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Simple-ribbon fusions and Alexander polynomials

Tsuneo Ishikawa, Kengo Kishimoto, Tetsuo Shibuya and Tatsuya Tsukamoto
Osaka Institute of Technology

1 Introduction

All knots and links are assumed to be ordered and oriented, and they are considered up to ambient isotopy in the oriented 3-sphere S^3.

A (m-)*ribbon fusion* on a link L is an m-fusion on L and an m-component trivial link $\mathcal{O}$ which is disjoint from L and each of whose component is attached by a unique band to L. Note that any ribbon link can be obtained from the trivial link by a ribbon fusion.

An m-ribbon fusion is called a (m-)*simple-ribbon fusion* (or an SR-fusion) if $\mathcal{O}$ bounds m mutually disjoint disks $\mathcal{D}$ which are split from L such that each disk of $\mathcal{D}$ intersects one of the bands $\mathcal{B}$ for the ribbon fusion exactly once and each band of $\mathcal{B}$ intersects one disk of $\mathcal{D}$ exactly once [3].

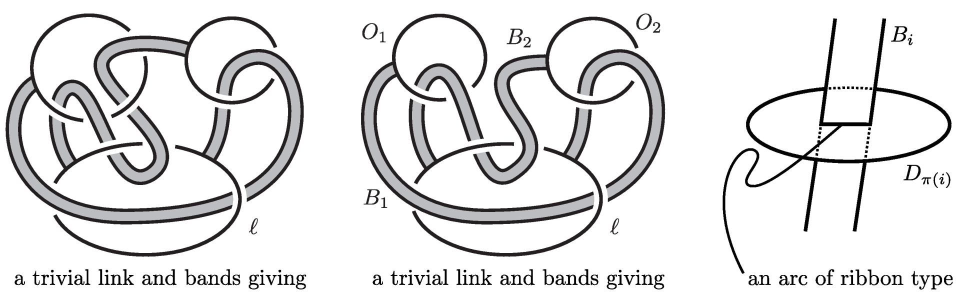
The following is the precise definition of the simple-ribbon fusion. Let L be a link and $\mathcal{O} = O_1 \cup \cdots \cup O_m$ the m-component trivial link which is split from L. Let $\mathcal{D} = D_1 \cup \cdots \cup D_m$ be a disjoint union of non-singular disks with $\partial D_i = O_i$ and $D_i \cap L = \emptyset$ $(i = 1, \cdots, m)$, and let $\mathcal{B} = B_1 \cup \cdots \cup B_m$ be a disjoint union of disks, called *bands*, for an m-fusion of L and $\mathcal{O}$ satisfying the following:

(i) $B_i \cap L = \partial B_i \cap L = \{$ a single arc $\}$;

(ii) $B_i \cap \mathcal{O} = \partial B_i \cap O_i = \{$ a single arc $\}$; and

(iii) $B_i \cap \operatorname{int} \mathcal{D} = B_i \cap \operatorname{int} D_{\pi(i)} = \{$ a single arc of ribbon type $\}$, where π is a certain permutation on $\{1, 2, \ldots, m\}$.

Let L' be a link obtained from a link L and $\mathcal{O}$ by the m-fusion along $\mathcal{B}$, i.e., $L' = (L \cup \mathcal{O} \cup \partial\mathcal{B})$ $-\text{int}(\mathcal{B} \cap L) - \text{int}(\mathcal{B} \cap \mathcal{O})$. Then we say that L' is obtained from L by a *simple-ribbon fusion* or an SR-fusion (*with respect to* $\mathcal{D} \cup \mathcal{B}$). If there exists a 3-ball X such that int$X$ contains $\mathcal{D}$ and each band of $\mathcal{B}$ intersects with ∂X in an arc (and thus $X \cap L = \emptyset$), then we call the SR-fusion an SR-*move* ([5], [6]).



a trivial link and bands giving a ribbon fusion on l

a trivial link and bands giving a simple-ribbon fusion on l

an arc of ribbon type

Figure 1:

