Fundamental Domains of Arithmetic Quotients of the General Linear Group and Humbert Forms

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1 Introduction

This is a résumé that covers the main results in the article Fundamental domains of arithmetic quotients of reductive groups over number fields (with appendix by Takao Watanabe) [7]. The paper mainly focuses on the determination and construction of fundamental domains associated to certain arithmetic quotients of reductive algebraic groups over an algebraic number field \( k \).

Definition. Let \( T \) be a locally compact Hausdorff space and \( \Gamma \) a discrete group with a properly discontinuous action on \( T \). A subset \( \Omega \) of \( T \) satisfying

(i) \( T = \Gamma \Omega^- \),

(ii) \( \Omega^+ \cap \gamma \Omega^- = \emptyset \) for all \( \gamma \in \Gamma \setminus \{e\} \)

is called a fundamental domain of \( T \) with respect to \( \Gamma \) or just a fundamental domain of \( \Gamma \backslash T \) (\( T/\Gamma \) in the case of a right action). Here \( \Omega^+, \Omega^- \) denote the interior and closure of \( \Omega \) in \( T \) respectively.

In particular we study arithmetic quotients of \( GL_n \), and the results of which are used to construct fundamental domains for \( P_n \), the cone of positive definite Humbert forms over \( k \), with respect to arithmetic subgroups of \( GL_n(k) \).

For the first part of the paper we consider a general connected reductive isotropic algebraic group \( G \) over \( k \) and investigate fundamental domains of the quotients \( G(k) \backslash G(A)^1 \) and \( \Gamma_i \backslash G(k_{\infty})^1 \) with arithmetic subgroups \( \Gamma_1, \ldots, \Gamma_{n_G} \) of \( G(k) \) (\( n_G \): the class number of \( G \)).

The results here are an extension of Watanabe's results in [9]. A maximal \( k \)-parabolic subgroup of \( G \), \( Q_n \), is taken and we define the Ryskhoi domain of \( G \) associated to \( Q_n \), \( R_n \). This was introduced in [9] for the purpose of constructing a fundamental domain for \( G(k) \backslash G(A)^1 \) well-matched with the Hermite function of \( Q_n \), \( m_n \). Watanabe also considered the case when \( G \) is of class number 1, and obtained a fundamental domain for \( G(k_{\infty}) \) with respect to \( G_0 = G(k) \cap G_{A_{\infty}} \) (this coincides with \( \Gamma_1 \)). Here however, we will consider algebraic groups of any general class number \( n_G \).

The second topic of interest in this paper is the special case when \( G \) is the general linear group \( GL_n \) defined over \( k \). It is well known that the class number of \( G \) in this case is equal to \( h \), the class number of \( k \). The \( \Gamma_i \) in this case are the subgroups of \( GL_n(k) \) stabilizing certain \( O \)-lattices in \( k^n \).

In the final section we proceed onto \( P_n \), the space of positive definite Humbert forms over \( k_{\infty} \), with the usual identification \( P_n = \prod_{\sigma} P_n(k_\sigma) \) where \( P_n(k_\sigma) \) denotes the set of \( n \) by \( n \) positive-definite real symmetric/complex Hermitian matrices depending on whether \( \sigma \) is a real/imaginary, the product taken over all infinite places \( \sigma \) of \( k \).

When \( k = \mathbb{Q} \), \( P_n \) is just the cone of positive-definite real symmetric matrices, and fundamental domains for \( P_n/GL_n(\mathbb{Z}) \) in this case have been historically constructed by Korkin and Zolotarev [6], Minkowski [8] and later on Grenier [4]. For \( P_n \) over a general number field, in [5] Humbert has previously provided a fundamental domain constructed with respect to the particular group \( GL_n(O) \). As \( GL_n(O) \) coincides with one of the \( \Gamma_i \) we study in this paper, the question can be raised about fundamental domains for \( P_n \) with respect to each of the groups \( \Gamma_i \) when \( n_G > 1 \).

As such we proceed in the final sections to provide a general way of constructing fundamental domains for \( P_n/\Gamma_i \) given any number field. The method of construction follows and generalizes the example given by Watanabe in [9] for the specific case \( k = \mathbb{Q} \). As already noted in [9], when \( k = \mathbb{Q} \) the fundamental domain for \( P_n/GL_n(\mathbb{Z}) \) resulting from this method coincides with Grenier's ([4]). It was observed by Dutour Sikirić and Schürmann that this fundamental domain is in fact equivalent to the one previously mentioned.
developed by Korkin and Zolotarev. Regarding $P_n/GL_n(O)$ for general number fields however, we note that the fundamental domain produced by the method here differs from Humbert’s construction which utilizes the matrix trace, whereas the domain here is defined using the adele norm of matrix determinants.

Notation

We fix $k$, an algebraic number field of finite degree over $Q$, and denote its ring of integers by $O$ and the adele ring by $A$. $p_\infty$ and $p_f$ denote the sets of infinite and finite places of $k$ respectively and we let $p = p_\infty \cup p_f$. $k_\infty$ denotes the usual étale $R$-algebra $k \otimes Q R$ which we identify with $\prod_{\sigma \in p_\infty} k_\sigma$.

2 Fundamental domains of $G(k) \backslash G(A)^1$ and $\Gamma_i \backslash G(k_\infty)^1$

2.1 The Ryshkov domain of $G$ associated to $Q$

Let $G$ be a connected reductive isotropic affine algebraic group defined over $k$. Fix a minimal $k$-parabolic subgroup of $G$ and let $Q$ be a proper maximal $k$-parabolic subgroup of $G$ containing it.

