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Several new characteristic properties of Appell polynomials

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1 Introduction

There are several aspects, definitions or characteristics of Applell polynomials. Let g, φ two fixed functions. The function $g:\mathbb{C}\to\mathbb{C}$ is holomorphic at 0, $g(0)=0$, $g'(0)\neq 0$, and $\varphi:\mathbb{N}\to\mathbb{C}\backslash\{0\}$ an arbitrary function. The function $g(t)$ is sometimes called delta, implying that there is no constant term. Let $(P_n(x))_n$ be a sequence of polynomials. We call $(P_n(x))_n$ a sequence of Appell polynomials of type (g, φ), if and only if there exists $f:\mathbb{C}\to\mathbb{C}$ holomorphic at 0 such that $f(0)\neq 0$ and

$$\sum_{n=0}^{\infty}P_n(x)\frac{t^n}{\varphi(0)\cdots\varphi(n)}=f(t)e^{xg(t)}. \tag{1}$$

The function $f(t)$ is sometimes called invertible. Then, in general, $P_n(x)$ is sometimes called Sheffer ([16]). The ordinary Appell sequence of polynomials corresponds to special type

$$g(t)=t,\quad \varphi(0)=1\quad\text{and}\quad \varphi(n)=n\quad (n\geq 1)\,.$$

Therefore, every Appell sequence is a Sheffer sequence, but most Sheffer sequences are not Appell sequences. There are some well-known Appell sequences of polynomials. In particular, for the above type, we have Bernoulli polynomials $B_n(x)$ corresponding to $f_B(t)=\frac{t}{e^t-1}$, and Euler polynomials corresponding to $f_E(t)=\frac{2}{e^t+1}$.

Hermite polynomials ([8]) and Laguerre polynomials (see e.g. [17]) also belong to Appell polynomials. However, we do not treat them in this paper, as we consider the polynomials in the view point of the symmetric relations in (2).

For a fixed analytic function g, we study the classes of sequences of Appell polynomials $(P_n(x))_n$ in (1) when they satisfy the symmetric relation

$$P_n(a-x) = (-1)^n P_n(x) \tag{2}$$

for a real parameter a.

1.1 Known characterizations of Appell polynomials

We review the known results concerning special cases of ordinary Appell sequences of polynomials, in particular, Bernoulli and Euler polynomials. According to Bernoulli, Euler, Appell, Hurwitz, Raabe and Lucas, there are several approaches to study Bernoulli and Euler polynomials. We refer to Lehmer's paper [11] for concise details about the first five approaches. These approaches can be generalized to any generalized Appell sequences of polynomials.

(i) Generating functions theory (Bernoulli [3]; Euler [6])

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{te^{xt}}{e^t-1} \quad (|t| < 2\pi). \tag{3}$$

$$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2e^{xt}}{e^t+1} \quad (|t| < \pi). \tag{4}$$

(ii) Appell sequence theory (Appell [1])

$$\frac{d}{dx}B_n(x) = n \cdot B_{n-1}(x).$$

$$\frac{d}{dx}E_n(x) = n \cdot E_{n-1}(x).$$

(iii) Umbral Calculus (Lucas [12])

$$B_n(x) = (B+x)^n.$$

(iv) Fourier Series (Hurwitz [9])

$$B_n(x) = \frac{-n!}{(2\pi i)^n} \sum_{0\neq k\in\mathbb{Z}} \frac{e^{2\pi i k x}}{k^n} \quad (0<x<1).$$

$$E_n(x) = \frac{2n!}{(2\pi i)^{n+1}} \sum_{k\in\mathbb{Z}} \frac{e^{2\pi i\left(k+\frac{1}{2}\right)x}}{\left(k+\frac{1}{2}\right)^{n+1}} \quad (0<x<1).$$

(v) Raabe multiplication theorem (Raabe [15])

$$\sum_{k=0}^{m-1} B_n\left(\frac{x+k}{m}\right) = m^{1-n}B_n(x) \quad (\forall m\geq 1,\ \forall n\in\mathbb{N}).$$

$$\sum_{k=0}^{m-1}(-1)^k E_n\left(\frac{x+k}{m}\right) = m^{-n}E_n(x) \quad (\forall m\geq 1 \text{ odd},\ \forall n\in\mathbb{N}).$$