Since every permutation is a product of cyclic permutations, we can rename the indices of the components of $\mathcal{O}$, $\mathcal{D}$, and $\mathcal{B}$ as

$$\mathcal{O} = \mathcal{O}^1 \cup \cdots \cup \mathcal{O}^n = (O_1^1 \cup \cdots \cup O_{m_1}^1) \cup \cdots \cup (O_1^n \cup \cdots \cup O_{m_n}^n),$$
$$\mathcal{D} = \mathcal{D}^1 \cup \cdots \cup \mathcal{D}^n = (D_1^1 \cup \cdots \cup D_{m_1}^1) \cup \cdots \cup (D_1^n \cup \cdots \cup D_{m_n}^n), \text{ and}$$
$$\mathcal{B} = \mathcal{B}^1 \cup \cdots \cup \mathcal{B}^n = (B_1^1 \cup \cdots \cup B_{m_1}^1) \cup \cdots \cup (B_1^n \cup \cdots \cup B_{m_n}^n), \text{ where}$$

$\partial D_i^k = O_i^k$, $B_i^k \cap \mathcal{O} = \partial B_i^k \cap O_i^k$, and $B_i^k \cap \operatorname{int} \mathcal{D} = B_i^k \cap \operatorname{int} D_{i+1}^k$ for any k $(1 \leq k \leq n)$. We consider the lower index modulo m_k. We call each $\mathcal{D}^k \cup \mathcal{B}^k$ the (k-th) *elementary component* of the SR-fusion, and m_k the *type* of the elementary component. The *type* of the SR-fusion is the ordered set $(m_1, m_2, \ldots, m_n)$. If $n = 1$, then we simply write $m = m_1$ instead of (m_1) and call the SR-fusion an *elementary SR-fusion.* If $m_k = 1$ (resp. $m_k \geq 2$) for any k, then we say that the SR-fusion is in class I (resp. class II).

In this paper, we survey some results about SR-fusions and genera, primeness and Alexander polynomials.

2 Simple ribbon fusions and genera

The *genus* of an oriented surface is the sum of genera of its connected components. A *Seifert surface* E for a link ℓ is a compact non-singular oriented surface in S^3 with no closed components such that $\partial E = \ell$. The *genus* $g(\ell)$ of a link ℓ is the minimal number of genera of all the Seifert surfaces for ℓ. The *disconnectivity number of* ℓ, denoted by $\nu(\ell)$, is the maximal number of connected components of all the Seifert surfaces for ℓ ([1]). For each integer r $(1 \leq r \leq \nu(\ell))$, the *$r$-th genus* of ℓ, denoted by $g_r(\ell)$, is the minimal number of genera of all the Seifert surfaces for ℓ with r connected components.

Note that there exists a Seifert surface E for ℓ with $\sharp(E) = r$ for each integer r $(1 \leq r \leq \nu(\ell))$, where $\sharp(E)$ is the number of the connected components of E. From the definition, we see that $g_1(\ell)$ coincides with the genus of ℓ, that $1 \leq \nu(\ell) \leq \sharp(\ell)$, and that $0 \leq g(\ell) = g_1(\ell) \leq g_2(\ell) \leq \cdots \leq g_{\nu(\ell)}(\ell)$, where $\sharp(\ell)$ is the number of components of ℓ. For the n-component trivial link $\mathcal{O}$, we have that $\nu(\mathcal{O}) = n$ and that $g_r(\mathcal{O}) = 0$ for each integer r $(1 \leq r \leq n)$.

An SR-fusion is *trivial* if $\mathcal{O}$ bounds mutually disjoint non-singular disks $\cup_i \Delta_i$ such that $\partial\Delta_i = O_i$ and intΔ_i does not intersect with $L \cup \mathcal{B}$ for each i $(1 \leq i \leq m)$. Here note that $\cup_i \Delta_i$ may intersect with int$\mathcal{D}$ (see Figure 2 for example). Since L is ambient isotopic to ℓ through $(\cup_i \Delta_i) \cup \mathcal{B}$, we know that a trivial SR-fusion does not change the link type.