Definition ([9, §4]). The Ryshkov domain of $G$ associated to $Q$ is defined by

$$R_Q := \{g \in G(A)^1 | m_Q(g) = H_Q(g)\}$$

where $H_Q : G(A) \to \mathbb{R}_{>0}$ and $m_Q : G(A)^1 \to \mathbb{R}_{>0}$ are respectively the height function and Hermite function associated to $Q$ given by

$$H_Q(umh) := |\chi_Q(m)|^{-1}_{A} (u \in U(A), m \in M(A), h \in K),$$

$$m_Q(g) := \min_{x \in Q(k) \backslash G(k)} H_Q(xg).$$

Here,

• $U$ and $M$: the unipotent radical and Levi subgroup of $Q$,
• $K$: maximal compact subgroup of $G(A)$,
• $\chi_Q$ the $k$-rational character of $M/(the$ maximal central $k$-split torus of $G)$ spanning the (rank 1) $Z$-module of all such characters.
• $G(A)^1 : = \{g \in G(A) | |\chi(g)|_{A} = 1 for all$ $k$-rational characters $\chi of $ $G\}$

The Ryshkov domain is useful to us because of the following theorem from [9].

Theorem 1. Let $\Omega$ be an open fundamental domain of $(R^0_Q)^-$ (closure of interior of $R_Q$ in $G(A)^1$) with respect to $Q(k)$. Then $\Omega^0$ is an open fundamental domain of $G(k) \backslash G(A)^1$.

Thus by starting with the Ryshkov domain, we can proceed to construct a fundamental domain for $G(k) \backslash G(A)^1$. The following subsection details this.

2.2 Constructing $R_Q$ and $\Omega$

Notation

• $K_f = \prod_{\sigma \in p_f} K_\sigma$ (the finite part of $K$),
• $G_{A,\infty} = G(k_\infty) \times K_f$, $G_{A,\infty}^1 = G_{A,\infty} \cap G(A)^1$,
• $G(k_\infty)^1 = G(k_\infty) \cap G(A)^1$. 
Also we will denote the class number of $G_i$, that is the finite number $|G(k)\backslash G(A)/G_{A,\infty}|$, by $n_{G_i}$. We note here that $|G(k)\backslash G(A)^1/G_{A,\infty}^1|$ is also equal to $n_{G_i}$.

First, take a complete set of representatives $\{\eta_i\}_{i=1}^{n_G}$ for $G(k)\backslash G(A)^1/G_{A,\infty}^1$. We then define the arithmetic subgroups $\Gamma_1, \ldots, \Gamma_{n_G}$ by

$$\Gamma_i = \eta_i G_{A,\infty}^1 \eta_i^{-1} \cap G(k).$$

Also for each $i = 1, \ldots, n_G$ take a complete set of representatives $\{\xi_{ij}\}_{j=1}^{h_i}$ for $Q(k)\backslash G(k)/\Gamma_i$ (where the number of double cosets $h_i$ is finite, see [2, §7]) and define groups

$$Q_{i,j} = Q \cap \xi_{ij} \Gamma_i \xi_{ij}^{-1} \subset Q(k)$$

and the sets

$$R_{i,j,\infty} = \{g \in G(k) : m_Q(\xi_{ij} \eta_i) = H_Q(\xi_{ij} \eta_i)\}$$

for $j = 1, \ldots, h_i$.

We can immediately verify that

- $G(A)^1 = \bigsqcup_{i=1}^{n_G} \bigsqcup_{j=1}^{h_i} Q(k) \Gamma_i \Gamma^{-1} \subset G(k)$,
- $R_Q = \bigcup_{i=1}^{n_G} \bigcup_{j=1}^{h_i} Q(k) R_{i,j,\infty} \subset G(k)^1$.

Also, by taking a complete set of representatives $\{\theta_{ijk}\}_k$ for $Q(k)/Q_{i,j}$, we obtain

$$R_Q = \bigcup_{i=1}^{n_G} \bigcup_{j=1}^{h_i} Q(k) R_{i,j,\infty} \theta_{ijk} \subset G(k)^1.$$  \hfill (1)

Denote $(R_{i,j,\infty}^*)^{-}$ by $R_{i,j,\infty}^*$ where the interior and closure is taken in $G(k)^1$. Similarly write $R_Q^*$ for $(R_Q^*)^{-}$ in $G(A)^1$. From (1) we have

$$R_Q^* = \bigcup_{i=1}^{n_G} \bigcup_{j=1}^{h_i} Q(k) \Omega_{i,j,\infty} \subset G(k)^1.$$  \hfill (2)

We have the following main result.

**Theorem 2.** For each $i = 1, \ldots, n_G$ and $j = 1, \ldots, h_i$, take open fundamental domains $\Omega_{i,j,\infty}$ of $R_{i,j,\infty}$ with respect to $Q_{i,j}$. Then the set

$$\Omega = \bigcup_{i=1}^{n_G} \bigcup_{j=1}^{h_i} \Omega_{i,j,\infty} \subset G(k)^1$$

is an open fundamental domain of $R_Q$ with respect to $Q(k)$.

**Corollary 3.** $\Omega^\circ$ (interior of $\Omega$ in $G(A)^1$) is an open fundamental domain of $G(A)^1$ with respect to $G(k)$.

**Proof.** From (2) we have

$$R_Q^* = \bigcup_{i=1}^{n_G} \bigcup_{j=1}^{h_i} Q(k) \Omega_{i,j,\infty} \subset G(k)^1.$$
Now suppose $\Omega \cap g\Omega^- \neq \emptyset$ for $g \in \mathcal{Q}(k)$. For some $i, i', j, j'$ we must have $g(\Omega_{i,j,\infty} \xi_{ij} \eta_i \mathcal{K}_f) \cap (\Omega_{i',j',\infty} \xi_{ij'} \eta_i \mathcal{K}_f) \neq \emptyset$. Writing $g = \theta_{ijk} g'$ with $g' \in Q_{i,j}$ and some $k$, we have

$$\theta_{ijk}(g')_m \Omega_{i,j,\infty} \xi_{ij} \eta_i \mathcal{K}_f \cap \Omega_{i',j',\infty}^{-} \xi_{ij'} \eta_i \mathcal{K}_f \neq \emptyset$$

since $(q')_m \xi_{ij} \eta_i \mathcal{K}_f \subset \xi_{ij} \eta_i \mathcal{K}_f$. Then (2) implies $i = i', j = j'$, and $\theta_{ijk} = e$. Thus $\Omega_{i,j,\infty} \cap (q')_m \Omega_{i,j,\infty}^{-} = \Omega_{i,j,\infty} \cap q' \Omega_{i,j,\infty}^{-}$ must be non-empty, which means $g' = e$ and hence $g = e$. This proves the theorem, and the corollary follows from Theorem 1.

