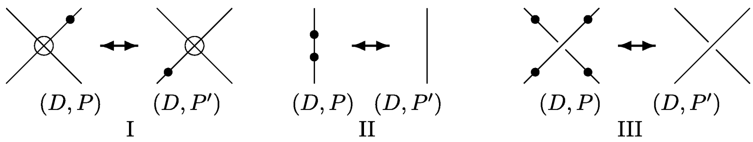

Figure 10: The cut point moves

$\phi = \psi \circ t$. Then we have

$$\phi : \{\text{virtual link diagrams with a cut system}\} \longrightarrow \{\text{virtual link diagrams}\}.$$

By Proposition 3.2, the diagram $\phi(D, P)$ is a normal virtual link diagram. We call it the *converted normal diagram* of (D, P). Thus we have

$$\phi(\{\text{virtual link diagrams with cut points}\}) \subset \{\text{normal virtual link diagrams}\}.$$

The local replacement of a virtual link diagram depicted in Figure 11 is called a *Kauffman flype* or a *K-flype*. If a virtual link diagram D' is obtained from D by a finite sequence of generalized Reidemeister moves and K-flypes, then they are said to be *K-equivalent*. For a virtual link diagram of D, if a virtual link diagram D' is obtained from D by a

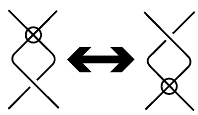

Figure 11: Kauffman flype

K-flype at a classical crossing c, then the sign of the corresponding classical crossing c' of D' is the same as that of c. If D is normal, then D' is normal.

We have the following theorem.

Theorem 4.1 **([7])** *Let (D, P) and (D', P') be virtual link diagrams with cut systems. If D and D' are equivalent (or K-equivalent), then the converted normal diagrams $\phi(D, P)$ and $\phi(D', P')$ are K-equivalent.*

The following lemma is a key lemma of the proof of Theorem 4.1.

Lemma 4.2 *Let D be a virtual link diagram. Suppose that P and P' are cut systems of D. Then the converted normal diagrams $\phi(D, P)$ and $\phi(D, P')$ are K-equivalent.*

5 Applications

In this section we give applications of our method.

For a 2-component virtual link diagram D, the half of the sum of signs of non-self classical crossings of D is said to be the *linking number* of D.

Proposition 5.1 **([7])** *The linking number is invariant under the generalized Reidemeister moves and K-flypes.*

We have the following theorems (Theorems 5.2 and 5.3).

Theorem 5.2 **([7])** *Let (D, P) be a virtual knot diagram with a cut system. Then $\phi(D, P)$ is a 2-component virtual link diagram and the linking number of $\phi(D, P)$ is an invariant of the virtual knot represented by D.*

The odd writhe is a numerical invariant of virtual knots [11]. A classical crossing c of D is said to be *odd* if there is an odd number of classical crossings of an arc of D whose two endpoints are c. The *odd writhe* of D is the sum of signs of odd crossings of D.

Theorem 5.3 **([7])** *Let (D, P) be a virtual knot diagram with a cut system. The linking number of $\phi(D, P)$ is equal to the odd writhe of D.*

H. Miyazawa, K. Wada and A. Yasuhara introduced the notion of an even virtual link. Let D be a virtual link diagram. If there is an even number of endpoints of chords on each circle of Gauss diagram of D, D is said to be *even*. If D and D' are equivalent and D is even, D' is even. A virtual link diagram of normal virtual link is even. A virtual knot is even.

Proposition 5.4 *Let D be an even n-component virtual link diagram D. For a cut system P of D, The double covering diagram $\phi(D, P)$ is a $2n$-component normal virtual link diagram, where each component of D corresponds to 2-component link diagram in $\phi(D, P)$.*

Let D be an n-component ordered virtual link diagram, where D_i is a i-th component of $D (i = 1, \ldots, n)$. For a cut system of D, P, we denote a 2-component of the converting diagram $\phi(D, P)$ which corresponds to D_i of D by $\widetilde{D}_i$ where each component of $\widetilde{D}_i$ is $\widetilde{D}_i^1$ or $\widetilde{D}_i^2$. Let $\tilde{\mathrm{lk}}(D_i^k, D_j^l)$ be the linking number between $\widetilde{D}_i^k$ and $\widetilde{D}_j^l$ for $k, l = 1, 2$.

The set $\{\tilde{\mathrm{lk}}(D_i^1, D_j^1), \tilde{\mathrm{lk}}(D_i^2, D_j^2), \tilde{\mathrm{lk}}(D_i^1, D_j^2), \tilde{\mathrm{lk}}(D_i^2, D_j^1)\}$ is denoted by $Q_{ij}(D) (i \neq j)$.

Corollary 5.5 *The set $Q_{ij}(D)$ is an invariant of an ordered even virtual link. The linking number $\tilde{lk}(D_i^1, D_i^2)$ is an invariant of an ordered even virtual link.*

References

[1] M. O. Bourgoin, *Twisted link theory*, Algebr. Geom. Topol. 8 (2008) 1249–1279.

[2] J. S. Carter, S. Kamada and M. Saito, *Stable equivalence of knots on surfaces and virtual knot cobordisms*, J. Knot Theory Ramifications 11 (2002) 311–322.

[3] F. Jaeger, L. H. Kauffman, H. Saleur, *The Conway polynomial in R^3 and in thickened surfaces: a new determinant formulation*, Combin. Theory Ser. B 61 (1994), 237–259.

[4] N. Kamada, *On the Jones polynomials of checkerboard colorable virtual knots*, Osaka J Math., 39 (2002), 325–333.

[5] N. Kamada, *Polynomial invariants and quandles of twisted links*, Topology and its Application, 159 (2012) 999–1006.

[6] N. Kamada, *A twisted link invariant derived from a virtual link invariant*, Journal of Knot Theory and its Ramifications, to appear, arXiv:1512.01053.

[7] N. Kamada, Converting virtual link diagrams to normal ones, Topology and its Application, to appear, arXiv:1606.00667.

[8] N. Kamada and S. Kamada, *Abstract link diagrams and virtual knots*, J. Knot Theory Ramifications 9 (2000) 93-106.

[9] N. Kamada and S. Kamada, *Double coverings of twisted links* J. Knot Theory Ramifications 25(2016) 1641011, 22 pages.

[10] L. H. Kauffman, *Virtual knot theory*, European J. Combin. 20 (1999) 663–690.

[11] L. H. Kauffman, *A self-linking invariant of virtual knots*. Fund. Math., 184 (2004), 135–158.

[12] J. Sawollek, *On Alexander-Conway polynomials for virtual knots and links*, preprint (1999, arXiv:9912173(math.GT)).

[13] O. Viro, *Virtual Links, Orientations of Chord Diagrams and Khovanov Homology*, Proceedings of 12th Gökova Geometry-Topology Conference, (2007) 184–209.

Graduate School of Natural Sciences
Nagoya City University
1 Yamanohata, Mizuho-cho, Mizuho-ku, Nagoya, Aichi 467-8501
JAPAN
E-mail address: `kamada@nsc.nagoya-cu.ac.jp`

名古屋市立大学大学院システム自然科学研究科　鎌田　直子