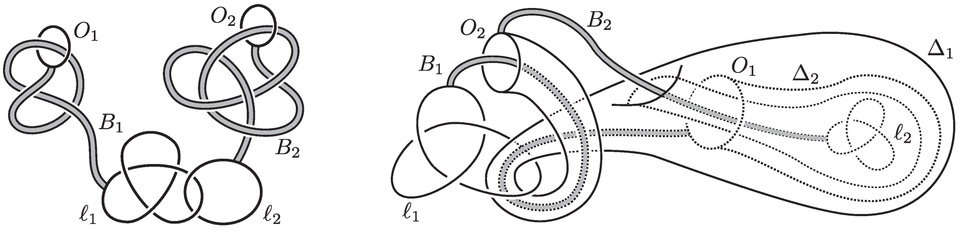


Figure 2:

We showed the following in [3].

Theorem 2.1. *Let L be a link obtained from a link ℓ by an SR-fusion. Then we have that $\nu(L) \leq \nu(\ell)$ and that $g_r(L) \geq g_r(\ell)$ for each integer r $(1 \leq r \leq \nu(L))$. Moreover, the following three conditions are equivalent :*

(1) the SR-fusion is trivial ;

(2) L is ambient isotopic to ℓ ; and

(3) $\nu(L) = \nu(\ell)$ and $g_{\nu(L)}(L) = g_{\nu(\ell)}(\ell)$.

Let $\dot{D}_i^k$ and $\dot{B}_i^k$ be disks and $f : \cup_{i,k} \left(\dot{D}_i^k \cup \dot{B}_i^k\right) \to S^3$ an immersion such that $f(\dot{D}_i^k) = D_i^k$ and $f(\dot{B}_i^k) = B_i^k$. In the following, we omit the upper index k unless it causes confusion.

Take an elementary component $\mathcal{D}^k \cup \mathcal{B}^k$. Denote the arc of int$D_i \cap B_{i-1}$ by α_i and let $B_{i,1}$ and $B_{i,2}$ be the subdisks of B_i such that $B_{i,1} \cup B_{i,2} = B_i$, $B_{i,1} \cap B_{i,2} = \alpha_{i+1}$, and $B_{i,1} \cap \partial D_i \neq \emptyset$.

Moreover, we denote the pre-images of α_i on $\dot{D}_i$ and $\dot{B}_{i-1}$ by $\dot{\alpha}_i$ and $\ddot{\alpha}_i$, respectively.

Take a point b_i on intα_i $(i = 1, \ldots, m_k)$ and an arc β_i on $D_i \cup B_{i,1}$ so that $\beta_i \cap (\alpha_i \cup \alpha_{i+1}) = \partial\beta_i = b_i \cup b_{i+1}$ (see Figure 3). Then $\beta^k = \cup_i \beta_i$ is a simple loop and we call $\mathcal{L} = \cup_k \beta^k$ an *attendant link* of the SR-fusion. We also call each β^k an (k-th) *component* of $\mathcal{L}$ and m_k

the *type* of β^k. Moreover, we denote the pre-images of α_i (resp. b_i) on $\dot{D}_i$ and $\dot{B}_{i-1}$ by $\dot{\alpha}_i$ and $\ddot{\alpha}_i$ (resp. $\dot{b}_i$ and $\ddot{b}_i$), respectively.

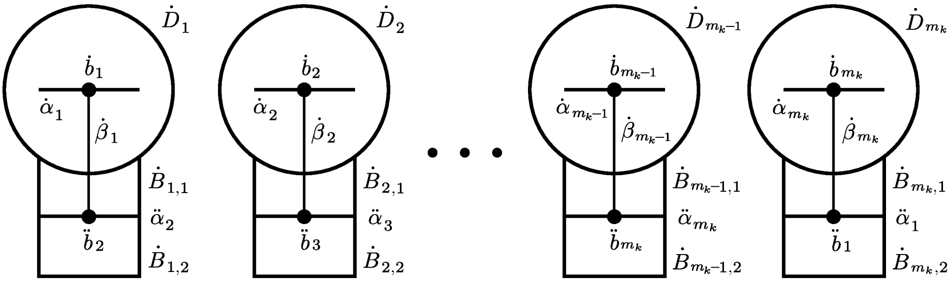
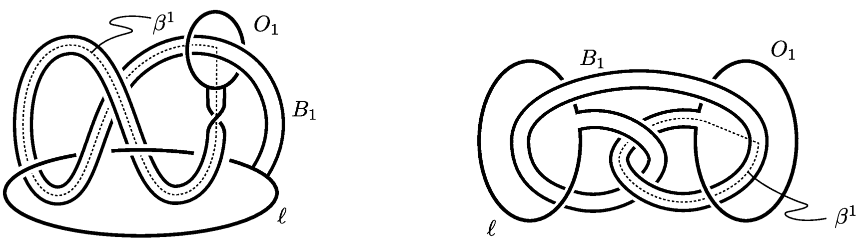