Additionally, for any fixed $1 \leq i \leq n_G$, we have the following theorem.

**Theorem 4.** The set $\Omega_{i,\infty} = \bigcup_{j=1}^{h_i} \xi_{ij}^{-1} \Omega_{i,j,\infty} \xi_{ij}$ is a fundamental domain of $G(k_{\infty})$ with respect to $\Gamma_i$.

**Proof.** To show that $G(k_{\infty}) = \Gamma_i \Omega_{i,\infty}$, consider an arbitrary $g \in G(k_{\infty})$. From corollary 3

$$G(A)^1 = G(k) \Omega^- = G(k) \bigcup_{\nu} h_i \xi_{ij}^{-1} \Omega_{i,j,\infty} \xi_{ij} \eta_i \mathcal{K}_f$$

$$= G(k) \bigcup_{\nu} h_i \xi_{ij} \Omega_{i,j,\infty}^{-} \xi_{ij} \eta_i \mathcal{K}_f$$

$$\subset G(k) \bigcup_{\nu} h_i \xi_{ij} \Omega_{i,j,\infty}^{-} \xi_{ij} \eta_i \mathcal{K}_f$$

so we may write $g \eta_i = g' \omega \eta_i h$, with $g' \in G(k)$, $\omega \in \Omega_{i,\infty}$, and $h \in K_f$. Rearranging we get $g' = (g \omega^{-1})(\eta_i h \eta_i^{-1})$ which belongs to $G(k_{\infty}) \eta_i K_{\infty}^{-1}$. Hence $g' \in \Gamma_i$. Since $g = (g' \omega)(\eta_i h \eta_i^{-1})$ and $g \in G(k_{\infty})$, $\eta_i h \eta_i^{-1}$ must necessarily be trivial. Thus $g \in \Gamma_i \Omega_{i,\infty}$.

Now suppose that $\Omega_{i,\infty} \cap g \Omega_{i,\infty}^{-}$ is non-empty for a $g \in \Gamma_i$. Then we must have $\Omega_{i,j,\infty} \xi_{ij}^{-1} \Omega_{i,j',\infty} \xi_{ij'} \neq \emptyset$ for some $j, j'$. Since $g \eta_i \mathcal{K}_f = \eta_i \mathcal{K}_f$,

$$\xi_{ij}^{-1} \Omega_{i,j,\infty} \xi_{ij} \cap g \xi_{ij} \Omega_{i,j',\infty} \xi_{ij'} \neq \emptyset$$

and thus $\xi_{ij}^{-1} \xi_{ij'} = e$ by Corollary 3. Hence $Q(k) \xi_{ij} \Gamma_i = Q(k) \xi_{ij} \Gamma_i$, which implies $j = j'$ whereby $g = \xi_{ij}^{-1} \xi_{ij'} = e$.

\section{The case $G = GL_n$}

In this section we will consider the case where $G$ is a general linear group $GL_n$ defined over $k$. Fixing an integer $1 \leq m < n$, we consider the maximal standard $k$-parabolic subgroup $Q$ defined by

$$Q(k) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in GL_m(k), b \in M_{m,n-m}(k), d \in GL_{n-m}(k) \right\}.$$

For the maximal compact subgroup $K$ of $G(A)$ let $K = K_{\infty} \times K_f$ where

$$K_{\infty} = \{ g \in GL_n(k_{\infty}) : {}^t g g = I_n \}, \quad K_f = \prod_{\sigma \in P_f} GL_n(\mathcal{O}_\sigma).$$

Here we identify $GL_n(k_{\infty})$ with $\prod_{\sigma \in P_{\infty}} GL_n(\mathcal{O}_\sigma)$, and for $g = (g_\sigma)_{\sigma \in P_{\infty}} \in GL_n(k_{\infty})$ we write $^t g$ for the element $(^t g_\sigma)_{\sigma \in P_{\infty}}$ of $GL_n(k_{\infty})$.

We shall see that in this case the number of double cosets of $Q(k) \setminus GL_n(k) / \Gamma_i$ for each $i$ is invariant and equal to $|GL_n(k) \setminus GL_n(A)^1 / G_{\lambda,\infty}|$, the class number of $GL_n$. 
Denote the set of all $O$-lattices in $k^r$ ($r \geq 1$) by $\mathcal{L}_r$, and the standard unit vectors of $k^r$ by $e^{(r)}_1, \ldots, e^{(r)}_r$.

For this section we simply write $\mathcal{L}$ for $\mathcal{L}_n$ and $e_k$ for $e^{(n)}_k$ ($1 \leq k \leq n$).

For $L \in \mathcal{L}_r$, and $g = (g_\sigma)_{\sigma \in \mathcal{P}} \in GL_r(A)$ put

$$gL = (k_{\infty}^r) \times \prod_{\sigma \in \mathcal{P}_f} g_\sigma L_\sigma \cap k^r \in \mathcal{L}_r. \quad (3)$$

This defines a transitive left action of $GL_r(A)^1$ on $\mathcal{L}_r$. Note that if $g \in GL_r(k)$ then $gL$ as defined above coincides with the usual image of $L$ under the linear transformation $v \mapsto gv$ of $k^r$. The subset of $\mathcal{L}$ consisting of all $O$-lattices of the form $gL$ with $g \in GL_n(k)$ will be referred to as the $O$-lattice class of $L$ or just the lattice class of $L$ in $\mathcal{L}$. Since every $O$-lattice in a lattice class has the same Steinitz class, we refer to the Steinitz class of any lattice representing the class as the Steinitz class for that lattice class.