(vi) Determinantal approach (Costabile et. al ([4, 5])

$B_0(x)=1$,

$$B_n(x) = \frac{(-1)^n}{(n-1)!}\begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^{n-1} & x^n \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 2 & 3 & \cdots & n-1 & n \\ 0 & 0 & 0 & \binom{3}{2} & \cdots & \binom{n-1}{2} & \binom{n}{2} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \binom{n-1}{n-2} & \binom{n}{n-2} \end{vmatrix} \quad (n=1,2,\dots).$$

$E_0(x)=1$,

$$E_n(x) = (-1)^n\begin{vmatrix} 1 & x & x^2 & x^3 & \cdots & x^{n-1} & x^n \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2}\binom{2}{1} & \frac{1}{2}\binom{3}{1} & \cdots & \frac{1}{2}\binom{n-1}{1} & \frac{1}{2}\binom{n}{1} \\ 0 & 0 & 1 & \frac{1}{2}\binom{3}{2} & \cdots & \frac{1}{2}\binom{n-1}{2} & \frac{1}{2}\binom{n}{2} \\ 0 & 0 & 0 & 1 & \cdots & \frac{1}{2}\binom{n-1}{3} & \frac{1}{2}\binom{n}{3} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\binom{n}{n-1} \end{vmatrix} \quad (n=1,2,\dots).$$

Consequently, a determinant expression of Bernoulli numbers B_n is given by

$$B_0 = 1,$$

$$B_n = \frac{(-1)^n}{(n-1)!}\begin{vmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} & \frac{1}{n+1} \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 2 & 3 & \cdots & n-1 & n \\ 0 & 0 & \binom{3}{2} & \cdots & \binom{n-1}{2} & \binom{n}{2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n-1}{n-2} & \binom{n}{n-2} \end{vmatrix} \quad (n = 1, 2, \dots).$$

A determinant expression of Euler numbers E_n is given by

$$E_0 = 1,$$

$$E_n(x) = (-1)^n \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2}\binom{2}{1} & \frac{1}{2}\binom{3}{1} & \cdots & \frac{1}{2}\binom{n-1}{1} & \frac{1}{2}\binom{n}{1} \\ 0 & 1 & \frac{1}{2}\binom{3}{2} & \cdots & \frac{1}{2}\binom{n-1}{2} & \frac{1}{2}\binom{n}{2} \\ 0 & 0 & 1 & \cdots & \frac{1}{2}\binom{n-1}{3} & \frac{1}{2}\binom{n}{3} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\binom{n}{n-1} \end{vmatrix} \quad (n = 1, 2, \dots).$$

More simplified forms by Glaisher ([7]) are given by

$$B_n = (-1)^n n! \begin{vmatrix} \frac{1}{2!} & 1 & & & \\ \frac{1}{3!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{n!} & \frac{1}{(n-1)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(n+1)!} & \frac{1}{n!} & \cdots & \frac{1}{3!} & \frac{1}{2!} \end{vmatrix}$$

and

$$E_{2n} = (-1)^n (2n)! \begin{vmatrix} \frac{1}{2!} & 1 & & & \\ \frac{1}{4!} & \frac{1}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{1}{(2n-2)!} & \frac{1}{(2n-4)!} & \cdots & \frac{1}{2!} & 1 \\ \frac{1}{(2n)!} & \frac{1}{(2n-2)!} & \cdots & \frac{1}{4!} & \frac{1}{2!} \end{vmatrix}.$$

Note that $E_{2n+1} = 0$ $(n = 0, 1, \dots)$.

It is immediate to obtain that

$$(-1)^n(n+1)!\begin{vmatrix} B_1 & 1 & & & \\ \frac{B_2}{2!} & B_1 & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{B_{n-1}}{(n-1)!} & \frac{B_{n-2}}{(n-2)!} & \cdots & B_1 & 1 \\ \frac{B_n}{n!} & \frac{B_{n-1}}{(n-1)!} & \cdots & \frac{B_2}{2!} & B_1 \end{vmatrix} = 1$$

and

$$(-1)^n(2n)!\begin{vmatrix} \frac{E_2}{2!} & 1 & & & \\ \frac{E_4}{4!} & \frac{E_2}{2!} & & & \\ \vdots & \vdots & \ddots & 1 & \\ \frac{E_{2n-2}}{(2n-2)!} & \frac{E_{2n-4}}{(2n-4)!} & \cdots & \frac{E_2}{2!} & 1 \\ \frac{E_{2n}}{(2n)!} & \frac{E_{2n-2}}{(2n-2)!} & \cdots & \frac{E_4}{4!} & \frac{E_2}{2!} \end{vmatrix} = 1\,.$$

