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ON THE SIEGEL SERIES AND INTERSECTION NUMBER

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ABSTRACT. This is a survey article to explain a main result of the author's recent preprint (joint work with T. Yamauchi) [CY].

CONTENTS

1. INTRODUCTION

A modular polynomial is a classical object to study in number theory, starting from Kronecker and Hurwitz. It also determines an affine model of a modular curve $Y_0(m)$. The degree of this polynomial was computed by Hurwitz in [Hur1885]. Gross and Keating in [GK93] reinterpreted it in terms of certain intersection number of a modular curve and a vertical line in the product of $Y_0(1)\times Y_0(1)$. In addition, they observed that this intersection number coincides with the Fourier coefficients of the Siegel-Eisenstein series of weight 2 and degree 1. They also generalized this observation into more general setting, by computing the intersection number of two modular curves inside $Y_0(1) \times Y_0(1)$ and by identifying it with the Fourier coefficients of the Siegel-Eisenstein series of weight 2 and degree 2.

Furthermore, motivated by the fact that the coefficients of a modular polynomial are integers, they computed the arithmetic intersection number of three of integral models of modular curves inside an integral model of $Y_0(1) \times Y_0(1)$. Remarkably, it was described purely in terms of certain invariant of an integral ternary quadratic form invented by themselves. This invariant is nowadays called the Gross-Keating invariant.

Gross and Keating already expected in the introduction of their paper that this arithmetic intersection number seems to coincide with the derivative of the Fourier coefficients of the Siegel-Eisenstein series of weight 2 and degree 3. It was confirmed by Kudla in his paper [Kud97].

Kudla, Kudla-Rapoport, and Kudla-Rapoport-Yang have extensively studied the relation between arithmetic intersection numbers of special cycles on $\mathrm{GSpin}(n,2)$ Shimura varieties and the derivative of the Fourier coefficients of the Siegel-Eisenstein series of weight 2 (or 3/2) and degree $n+1$ in a series of papers [Kud97], [KR1], [KR2], [KRY1], and [KRY2], with restriction $n \leq 3$. A method of comparison between them used in the above works is to compute both sides precisely and then compare directly.

A main difficulty in these works is the computation of the local intersection multiplicities at a given closed point. Their idea to compute the intersection multiplicites in all the works listed

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above is first to reduce the problems to the case of supersingular elliptic curves. Then they used the result of Gross and Keating in [GK93].

On the other hand, the Siegel series is a local factor of the Fourier coefficient of the Siegel-Eisenstein series. Its explicit form, over $\mathbb{Z}_p$ with any p, was obtained by Katsurada in [Kat99]. But, His formula was based on certain symmetric matrix and so it was not clear which invariants of a quadratic form determine the Siegel series. It was Ikeda and Katsurada who found the answer to this question in [IK2]. They showed that the Siegel series is completely determined by the (extended) Gross-Keating datum.

A main contribution of [CY] (joint work with Yamauchi) is to study the Siegel series conceptually and to find that both intersection multiplicity side and the Siegel series side have the same inherent structures. This yields a new identity between the mod p intersection number on the special fiber of Gross-Keating's setting and the Fourier coefficients of the Siegel-Eisenstein series of weight 2 and degree 2, which is independent of the characteristic of a finite field (at least with $p > 2$).

In this article, we will survey a main result of [CY]. The organization of this article is as follows. In Section 2, we explain Gross-Keating's result of [GK93]. In Section 3, we explain Ikeda-Katsurada's work of [IK2]. In Section 4, we explain a main result of [CY].

2. Gross-Keating's intersection number

We first recall the definition of a modular polynomial and its properties.

Definition 2.1 (The second paragraph of page 14 in [Vog07]). A modular polynomial of degree m for $m \in \mathbb{Z}_{>0}$ is a complex-valued function with two variables defined as follows:

$$\phi_m : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}, (j, j') \mapsto \prod_{\tilde{E}' \to E'} (j - j(\tilde{E}')),$$

where E' is an isomorphic class of ellpitic curves over $\mathbb{C}$ whose j-invariant is j' and $\tilde{E}'$ runs over all isomorphic classes over $\mathbb{C}$ having an isogeny to E' of degree m.

Remark 2.2 (Theorem 1.2 in [Vog07]). A modular polynomial ϕ_m has important properties listed below:

(1) ϕ_m is a polynomial with two variables having coefficients in $\mathbb{Z}$ so that

$$\phi_m \in \mathbb{Z}[j, j'].$$

(2) $\phi_m(j, j') = \pm\phi_m(j', j)$, ("$-$" precisely if m is a square).