From the previous section, we will require a complete set representing $GL_n(k)\backslash GL_n(A)^1/G_{A,\infty}^1$. Take $\{\eta_1, \ldots, \eta_h\}$ to be such a set of matrices. Then for each $i = 1, \ldots, h$ put $L_i = \eta_i(Oe_1 + \cdots + Oe_n) \in \mathcal{L}$. We then have a one-to-one correspondence between $GL_n(k)\backslash GL_n(A)^1/G_{A,\infty}^1$ and the set of $O$-lattices classes in $\mathcal{L}$ by mapping each $\eta_i$ to the lattice class of $L_i$. That this is a bijection follows from $G_{A,\infty}^1$ being the stabilizer group of the $O$-lattice $Oe_1 + \cdots + Oe_n$ under the action of $GL_n(A)^1$ on $\mathcal{L}$.

Continuing the map to $St(L_i)$, the Steinitz class of $L_i$ gives us a bijection from $GL_n(k)\backslash GL_n(A)^1/G_{A,\infty}^1$ to $Cl(k)$. As a result the class number of $GL_n(k)$ is equal to the class number of $k$, which we write as $h$. We can proceed on to our next main results, that $h_i = |Q(k)\backslash GL_n(k)/G_{i}|$ is also equal to $h$ for every $i = 1, \ldots, h$.

Identify $Q(k)\backslash GL_n(k)$ with the set of all $m$-dimensional linear subspaces of $k^n$ denoted by $Gr_m$ (the Grassmanian) via the bijection

$$Q(k)\backslash GL_n(k) \ni Q(k)g \mapsto g^{-1}(\sum_{k=1}^{m} ke_k) \in Gr_m. \quad (4)$$

Fix $i \in \{1, \ldots, h\}$. Considering the left action of $G_i \subset GL_n(k)$ on $Gr_m$, the map (4) gives rise to the bijection

$$Q(k)\backslash GL_n(k)/G_i \ni Q(k)gG_i \mapsto g^{-1}(\sum_{k=1}^{m} ke_k) \in G_i\backslash Gr_m \quad (5)$$

which lets us identify $Q(k)\backslash GL_n(k)/G_i$ with $G_i\backslash Gr_m$.

**Lemma 5.**

$$G_i = \{g \in GL_n(k) : gL_i = L_i\}$$

i.e. $G_i$ is the stabilizer of $L_i$ in $GL_n(k)$, under the action of $GL_n(A)^1$ on $\mathcal{L}$.

**Theorem 6.** The map

$$\lambda_i : G_i\backslash Gr_m \rightarrow Cl(k), \quad \lambda_i(G_iV) = St(L_i \cap V) \quad (V \in Gr_m) \quad (6)$$

is a well-defined bijection, and thus $h_i = h$.

The above bijections give us an explicit way to find candidates for $\{\eta_1\}_{i=1}^h$ and $\{\xi_i\}_{j=1}^h$ as follows. Let $\{a_1, \ldots, a_h\}$ be a complete set of fractional ideals representing the ideal class of $k$. For each $i = 1, \ldots, h$, we shall require an element $\eta_i \in GL_n(A)^1$ such that the Steinitz class of the resulting lattice $L_i = \eta_i(\sum_{k=1}^{n} Oe_k)$ is the ideal class represented by $a_i$.

Let $D_n(x)$ ($x \in A$) denote the unit matrix of size $n$ with bottom-most diagonal entry replaced by $x$. For each $1 \leq i \leq h$ we can choose $a_i \in A^X$ such that $a_i$ generates the principal ideal $a_iO_x$ for every finite $x$ and $|a_i|_{\infty} = N(a_i)$, the ideal norm of $a_i$. Then $D_n(a_i) \in GL_n(A)^1$ since $|\det D_n(a_i)|_A = |a_i|_A = 1$, and

$$D_n(a_i)\left(\sum_{k=1}^{n} Oe_k\right) = \sum_{1 \leq k \leq n} Oe_k + a_i e_n.$$

Hence putting $\eta_i = D_n(a_i)$ ($1 \leq i \leq h$) gives us our required set of representatives for $GL_n(k)\backslash GL_n(A)^1/G_{A,\infty}^1$. The corresponding $O$-lattice $L_i$ and its stabilizer group $G_i$ will be denoted by $L_n(a_i)$ and $G_n(a_i)$ respectively whenever we want to call to attention the fractional ideal $a_i$ or the dimension $n$. 
We can also proceed similarly to find for a fixed $i$ a suitable set of representatives for $Q(k)\backslash GL_n(k)/\Gamma_i$. We do this using the bijection

$$Q(k)\backslash GL_n(k)/\Gamma_i \ni Q(k)g\Gamma_i \mapsto St(L_i \cap g^{-1}V_m) \in Cl(k)$$

formed by composing $\lambda_i$ with the bijection (5), where $V_m = \sum_{k=1}^{m} k e_k$.

For each $j \in \{1, \ldots, h\}$ the ideal $a_j a_j^{-1}$ shares the same ideal class as a unique $a_{j'}$ ($j' \in \{1, \ldots, h\}$), that is $[a_j][a_{j'}] = [a_j]$. Putting $\tau_i(j) := j'$ defines a permutation $\tau_i$ on $\{1, \ldots, h\}$.

Call a set of matrices $\{\xi_1, \ldots, \xi_h\} \subset GL_n(k)$ an $(m, n)$-splitting set for $L_n(a_i)$ if for each $j = 1, \ldots, h$

$$\xi_j L_n(a_i) = \left( \sum_{1 \leq k < m} O e_k + a_j e_m \right) + \left( \sum_{m < k < n} O e_k + a_{\tau_i(j)} e_n \right)$$

$$\simeq L_m(a_j) \oplus L_{n-m}(a_{\tau_i(j)}).$$

(7)

Since $St(L_i \cap \xi_j^{-1}V_m) = St(\xi_j L_i \cap V_m) = [a_j]$ ($i \leq j \leq h$), such a set of matrices completely represents $Q(k)\backslash GL_n(k)/\Gamma_i$.