Therefore, for $n \geq 1$

$$B_n = n! \sum_{t_1+2t_2+\cdots+nt_n=n} \frac{(t_1+\cdots+t_n)!}{(t_1)!\ldots(t_n)!}\left(\frac{-1}{2!}\right)^{t_1}\left(\frac{-1}{3!}\right)^{t_2}\cdots\left(\frac{-1}{(n+1)!}\right)^{t_n}$$

and

$$E_{2n} = (2n)! \sum_{t_1+2t_2+\cdots+nt_n=n} \frac{(t_1+\cdots+t_n)!}{(t_1)!\ldots(t_n)!}\left(\frac{-1}{2!}\right)^{t_1}\left(\frac{-1}{4!}\right)^{t_2}\cdots\left(\frac{-1}{(2n)!}\right)^{t_n}$$

(*Cf.* [10, 13, 14]).

2 Main results

For this section we consider fixed type (g, φ) Appell sequences of polynomials.

2.1 Characterization of Appell sequences polynomials of type (g, φ)

We state our first main result.

Theorem 1. *Let a be a real parameter. We set*

$$h(t) := f(t)e^{\frac{a}{2}g(t)}$$

and denote by

$V(a) := \{(P_k)_k$ *Appell polynomials of sequences* (1) $\mid P_k(a-x) = (-1)^k P_k(x)\}$.

We have

$$V(a) \neq \emptyset \iff g \text{ is odd and } h \text{ is even.}$$

Corollary 1. *For $a = 0$, we have the function $h(t) = f(t)$. Then $V(0) \neq \emptyset$ if and only if g is odd and f is even.*

Theorem 2. *Let $a \neq 0$ be a real parameter. We have the following characterization for the set $V(a)$ of Appell sequences of polynomials. We have $V(a) \neq \emptyset$ if and only if the functions g and $t \to (e^{ag(t)} - 1)f(t)$ are odd.*

2.2 Application to type $g(t) = t$

In this section, we fix the type $g(t) = t$ and $\varphi(0) = 1, \varphi(n) = n$ $(n \geq 1)$. Next, we shall describe the set $V(a)$ explicitly, by truncating the Appell sequences (1). Denote by

$$V_n(a) = \{P \in \mathbb{C}_n[x] | \exists (P_k)_k \in V(a) : P = P_{k_0} \text{ for some } k_0 \in \mathbb{N}\}.$$

Theorem 3. *Let n be a positive integer, and $a \neq 0$ a real parameter. We have*

$$V_n(a) = \ Vect(B_{n-2k}(x/a); 0 \leq k \leq n/2),$$

which is the subspace spanned by $\{B_{n-2k}(x/a); 0 \leq k \leq n/2\}$, with dimension equal to $[n/2] + 1$. Alternatively, we have

$$V_n(a) = \ Vect(E_{n-2k}(x/a); 0 \leq k \leq n/2).$$

By symmetric properties of Bernoulli and Euler polynomials, $V_n(a)$ contains $\mathrm{Vect}(B_{n-2k}(x/a); 0 \leq k \leq n/2)$ and $\mathrm{Vect}(E_{n-2k}(x/a); 0 \leq k \leq n/2)$.

Theorem 4. *Let a be a nonzero real parameter, and $(P_n(x))_n$ be a sequence of Appell polynomials of type (g, φ) such that (2) holds. Let $(a_k)_{k \in \mathbb{N}}$ be a sequence of real numbers such that the function F has the following property:*

$$F : t \to f(t) - \sum_k a_k \frac{t^k}{k!} \text{ is odd or even} \tag{5}$$

Then, we obtain

$$P_n(x) = \sum_{k \text{ even}} a_k \binom{n}{k} a^{n-k} E_{n-k}\left(\frac{x}{a}\right), \quad \textit{if } F \textit{ is odd} \tag{6}$$

$$P_n(x) = -2 \sum_{k \text{ odd}} a_k \, \frac{1}{k} \binom{n}{k-1} a^{n-k+1} B_{n-k+1}\left(\frac{x}{a}\right), \quad \textit{if } F \textit{ is even} . \tag{7}$$