Due to (2) of the above remark, it is natural to consider the 'degree' of a polynomial ϕ_m when the second variable j' (or the first variable j) is fixed by a nonzero constant. The degree of a polynomial $\phi_m(j, c)$ for a nonzero value c is computed by Hurwitz in 19^{th} century.

On the other hand, it can be interpreted in terms of the intersection number of a modular curve (which is the solution set of $\phi_m = 0$) and the horizontal line (which is the solution set of $j' = c$) in $\mathbb{C}^2$. Note that this is the same as the dimension of $\mathbb{C}[j, j']/(\phi_m, j' - c)$ as a vector space over $\mathbb{C}$. Using the idea of Hurwitz's formula on the degree of $\phi_m(j, j)$ in [Hur1885], Gross and Keating further computed the intersection number of two curves given by $\phi_m = 0$ and $\phi_{m'} = 0$, when mm' is not a perfect square, which is also the same as the dimension of $\mathbb{C}[j, j']/(\phi_m, \phi_{m'})$ as a vector space over $\mathbb{C}$. It turns out to be

$$c \cdot \sum_{\substack{M>0 \\ \mathrm{diag}M=(m_1,m_2)}} c(M),$$

where $c(M)$ is the Fourier coefficient of the Siegel-Eisenstein series of weight 2 and degree 2 for a positive definite symmetric matrix M of size 2×2, by Proposition 2.4 of [GK93].

Since all coefficients of $\phi_m \in \mathbb{Z}[j, j']$ are in $\mathbb{Z}$, it is natural to consider the integral (or arithmetic) version of the above discussion, namely $\log \#\mathbb{Z}[j, j']/(\phi_{m_1}, \phi_{m_2}, \phi_{m_3})$. This question can also be formulated in terms of an arithmetic intersection number as follows.

Let $\mathcal{M}$ be the moduli stack over $\mathbb{Z}$ of elliptic curves and let $\mathcal{T}_m$ be the moduli stack over $\mathbb{Z}$ of isogenies of elliptic curves of degree m. These are Deligne-Mumford stacks.

Let $\mathcal{S} = \mathcal{M} \times_{\mathbb{Z}} \mathcal{M}$. Then the arithmetic intersection of three 'divisors' $\mathcal{T}_{m_i}$ is formulated as

$$\mathcal{X} = \mathcal{T}_{m_1} \times_{\mathcal{S}} \mathcal{T}_{m_2} \times_{\mathcal{S}} \mathcal{T}_{m_3}.$$

Proposition 2.3 (Proposition 2.3 in [Gör07]). *The arithmetic intersection number of $\mathcal{X}$ is defined by the following identity:*

$$\log \#\mathbb{Z}[j, j']/(\phi_{m_1}, \phi_{m_2}, \phi_{m_3}) = \sum_p \log(p) \cdot \sum_{x \in \mathcal{X}(\bar{F}_p)} \frac{1}{\#\mathrm{Aut}(x)} \lg \hat{\mathcal{O}}_{\mathcal{X},x}.$$

Here, $\hat{\mathcal{O}}_{\mathcal{X},x}$ is the completion of the stalk of $\mathcal{X}$ at a given closed point x and $\lg \hat{\mathcal{O}}_{\mathcal{X},x}$ is the length of the ring $\hat{\mathcal{O}}_{\mathcal{X},x}$.

If $\mathcal{X}$ is of dimension 0 so that the arithmetic intersection number is finite, then the support of $\mathcal{X}$ is contained in the locus of pairs of supersingular elliptic curves, whose dimension is 0. Gross and Keating found exactly when the intersection is of dimension 0 in terms of certain properties of a ternary quadratic form. For more discussion, we refer to Proposition 3.2 of [Gör07].

From now on, we assume that $\mathcal{X}$ is of dimension 0. A closed point of $\mathcal{X}(\bar{F}_p)$ is of the form (E, E', f_1, f_2, f_3), where E and E' are supersingular elliptic curves over $\bar{F}_p$ and $f_i : E \to E'$ is an isogeny of degree m_i. In the formula of the arithmetic intersection number given in Proposition 2.3, a major difficulty is the computation of $\lg \hat{\mathcal{O}}_{\mathcal{X},x}$, which is the intersection multiplicity at x. This can be formulated in terms of a deformation problem, summarized below.