One such set is given as follows. For each $j = 1, \ldots, h$, first take $\kappa_j \in k$ such that $a_j a_{\tau_i(j)} = \kappa_{ij} a_i$. Then choose elements $\alpha_{ij} \in a_j$, $\alpha_{ij}' \in a_{\tau_i(j)}$, $\beta_{ij} \in a_j^{-1}$ and $\beta_{ij}' \in a_{\tau_i(j)}^{-1}$ satisfying

$$\alpha_{ij} \beta_{ij} - \alpha_{ij}' \beta_{ij}' = 1$$

(see [3, §1, Prop. 1.3.12 or Algorithm 1.3.16]) and define the matrix

$$\xi_{ij} := \begin{bmatrix} I_{m-1} & \kappa_{ij} \beta_{ij}' \\ \alpha_{ij} & \alpha_{ij}' \end{bmatrix} \in GL_n(k).$$

By direct calculation it is easily verified that $\{\xi_{ij}\}_{j=1}^{h} \subset GL_n(k)$ is indeed an $(m, n)$-splitting set for $L_n(a_i)$ and thus fully represents $Q^{n,m}(k)\backslash GL_n(k)/\Gamma_n(a_i)$.

4 Fundamental domains of $GL_n(k)\backslash GL_n(A)^1$ and $P_n/\Gamma_i$

We will apply the general results of section 2 to $GL_n$, before proceeding to $P_n$. The matrices $\{\eta_i\}_{i=1}^{h}$ and $\{\xi_{ij}\}_{j=1}^{h}$ used from here on are the same ones chosen in the end of the previous section.

4.1 The height function

The height function associated to the parabolic subgroup $Q$ used in the previous section is given by

$$H_Q \left( \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} h \right) = |\det a|_{\mathbb{A}}^{-(n-m)/l} |\det d|_{\mathbb{A}}^{m/l} \quad (u \in U(\mathbb{A}), \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \in M(\mathbb{A}), h \in K)$$

where $l$ is the greatest common divisor of $n - m$ and $m$.

Definition. For each $\sigma \in \mathfrak{p}$ define $H_\sigma : \mathcal{O}_\sigma^n \to \mathbb{R}_{>0}$ by

$$H_\sigma \left( \sum_I a_I (e_{i_1} \wedge \cdots \wedge e_{i_m}) \right) = \left( \sup_I |a_I|_{\sigma} \right)^{k_{\sigma,n}/2} \quad (\sigma \in \mathfrak{p}_\mathbb{R}),$$

the sum and the supremum taken over all $I = \{i_1 < \cdots < i_m\} \subset \{1, \ldots, n\}$. We call this the **local height function** at $\sigma$. $H_\sigma$ can be extended to a function of $GL_n(\mathbb{k}_\sigma)$ by defining

$$H_\sigma(\gamma) = H_\sigma(\gamma e_1 \wedge \cdots \wedge \gamma e_m), \quad \gamma \in GL_n(\mathbb{k}_\sigma).$$
The following lemma allows us to express the height function $H_Q$ (restricted to $GL_n(A)^1$) in terms of these local heights.

**Lemma 7.**

$$H_Q(g) = \prod_{\sigma \in p} H_\sigma(g_\sigma^{-1})^{n/l}$$

for $g = (g_\sigma)_{\sigma \in p} \in GL_n(A)^1$.

We proceed to describe the sets $R_{i,j,\infty}$ using the matrices $\eta_i$ and $\xi_{ij}$ chosen at the end of the previous section. For the rest of this paper, for a square matrix $A$ with entries in $A$ or $k_\infty$, we will write $|A|$ and $|A|_{\infty}$ to denote $|\det A|$ and $|\det A|_{\infty}$ respectively. When the size of $A$ is at least $m$, we write $A^{[m]}$ for the top-left $m \times m$ submatrix of $A$, and use $|A|_{\infty}^{[m]}$ to denote $|A^{[m]}|_{\infty}$.

**Theorem 8.** Let $X_{ij}$ denote the $n \times m$ matrix formed by the first $m$ columns of $\xi_{ij}^{-1}$. Then

$$H_Q(\xi_{ij}^{-1}g\eta_i) = N(a_i)^{n/l} \left|^{t}\overline{X}_{ij}^{t}\overline{\gamma}^{-1t}\overline{g}^{-1}(\eta_i)^{-2}g^{-1}\gamma^{-1}X_{ij}\right|_{\infty}^{n/2l}$$

for any $1 \leq i, j \leq h$, $\gamma \in \Gamma_i$ and $g \in GL_n(k_\infty)^1$.

**Proof.** This can be proved by verifying that

- $\prod_{\sigma \in p_f} H_\sigma((\xi_{ij}^{-1}g\eta_i)_\sigma) = N(a_i)$ from our choices of $\eta_i$ and $\xi_{ij}$,
- $\prod_{\sigma \in p} H_\sigma((\xi_{ij}^{-1}g\eta_i)_\sigma) = \left|^{t}\overline{X}_{ij}^{t}\overline{\gamma}^{-1t}\overline{g}^{-1}(\eta_i)^{-2}g^{-1}\gamma^{-1}X_{ij}\right|_{\infty}$ which can be shown using the Cauchy-Binet formula.

The result follows from the previous lemma. $\square$

Now fix $1 \leq i, j \leq h$ and first consider the set $\xi_{ij}^{-1}R_{i,j,\infty}\xi_{ij}$. It is easy to directly verify that

$$\xi_{ij}^{-1}R_{i,j,\infty}\xi_{ij} = \{g \in G(k_\infty)^1 : H_Q(\xi_{ij}g\eta_i) = m_Q(g\eta_i)\}$$

Hence for $g \in \xi_{ij}^{-1}R_{i,j,\infty}\xi_{ij}$ we have

$$H_Q(\xi_{ij}g\eta_i) = m_Q(g\eta_i) = \min_{x \in G(k) \setminus GL_n(k)} H_Q(xg\eta_i) = \min_{1 \leq k \leq h} H_Q(\xi_{ik}g\eta_i)$$

which in this case can be written using (8) as

$$\left|^{t}\overline{X}_{ik}^{t}\overline{\gamma}^{-1t}\overline{g}^{-1}(\eta_i)^{-2}g^{-1}\gamma^{-1}X_{ik}\right|_{\infty} \leq N(a_k)^{2} \left|^{t}\overline{X}_{ik}^{t}\overline{\gamma}^{-1t}\overline{g}^{-1}(\eta_i)^{-2}g^{-1}\gamma^{-1}X_{ik}\right|_{\infty}$$

for all $k = 1, \ldots, h$ and $\gamma \in \Gamma_i$.