2.3 Fourier expansions for Appell polynomials of type $g(t) = t$

For $0 < x < 1$ if $n = 1$, and $0 \leq x \leq 1$ if $n \geq 2$. It is well-known that

$$B_n(x) = \frac{-n!}{(2\pi i)^n} \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{2\pi i k x}}{k^n}. \tag{8}$$

Concerning Euler polynomials, for $0 < x < 1$ if $n = 0$, and $0 \leq x \leq 1$ if $n \geq 1$, we have

$$E_n(x) = \frac{2n!}{(2\pi i)^{n+1}} \sum_{k \in \mathbb{Z}} \frac{e^{2\pi i \left(k - \frac{1}{2}\right) x}}{\left(k - \frac{1}{2}\right)^{n+1}}. \tag{9}$$

Theorem 5. *Let a be a nonzero real parameter, and $(P_n(x))_n$ be a sequence of Appell polynomials of type (g, φ) such that (2) holds. Let $(a_k)_{k \in \mathbb{N}}$ be a sequence of real numbers such that the function*

$$F : t \to f(t) - \sum_k a_k \frac{t^k}{k!} \quad \textit{is odd or even.} \tag{10}$$

(i) If F odd and $0 < x/a < 1$ $(n = 0)$; $0 \leq x/a \leq 1$ $(n \geq 1)$, then we have

$$P_n(x) = \frac{2a^n (n!)}{(2\pi i)^{n+1}} \sum_{m \in \mathbb{Z}} c_m^-(a) \frac{e^{2\pi i \left(m - \frac{1}{2}\right) x}}{\left(m - \frac{1}{2}\right)^{n+1}}, \tag{11}$$

where

$$c_m^-(a) := \sum_{k \text{ even}} \frac{a_k}{k!} \left(\frac{\pi i}{a}\right)^k (2m-1)^k.$$

(ii) If F is even and $0 < x/a < 1$ $(n = 1)$; $0 \leq x/a \leq 1$ $(n \geq 2)$, then we have

$$P_n(x) = -\frac{2a^n(n!)}{(2\pi i)^{n+1}} \sum_{m \in \mathbb{Z}} c_m^+(a) \frac{e^{2\pi i\, mx}}{m^n}, \tag{12}$$

where

$$c_m^+(a) := \sum_{k \text{ odd}} \frac{a_k}{k!} \left(\frac{\pi i}{a}\right)^{k-1} m^{k-1}.$$

3 New results on Bernoulli and Euler polynomials of higher order

In this section, we give two applications of our results. We obtain new explicit formulas and Fourier series for Bernoulli and Euler polynomials of higher order.

3.1 Bernoulli polynomials

We start with two applications. We obtain new characterizations of Bernoulli and Euler polynomials. Note that for $\varphi(0) = 1$, $\varphi(k) = k$ and $a_k = E_k(0)$ $(k \geq 1)$, it is well-known that $E_0(0) = 1$ and $E_k(0) = 0$ for $k \neq 0$ *even.* Then, we obtain

$$f(t)e^{xt} = \sum_{k \text{ even}} E_k(0)\frac{t^k}{k!} \cdot \frac{2e^{xt}}{e^t+1} = \frac{2e^{xt}}{e^t+1}.$$

We get $P_n(x) = E_n(x)$. It means, in particular, that if the function $F : t \to f(t) - 1$ is *odd* and $P_n(1-x) = (-1)^n P_n(x)$, then $P_n(x) = E_n(x)$. Similarly, one can apply it to Bernoulli polynomials $B_n(x)$.
We have the following general formulation.