Figure 3:

Let L be a link obtained from a non-split link ℓ by an SR-fusion with an attendant link $\mathcal{L}$. We divide $\mathcal{L}$ into three classes $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$; $\mathcal{L}_1 = \beta^1 \cup \cdots \cup \beta^s$ such that each b^k has type $m_k \geq 2$, $\mathcal{L}_2 = \beta^{s+1} \cup \cdots \cup \beta^{s+t}$ such that each b^k has type $m_k = 1$ and is non-split from ℓ, and $\mathcal{L}_3 = \beta^{s+t+1} \cup \cdots \cup \beta^{s+t+u(=n)}$ such that each b^k has type $m_k = 1$ and is split from ℓ (here we rename the index for the components if necessary).



$\mathcal{L} \cup \ell$ is non-split $\qquad$ $\mathcal{L} \cup \ell$ is split

Figure 4:

Then we have the following, where note that if ℓ is a knot, then $\nu(L) = \nu(\ell) = 1$, and thus $g_{\nu(L)}(L) = g(L)$ and $g_{\nu(\ell)} = g(\ell)$.

Theorem 2.2. *Let L be a link obtained from a non-split link ℓ by an SR-fusion with an attendant link $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$. If $\nu(L) = \nu(\ell)$, then we have that*

$$g_{\nu(L)}(L) \geq g_{\nu(\ell)}(\ell) + \sum_{k=1}^{s} \left[\frac{m_k+1}{2}\right] + t,$$

where $[x]$ is the greatest integer not greater than x.

3 Simple ribbon fusions and primeness of knots

A *decomposing sphere* Σ for a knot K is a 2-sphere in S^3 which intersects with K at exactly two points. Then K is decomposed into two knots K_1 and K_2 by Σ, where we note that K_1 and K_2 may be trivial. A decomposing sphere Σ for K is *non-trivial* if K_1 and K_2 are non-trivial. A knot K is *composite* if there is a non-trivial decomposing sphere of K. Otherwise it is called *prime*.

An SR-fusion is *reducible* if there exists a trivial elementary component. Otherwise, we say that the SR-fusion is *irreducible*.

We say that the SR-fusion is *decomposable* if there exists a union of elementary components $\mathcal{D}' \cup \mathcal{B}'$ of the SR-fusion with respect to $\mathcal{D} \cup \mathcal{B}$ and a non-trivial decomposing sphere for K' bounding a 3-ball B^3 containing $\mathcal{D}' \cup \mathcal{B}'$ such that $B^3 \cap K$ is a trivial arc. Otherwise it is called *indecomposable*.

We give some sufficient conditions for the primeness of the knot obtained by an SR-fusion in [4].

Theorem 3.1. *Let K be a knot obtained from a prime knot k by an indecomposable SR-fusion. Then K' is prime.*

Theorem 3.2. *Let K be a non-trivial knot obtained from a trivial knot O by an indecomposable SR-fusion. If K is neither the square knot nor the connected sum of two figure-eight knots, then K is prime.*

Remark 3.3. Figure 5 shows the irreducible and indecomposable SR-fusions on the trivial knot O such that K is the square knot and the connected sum of two figure-eight knots, respectively. We note that the SR-fusion in the center is a simple-ribbon move [6].