Now $^{t}\overline{X}_{ik}^{t}\overline{\gamma}^{-1t}\overline{g}^{-1}(\eta_i)^{-2}g^{-1}\gamma^{-1}X_{ik} = (^{t}\overline{\xi}_{ik}^{t}\overline{\gamma}^{-1t}\overline{g}^{-1}(\eta_i)^{-2}\gamma^{-1}X_{ik})^{[m]}$, which by letting $g_{[ij]} = \xi_{ij}g\xi_{ij}^{-1}$ can be rewritten as

$$\left|^{t}(\xi_{ij}^{-1}g_{[ij]}\xi_{ij}^{-1})^{[m]}\right|_{\infty} \leq N(a_k)^{2} \left|^{t}(\xi_{ij}^{-1}g_{[ij]}\xi_{ij}^{-1})^{[m]}\right|_{\infty}$$

This lets us express the set $R_{i,j,\infty}$ as follows. For $g \in GL_n(k_\infty)$ let $\pi_{ij}(g)$ denote $^{t}\overline{\gamma}^{-1t}(^{t}\overline{\xi}_{ij}^{-1}(\eta_i)^{-2}\gamma^{-1}X_{ik})g^{-1}$. Then $g \in R_{i,j,\infty}$ if and only if

$$|\pi_{ij}(g)|_{\infty}^{[m]} \leq N(a_k)^{2} \left|^{t}(\xi_{ij}^{-1}g_{[ij]}\xi_{ij}^{-1})^{[m]}\right|_{\infty}$$

for all $k = 1, \ldots, h$ and $\gamma \in \Gamma_i$. 

\[41\]
4.2 Fundamental domains of $P_n/\Gamma_i$

For each infinite place $\sigma$ of $k$ let $P_n(k_\sigma)$ denote the subset of $GL_n(k_\sigma)$ consisting of all positive definite real symmetric matrices when $\sigma$ is real and positive definite Hermitian matrices when $\sigma$ is imaginary. We consider the subset $P_n(k_\sigma)$ defined by $P_n = \prod_{\sigma \in P_m} P_n(k_\sigma)$. This is the space of positive definite Humbert forms over $k_\infty$.

We have the following right action of $GL_n(k_\infty)$ on $P_n$.

$$A \cdot g = \overline{g}A g \ (g \in GL_n(k_\infty), \ A \in P_n).$$

(10)

To determine fundamental domains in $P_n$ with respect to subgroups of $GL_n(k)$, we consider instead the induced action $A \cdot gZ = \overline{g}Ag$ of $GL_n(k)/Z$ on $P_n$. Here $Z = \{z \in k : z\bar{z} = 1\}$, the set of roots of unity in $k$. Here $\{zI_n : z \in Z\}$ is naturally seen to be the intersection of $K_\infty$ and the center of $GL_n(k)$.

Now for each $1 \leq i,j \leq h$, put

$$K_{i,j,\infty} = (\xi_{ij} \eta_{i})\infty (\eta_{i} \xi_{ij})^{-1}, \quad P_{n}^{ij} = \{A \in P_n : |A|_\infty = N(\kappa_{ij}a_i)^{-2}\},$$

and define the map $\pi_{ij} : G(k_\infty) \ni g \mapsto \overline{g}g^{-1}((\xi_{ij}^{-1} \eta_{i})_\infty^{-2} \xi_{ij}^{-1})^{-1} \in P_n$. Note that $K_{i,j,\infty}$ is the stabilizer of $(\xi_{ij}^{-1} \eta_{i})_\infty^{-2} \xi_{ij}^{-1}$ under the action of $GL_n(k_\infty)$ on $P_n$ and that $\pi_{ij}$ preserves this action. Thus the surjective map $\pi_{ij}$ gives us the isomorphisms

$$GL_n(k_\infty)/K_{i,j,\infty} \simeq P_n \quad \text{and} \quad GL_n(k_\infty)^1/K_{i,\infty} \simeq \pi_{ij}(GL_n(k_\infty)^1) = P_n^{ij}.$$ 

since $(\xi_{ij}^{-1} \eta_{i})_\infty^{-2} \xi_{ij}^{-1} = N(\kappa_{ij}a_i)^{-2}$.

Lastly let $F_{i,j}$ denote the following closed subset of $P_n$:

$$\{A \in P_n : |A|^{[m]}_\infty \leq \left(\frac{N(a_k)}{N(a_j)}\right)^2 |(\xi_{ij}^{-1} \eta_{i})_\infty^{-2} \xi_{ij}^{-1}|_\infty, 1 \leq k \leq h, \gamma \in \Gamma_i\}.$$ 

From (9), $\pi_{ij}$ maps $R_{i,j,\infty}$ onto $F_{i,j} \cap P_{n}^{ij}$. We also note that $F_{i,j}$ is right $Q_{i,j}$-invariant under the action (10).

Thus the subgroup $Q_{i,j}$ of $GL_n(k_\infty)$ acts on $R_{i,j,\infty}$ from the left and on $F_{i,j}$ from the right, and $\pi_{ij}$ preserves this. Hence by constructing a fundamental domain for $F_{i,j}/Q_{i,j}$, we can find one for $Q_{i,j} \setminus R_{i,j,\infty}$ by taking inverses under $\pi_{ij}$.