Theorem 6 (Bernoulli polynomials by symmetry). *Let $(P_n(x))_n$ be a sequence of Bernoulli polynomials of type* (1) *such that*

$$P_n(1-x) = (-1)^n P_n(x). \tag{13}$$

Let N be a positive integer. If the function $F : t \to f(t) - \sum_{k=0}^{N} B_k(0)\frac{t^k}{k!}$ is even, then we obtain

$$P_n(x) = B_n(x) \quad \text{and} \quad f(t) = \frac{t}{e^t - 1}. \tag{14}$$

3.2 Euler polynomials

Theorem 7 (Euler polynomials by symmetry). *Let $(P_n(x))_n$ be a sequence of Euler polynomials of type* (1) *such that (13) holds. Let N be a positive integer. If the function $F : t \to f(t) - \sum_{k=0}^{N} E_k(0)\frac{t^k}{k!}$ is odd, then we obtain*

$$P_n(x) = E_n(x) \quad \text{and} \quad f(t) = \frac{2}{e^t + 1}. \tag{15}$$

3.3 Bernoulli polynomials of order r

Let r be a positive integer. The Bernoulli polynomials and numbers of order r are given by the equations

$$\sum_{n \geq 0} B_n^{(r)}(x)\frac{t^n}{n!} = \left(\frac{t}{e^t - 1}\right)^r e^{xt}$$

and $B_n^{(r)} = B_n^{(r)}(0)$.
The function $f(t) = \left(\frac{t/r}{e^{t/r}-1}\right)^r$ $(r \geq 1)$ satisfy $f(t)e^t = f(-t)$ and the function

$$F(t) = f(t) - \sum_{k \geq 0} r^{-2k-1} B_{2k+1}^{(r)} \frac{t^{2k+1}}{(2k+1)!}$$

is an even function. So, we obtain

$$r^{-n} B_n^{(r)}(rx) = -2 \sum_{0 \leq k \leq n/2} r^{-2k-1} \frac{B_{2k+1}^{(r)}}{2k+1} \binom{n}{2k} B_{n-2k}(x).$$

We have a new formula for the generalized Bernoulli polynomials $B_n^{(r)}(x)$

$$B_n^{(r)}(x) = -2 \sum_{0\le k\le n/2} r^{n-2k-1} \frac{B_{2k+1}^{(r)}}{2k+1} \binom{n}{2k} B_{n-2k}(x/r).$$

Thanks to the relations in [2, (1.10) and Corollary 1.8], the coefficients $B_{2k+1}^{(r)}$ are given by the formula

$$B_n^{(r)} = \begin{cases} n\binom{n-1}{r-1} \sum_{k=1}^{r} (-1)^{k-1} s(r,k) \frac{B_{n-r+k}}{n-r+k}\ , & n \ge r \\ \\ \frac{1}{\binom{r-1}{n}} s(r, r-n), & 0 \le n \le r-1\ . \end{cases} \tag{16}$$

where $s(n,l)$ is the Stirling number of the first kind.
Therefore, we obtain the formulas.

Theorem 8. *For $n \ge r \ge 2$, we have*

$$\begin{aligned} B_n^{(r)}(x) = -2\ r^{n-1} & \sum_{0\le k\le r/2-1} r^{-2k}\ \frac{s(r, r-2k-1)}{2k+1} \frac{\binom{n}{2k}}{\binom{r-1}{2k+1}} B_{n-2k}(x/r) \\ & -2\ r^{n-1} \sum_{r/2\le k\le n/2} r^{-2k} \frac{B_{2k+1}^{(r)}}{2k+1} \binom{n}{2k} B_{n-2k}(x/r), \end{aligned} \tag{17}$$

with

$$\frac{B_{2k+1}^{(r)}}{2k+1} = \binom{2k}{r-1} \sum_{j=1}^{r} (-1)^{j-1} s(r,j) \frac{B_{2k+1-r+j}}{2k+1-r+j} \quad (j \ge r/2).$$

We give details for $r = 2, 3$, which give us new explicit formulas for $B_n^{(2)}(x)$ and $B_n^{(3)}(x)$.
Thanks to $B_1^{(2)} = s(2,1) = -1, B_{2k+1}^{(2)} = -(2k+1)B_{2k}$ $(k \ge 1)$, then we have

$$\begin{aligned} B_n^{(2)}(x) &= \sum_{0\le k\le n/2} 2^{n-2k} \binom{n}{2k} B_{2k} B_{n-2k}(x/2), \\ B_n^{(3)}(x) &= 3^n B_n(x/3) - 2 \sum_{1\le k\le n/2} 3^{n-2k-1} \frac{B_{2k+1}^{(3)}}{2k+1} \binom{n}{2k} B_{n-2k}(x/3) \end{aligned}$$

and then for $n \geq 4$ we have

$$\begin{aligned} B_n^{(3)}(x) &= 3^n B_n(x/3) + \frac{1}{2}3^{n-2}\binom{n}{2}B_{n-2}(x/3) \\ &\quad -2\sum_{2\leq k\leq n/2} 3^{n-2k}(2k-1)\binom{n}{2k}B_{2k}B_{n-2k}(x/3). \end{aligned}$$