Let W be the ring of Witt's vectors for $\bar{F}_p$ so that W can be considered as the completion of the maximal unramified extension of $\mathbb{Z}_p$. We consider a deformation functor Def_x defined on the category of local Artin W-algebras such that $\mathrm{Def}_x(R)$ is the set of (isomorphism classes of) three isogenies of a pair of abelian schemes over R, whose reduction to $\bar{F}_p$ is the same as the given x. This deformation functor is represented by a local Artinian ring $\hat{\mathcal{O}}_{\mathcal{X},x}$ since the dimension of $\mathcal{X}$ is 0.

On the other hand, three isogenies (f_1, f_2, f_3) of a point $x \in \mathcal{X}(\bar{F}_p)$ define an integral ternary quadratic form

$$Q : \mathbb{Z}^3 \longrightarrow \mathbb{Z}, (a, b, c) \mapsto \deg(af_1 + bf_2 + cf_3).$$

Gross and Keating defined certain invariant of a ternary quadratic form $Q \otimes \mathbb{Z}_p$ over $\mathbb{Z}_p$, which is called the Gross-Keating invariant (cf. Definition 3.2). Then the intersection multiplicity $\lg \hat{\mathcal{O}}_{\mathcal{X},x}$, computed by Gross and Keating, is described purely in terms of the Gross-Keating invariant by Proposition 5.4 of [GK93].

3. Ikeda-Katsurada's result about the Siegel series

From Kudla's insight, the local intersection multiplicities are related to the derivative of the Siegel series. In particular, in the Gross-Keating's setting, the intersection multiplicities are completely determined by the Gross-Keating invariant of a ternary quadratic form. Thus, it is natural to ask whether or not the Siegel series in the general case can be determined by the Gross-Keating invariant. It is Ikeda and Katsurada who had the entire answer to this question in their recent preprints [IK1] and [IK2]. In this section, we briefly explain their result.

Let A be the ring of integers of a finite field extensoin of $\mathbb{Q}_p$ for any prime p. Let π be a uniformizer of A and let f be the cardinality of the residue field $A/(\pi)$.

Let B be a non-degenerate half-integral symmetric matrix over A of size $n \times n$. Here by an half-integral matrix over A, we mean that each non-diagonal entry multiplied by 2 and each diagonal entry of B are in A.

We first give the definition of the Siegel series.

Definition 3.1 (Definitions 4.1 and 4.10 in [CY]). Let (L, q_L) be a quadratic A-lattice and let H be the hyperbolic plane whose associated symmetric matrix is $\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$. Let H_k be the k-copies of H.

(1) The local density associated to the pair of two quadratic lattices L and H_k, denoted by $\alpha(L, H_k)$, is

$$\alpha(L, H_k) = \lim_{N \to \infty} f^{-N \ dim \mathrm{O}_{A\otimes\mathbb{Q}_p}(L\otimes\mathbb{Q}_p, H_k\otimes\mathbb{Q}_p)} \#\mathrm{O}_A(L, H_k)(A/\pi^N A).$$

Here, $\mathrm{O}_A(L, H_k)(A/\pi^N A)$ is the set of linear maps from $L \otimes A/\pi^N A$ to $H_k \otimes A/\pi^N A$ preserving the associated quadratic forms.

(2) For a given quadratic lattice (L, q_L), the Siegel series is defined to be the polynomial $\mathcal{F}_L(X)$ of X such that

$$\mathcal{F}_L(f^{-k}) = \alpha(L, H_k),$$

where $k/2 \geq$ *the rank of* L.

We will define the Gross-Keating invariant for B below. The definition is taken from [IK1].

Definition 3.2 (Definitions 0.1 and 0.2 in [IK1]). (1) Let $B = (b_{ij})$ be a non-degenerate half-integral symmetric matrix over A of size $n \times n$. Let $S(B)$ be the set of all non-decreasing sequences $(a_1, \cdots, a_n) \in \mathbb{Z}^n_{\geq 0}$ such that

$$\begin{array}{ll} \mathrm{ord}(b_{ii}) \geq a_i & (1 \leq i \leq n); \\ \mathrm{ord}(2b_{ij}) \geq (a_i + a_j)/2 & (1 \leq i \leq j \leq n). \end{array}$$