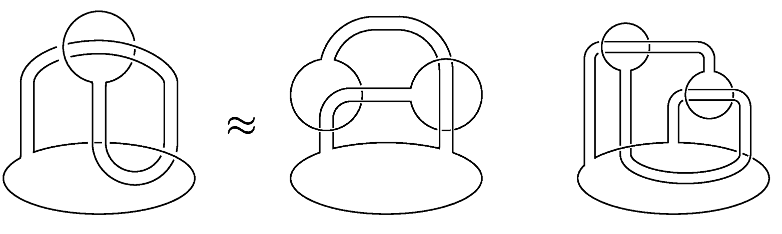


Figure 5: The square knot and the connected sum of two figure-eight knots.

4 Simple ribbon fusions and Alexander polynomials of knots

Let K be a knot obtained from a knot k by an elementary SR-fusion with respect to $\mathcal{D} \cup \mathcal{B}$. We call a disk D_i is *positive* (or *negative*) if D_i intersects B_{i-1} as in Figure 6.

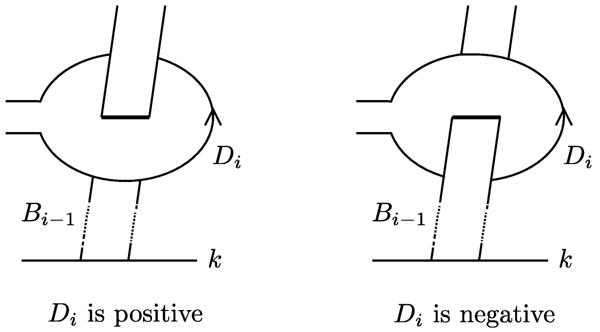


Figure 6:

Then we obtain the following result [2].

Theorem 4.1. *Let K be a knot obtained from a knot k by an elementary SR-fusion of type m with an attendant knot $\mathcal{L}$. Then*

$$\Delta_K(t) = f(t)f(t^{-1})\Delta_k(t),$$

where $f(t) = (1-t)^m - t^{lk(\mathcal{L},k)}(-t)^p$, and p is the number of positive disks.

A *simple ribbon knot* is a knot obtained from a trivial knot by SR-fusions. For example, all knots with up to 9 crossings are simple-ribbon knots. By definition, a simple-ribbon knot is ribbon. But the converse does not hold as follows.

Example 4.2. We show that ribbon knots 10_{123} and $5_2\#5_2$ are not simple-ribbon. By Theorem 4.1, for a simple-ribbon knot K, $\Delta_K(-1) = \prod_i(2^{m_i} + \varepsilon_i)$ for positive integers m_i and $\varepsilon_i = \pm 1$. Since $\Delta_{10_{123}}(-1) = 11^2$, 10_{123} is not simple-ribbon.

We assume that $5_2\#5_2$ is simple-ribbon. By Theorem 2.2 and Theorem 4.1, $5_2\#5_2$ should be obtained from a trivial knot by an elementary SR-fusion of type 3, because $g(5_2\#5_2) = 2$ and $\Delta_{5_2\#5_2}(-1) = (2^3-1)^2$. By Theorem 3.2, a non-prime knot obtained from a trivial knot by an elementary SR-fusion is the square knot nor the connected sum of two figure-eight knots, which is a contradiction. Then $5_2\#5_2$ are not simple-ribbon.

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Department of Mathematics
Osaka Institute of Technology
Osaka 535-8585
Japan
E-mail address: `tsuneo.ishikawa@oit.ac.jp`

大阪工業大学　石川　恒夫

Department of Mathematics
Osaka Institute of Technology
Osaka 535-8585
Japan
E-mail address: `kengo.kishimoto@oit.ac.jp`

大阪工業大学　岸本　健吾

Department of Mathematics
Osaka Institute of Technology
Osaka 535-8585
Japan
E-mail address:

大阪工業大学　渋谷　哲夫

Department of Mathematics
Osaka Institute of Technology
Osaka 535-8585
Japan
E-mail address: `tatsuya.tsukamoto@oit.ac.jp`

大阪工業大学　塚本　達也