3.4 Euler polynomials of order r

Let r be a positive integer. The Euler polynomials and numbers of order r are given by the equations

$$\sum_{n\geq 0} E_n^{(r)}(x)\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^r e^{xt}$$

and $E_n^r = E_n^{(r)}(0)$.

Let $f(t) = \left(\frac{2}{e^{t/r}+1}\right)^r$. The function f satisfy $f(t)e^t = f(-t)$ and the function

$$F(t) = f(t) - \sum_{k\geq 0} r^{-2k}E_{2k}^{(r)}\frac{t^{2k}}{(2k)!}$$

is an odd function. So, we obtain

$$r^{-n}E_n^{(r)}(rx) = \sum_{0\leq k\leq n/2} r^{-2k}E_{2k}^{(r)}\binom{n}{2k}E_{n-2k}(x).$$

Then, we get a new formula for $E_n^{(r)}(x)$:

$$E_n^{(r)}(x) = \sum_{0\leq k\leq n/2} r^{n-2k}\binom{n}{2k}E_{2k}^{(r)}E_{n-2k}(x/r).$$

On the other hand, it is well-known that the numbers $E_n^{(r)}$ can be express explicitly in terms of Stirling numbers of first kind and Euler numbers $E_k := E_k(0)$. Precisely, we have

Lemma 1. *Let r be a positive integer. We have*

$$E_n^{(r)} = \frac{2^{r-1}}{(r-1)!}\sum_{j=0}^{r-1}(-1)^j s(r,r-j)E_{n+r-j-1}. \tag{18}$$

Hence, we get the result

Theorem 9. *Let r be a positive integer. We have*

$$E_n^{(r)}(x) = \frac{2^{r-1}}{(r-1)!} \sum_{\substack{0\le j\le r-1\\ 0\le k\le n/2}} (-1)^j s(r, r-j) \binom{n}{2k} r^{n-2k} E_{2k+r-j-1} E_{n-2k}(x/r). \tag{19}$$

The relation is obvious for $r = 1$. For $r = 2$ and $n \geq 2$, we obtain

$$E_n^{(r)}(x) = 2^n E_n(x/2) + \sum_{1\le k\le n/2} \binom{n}{2k} 2^{n+1-2k} E_{2k+1} E_{n-2k}(x/r). \tag{20}$$

3.5 Fourier expansions for higher Bernoulli and Euler polynomials

We apply our main results to get Fourier series for Bernoulli and Euler polynomials of order r. From our Theorem 8 and Theorem 9, we can obtain the Fourier expansions for the polynomials $B_n^{(r)}(x)$ and $E_n^{(r)}(x)$.

Theorem 10 (Fourier expansion). *For $x \in (0, r)$, we have Fourier expansion for the Euler polynomials of order $r \geq 1$ given by*

$$E_n^{(r)}(x) = \frac{2^r}{(r-1)!} \frac{n!}{(2\pi i)^{n+1}} \sum_{m\in\mathbb{Z}} c_m(n, r) \frac{e^{2\pi i(m-1/2)\frac{x}{r}}}{(m-1/2)^{n+1}}, \tag{21}$$

where

$$c_m(n, r) = \sum_{\substack{0\le j\le r-1\\ 0\le k\le n/2}} (-1)^j s(r, r-j)(\pi i)^{2k}(2m-1)^{2k} E_{2k+r-j-1}. \tag{22}$$

For example,

(i) for $r = 1$, we have $c_m(n, 1) = 1$ for any $m \in \mathbb{Z}$. We recover the known result about periodic Euler functions.

(ii) for $r = 2$,

$$c_m(n, 2) = 1/2 + \sum_{1\le k\le n/2} (\pi i)^{2k}(2m-1)^{2k} E_{2k+1} \quad (n \geq 2). \tag{23}$$

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