We start by observing that $\xi_{ij}^\Gamma_1 \xi_{ij}^{-1}$ is the stabilizer in $GL_n(k)$ of the $O$-lattice $\xi_{ij}L_i$ described in (7). This gives us the following expression for $Q_{i,j}$

$$Q_{i,j} = (k) \cap \xi_{ij}^\Gamma_1 \xi_{ij}^{-1}.$$ 

Any $A \in P_n$ can be written uniquely in the form

$$A = \begin{bmatrix} I_m & 0 \\ u_{A,m} & I_{n-m} \end{bmatrix} \begin{bmatrix} A^{[m]} & 0 \\ 0 & A_{[n-m]} \end{bmatrix} \begin{bmatrix} I_m & u_{A,m} \\ 0 & I_{n-m} \end{bmatrix}.$$ 

(11)

with $A^{[m]} \in P_m, A_{[n-m]} \in P_{n-m}$ and $u_{A,m} \in M_{m,n-m}(k_\infty)$. (The symbol $A^{[m]}$ here coincides with its prior use to denote the top left $m \times m$ submatrix of $A$). It is easy to verify that the action of $g = \begin{bmatrix} \overline{a} & b \\ 0 & d \end{bmatrix} \in Q_{i,j}$ on $A$ result in

$$(t^qAg)^{[m]} = t^qA^{[m]}a, \quad (t^qAg)_{[n-m]} = t^qA_{[n-m]}d,$$

$$u_{t^qAg,m} = a^{-1}(u_{A,m}d + b).$$

These equations will determine the necessary form of our fundamental domain.

For each $k = 1, \ldots, h$ choose sets $\partial_k, \partial'_k$ and $\partial_k$ that are fundamental domains for $k_\infty$ with respect to addition by $a_k, a_k^{-1}$ and $a_k a_k^{-1}$ respectively. We require each of these sets are closed under multiplication by $Z$. Then choose also a subset $\partial_{ik}$ of $\partial_k$ that is a fundamental domain for $\partial_k$ with respect to multiplication by $Z$. Also if necessary (which will be the case when $m > 1$ and $n - m > 1$) take a fundamental domain $\partial_O$ of $k_\infty$ with respect to addition by $O$. 
Using these, we define for $1 < i, j < h$ the sets

$$\mathcal{D}_{i,j} = \begin{cases} [d_{11}, \ldots, d_{1,n-m}] : d_{m,n-m} \in \delta_{ij}, & d_{rs} \in \begin{cases} \mathcal{D}_{\Omega} & r < m, s < n - m \\ \mathcal{D}_{\tau_j(i)} & r < m, s = n - m \\ \mathcal{D}_j & r = s = n - m \end{cases} \end{cases}.$$ 

By observing the action of $Q_{i,j}$ on $F_{i,j}$, we establish the following result.

**Theorem 9.** Let $\mathfrak{B}$ and $\mathfrak{C}$ be fundamental domains for $P_m/T_m(a_i)$ and $P_{n-m}/T_{n-m}(a_{\tau_j(i)})$ respectively. Then $F_{i,j}(\mathfrak{B}, \mathfrak{C}) = \{ A \in F_{i,j} : A^{[m]} \in \mathfrak{B}, A_{[n-m]} \in \mathfrak{C}, \tau_{A_{[m]}} \in \mathfrak{D}_{i,j} \}$ is a fundamental domain of $F_{i,j}/Q_{i,j}$.

As a result, the inverse image of $F_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\tau_j(i)}) \cap P_{n,j}^{ij}$ under $\pi_{ij}$ is a fundamental domain of $Q_{i,j} \backslash R_{i,j,\infty}$. Also, if we have fundamental domains $\mathfrak{B}_k$ for $P_m/T_m(a_k)$, as well as fundamental domains $\mathfrak{C}_k$ of $P_{n-m}/T_{n-m}(a_k)$ for each $k = 1, \ldots, h$, we can then construct the sets $F_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\tau_j(i)}) (1 \leq i, j \leq h)$. Then by Corollary 3 a fundamental domain for $GL_n(\mathfrak{k}) \backslash GL_n(A)^1$ is given by the set

$$\bigcup_{1 \leq i, j \leq h} \pi_{ij}^{-1}(F_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\tau_j(i)}) \cap P_{n,j}^{ij}) \xi_{ij} K_f.$$ 

Also Theorem 4 shows us that $\bigcup_{j=1}^{h} \pi_{ij}^{-1} F_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\tau_j(i)}) \cap P_{n,j}^{ij} \xi_{ij}$ is a fundamental domain for $GL_n(\mathfrak{k}_\infty)^1$ with respect to $\Gamma_i$.

Now let

$$\Omega_i(\mathfrak{B}_1, \ldots, \mathfrak{B}_h, \mathfrak{C}_1, \ldots, \mathfrak{C}_h) = \bigcup_{j=1}^{h} \xi_{ij}^{-1} F_{i,j}(\mathfrak{B}_j, \mathfrak{C}_{\tau_j(i)}) \xi_{ij}.$$ 

**Theorem 10.** $\Omega_i(\mathfrak{B}_1, \ldots, \mathfrak{B}_h, \mathfrak{C}_1, \ldots, \mathfrak{C}_h) \cap P_{n,j}^{ij}$ is a fundamental domain of $P_{n,j}^{ij}$ with respect to $\Gamma_i$. In addition, if we assume that each of the $\mathfrak{B}_k$ and $\mathfrak{C}_k$ are closed under positive multiplication (viewing $\mathbb{R}_{>0}$ as a subset of $\mathfrak{k}_\infty$ via the usual diagonal embedding), then

$$\mathbb{R}_{>0} \mathfrak{B}_k = \mathfrak{B}_k, \quad \mathbb{R}_{>0} \mathfrak{C}_k = \mathfrak{C}_k,$$

then $\Omega_i(\mathfrak{B}_1, \ldots, \mathfrak{B}_h, \mathfrak{C}_1, \ldots, \mathfrak{C}_h)$ is a fundamental domain of $P_n/\Gamma_i$.

Using the theorem, we can construct fundamental domains for $P_n$ with respect to $\Gamma_i$ for each $i$ and $n \geq 1$. Since $\Gamma_i = \mathbb{O}^*$ for any $i$ when $n = 1$, we can start by choosing a fixed fundamental domain, $\Omega^1$, for $P_1$ with respect to $\mathbb{O}^*/Z$ that is closed under multiplication by $\mathbb{R}_{>0}$ (The existence of such a set can be shown using Voronoï reduction, as demonstrated in the appendix of [7]). Then for each $i = 1, \ldots, h$, let $\Omega_i^n = \Omega_i^1$ and define

$$\Omega_i^n = \Omega_i^{n-1} \Omega_i^{n-1} \cdots \Omega_i^2 \Omega_i^1 \cdots \Omega_i$$

inductively for $n \geq 2$. By construction $\mathbb{R}_{>0} \Omega_i^n = \Omega_i^n$ so for each $1 \leq i \leq h$ and $n \geq 1$, $\Omega_i^n$ gives us a fundamental domain for $P_n/\Gamma_i$.

### 4.3 An example ($k = Q(\sqrt{-5})$)

When $k$ is an imaginary quadratic field, we have $k_\infty = \mathbb{C}$. For $n = 1$ we have $P_1 = \mathbb{R}_{>0}(\subset \mathbb{C})$ and $\Gamma_i = \mathbb{O}^* = \mathbb{Z}$ acts trivially on $P_1$; hence $P_1$ itself is a fundamental domain for $P_1/\Gamma_1(a_i)$.

Consider in particular $k = Q(\sqrt{-5})$ of class number $h = 2$. We can choose representatives $a_1, a_2$ for $Cl(k)$ by putting $a_1 = \mathcal{O}$ and $a_2 = (\sqrt{2}, 1 + \sqrt{-5})$. Then following the procedure at the end of section 4, we see that

$$a_1^2 = a_1, \quad a_2^2 = 2a_1, \quad (\tau_1 = (1, 2, 1), \quad \kappa_{11} = 1, \kappa_{12} = 2),$$

$$a_1a_2 = a_2, \quad a_2a_1 = a_2, \quad (\tau_2 = (1, 2, 1), \quad \kappa_{21} = \kappa_{22} = 1).$$

$(2, 1)$—splitting sets for $L_2(a_i)$ are given by

$$\begin{cases} \xi_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \xi_{12} = \begin{bmatrix} 2 & 2 + \sqrt{-5} \\ 2 & 3 + \sqrt{-5} \end{bmatrix} \quad (i = 1), \\
\xi_{21} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & \xi_{22} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (i = 2). \end{cases}$$
For $1 \leq i, j, k \leq 2$ denote by $\Xi_{i,j,k}$ the set of the first columns of the matrices $\xi_{ij} \gamma \xi_{ik}^{-1}$ as $\gamma$ ranges over $\Gamma(a_i)$. Then for $A \in P_2$

$$\min_{\gamma \in \Gamma_i} \| (\xi_{ij} \gamma \xi_{ik}^{-1}) A (\xi_{ij} \gamma \xi_{ik}^{-1}) \|_2 = \min_{\gamma \in \Xi_{i,j,k}} \| A \mathbf{x} \|_2$$

and so $F_{i,j}^{2,1}$ can be expressed as

$$F_{i,j}^{2,1} = \left\{ \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} : \frac{b, c \in \mathbb{R}_{>0}, \ d \in \mathbb{C}}{\| e + d f \|^2 + \frac{c}{d} |f|^2 \geq 1, \ [f] \in \frac{1}{N(a_j)} \Xi_{i,j,1} \cup \frac{1}{-i,j,2} \Xi_{i,j,2}} \right\}.$$

Now for $\alpha, \beta \in k$ let $\mathcal{O}(\alpha, \beta) = \{ x + y \beta : -1/2 < x, y \leq 1/2 \}$. When $\alpha$ and $\beta$ generate a fractional ideal $\mathfrak{a}$, $\mathcal{O}(\alpha, \beta)$ is a fundamental domain for $\mathbb{C}$ with respect to addition by $\mathfrak{a}$. Also if we let $\mathcal{O}(\alpha, \beta)$ denote the subset of $\mathcal{O}(\alpha, \beta)$ where the range of $y$ is restricted to $0 \leq y \leq 1/2$, this gives us a fundamental domain for $\mathcal{O}(\alpha, \beta)$ with respect to multiplication by $Z = \{ \pm 1 \}$.

In particular $\mathcal{O}(1, \sqrt{-5}), \mathcal{O}(2, 1 + \sqrt{-5}), \mathcal{O}(1, \frac{1-\sqrt{-5}}{2})$ are fundamental domains for $\mathbb{C}$ with respect to addition by $O, a_2$ and $a_2^{-1}$ respectively, and we can put $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_{12} = \tilde{\mathcal{O}}(1, \sqrt{-5})$, $\tilde{\mathcal{O}}_{21} = \tilde{\mathcal{O}}(1, \frac{1-\sqrt{-5}}{2})$ and $\tilde{\mathcal{O}}_{22} = \tilde{\mathcal{O}}(2, \sqrt{-5})$. Then

$$F_{i,j}^{2,1}(P_1, P_1) = \left\{ \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix} \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} : \frac{b, c \in \mathbb{R}_{>0}, \ d \in \tilde{\mathcal{O}}_{ij}}{\| e + d f \|^2 + \frac{c}{d} |f|^2 \geq 1, \ [f] \in \frac{1}{N(a_j)} \Xi_{i,j,1} \cup \frac{2}{N(a_j)} \Xi_{i,j,2}} \right\}.$$

Writing $F_{i,j}^{2,1}(P_2, P_2)$ as $F_{ij}$ for short, we obtain the fundamental domains $\Omega_1^2 = F_{1,1} \cup \xi_{12} F_{1,2} \xi_{12}$ for $P_2/\Gamma_2(a_1)$ and $\Omega_2^2 = F_{1,1} \cup \xi_{21} F_{2,2} \xi_{22}$ for $P_2/\Gamma_2(a_2)$.

References


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