Put

$$\mathbf{S}(\{B\}) = \bigcup_{U \in \mathrm{GL}_n(\mathfrak{o})} S(B[U]).$$

The Gross-Keating invariant $\mathrm{GK}(B)$ of B is the greatest element of $\mathbf{S}(\{B\})$ with respect to the lexicographic order $\succeq$ on $\mathbb{Z}^n_{\geq 0}$. Here, the lexicographic order $\succeq$ on $\mathbb{Z}^n_{\geq 0}$ is the following (cf. the paragraph following Definition 0.1 of [IK1]). Choose two elements $(a_1, \cdots, a_n)$ and $(b_1, \cdots, b_n)$ in $\mathbb{Z}^n_{\geq 0}$. Let i be the first integer over which a_i differs from b_i (so that $a_j = b_j$ for any $j < i$). If $a_i > b_i$, then we say that $(a_1, \cdots, a_n) \succ (b_1, \cdots, b_n)$. Otherwise, we say that $(a_1, \cdots, a_n) \prec (b_1, \cdots, b_n)$.

(2) The symmetric matrix B is called *optimal* if $\mathrm{GK}(B) \in S(B)$.

(3) If B is a symmetric matrix associated to a quadratic lattice (L, q_L), then $\mathrm{GK}(L)$, called the Gross-Keating invariant of (L, q_L), is defined by $\mathrm{GK}(B)$. $\mathrm{GK}(L)$ is independent of the choice of the matrix B.

Ikeda and Katsurada imposed more conditions in addition to $\mathrm{GK}(L)$ and call it *the extended Gross-Keating datum*, denoted by $\mathrm{EGK}(L)$. Note that $\mathrm{GK}(L) = \mathrm{EGK}(L)$ if B is anisotropic and $A = \mathbb{Z}_p$. Thus for our purpose, $\mathrm{GK}(L)$ is enough to study a connection between local intersection multiplicities of [GK93] and the Siegel series. We refer to [IK2] for a precise definition of $\mathrm{EGK}(L)$. The main result of loc. cit. is the following theorem:

Theorem 3.3 (Theorem 1.1 in [IK2]). *The Siegel series $\mathcal{F}_L(X)$ is completely determined by* $\mathrm{EGK}(L)$. *In other words, for given two quadratic lattices (L, q_L) and (M, q_M),*

$$\mathcal{F}_L(X) = \mathcal{F}_M(X) \text{ if and only if } \mathrm{EGK}(L) = \mathrm{EGK}(M).$$

4. Main results of [CY]

4.1. **Siegel series and intersection multiplicities.** Ikeda and Katsurada introduced so-called 'reduced form' associated to B in [IK1] and showed that it is optimal. For a detailed definition of a reduced form, we refer to Section 3 of [IK1].

Let L be a quadratic lattice. We choose a basis $(e_1, \cdots, e_n)$ of a reduced form of L. Let l be the integer such that $a_l < a_{l+1} = \cdots = a_n$. We denote the lattice ($\supset L$) having basis $(e_1, \cdots, e_l, \frac{1}{\pi}\cdot e_{l+1}, \cdots, \frac{1}{\pi}\cdot e_n)$ by

$$\begin{cases} L^{(l+1,n)} & \textit{if } l < n-1; \\ L^{(n)} \ (\textit{or } L^{(n,n)}) & \textit{if } l = n-1. \end{cases}$$

Let $\mathcal{G}_{L,l,d}$ be the set of sublattices of $L^{(l+1,n)}$ containing L such that the dimension L'/L as a vector space is d.

In [CY], we first formulate the local density as a weighted sum of certain 'primitive local density' for an integral quadratic lattice L' containing L (cf. Equation (4.2) in [CY]). Then using scheme theory about smoothness, we show that the primitive local density for L' is (roughly) determined by two ingredients: the degree $[L' : L]$ and the dimension of a nonsingular part of L' modulo π (cf. Theorem 4.7 and the paragraph just before Proposition 4.8 in [CY]). This yields a description of the Siegel series in terms of certain 'weighted' lattice counting problem (cf. Theorem 4.9 and Remark 4.11 of [CY]).

By combining the above with Theorem 3.3, we obtain the following inductive formula:

Theorem 4.1 (Theorems 5.5 and 6.8 in [CY]). *Assume that $L^{(l+1,n)}$ is an integral quadratic lattice. Then we have the following inductive formula, with respect to the Gross-Keating invariant, of the Siegel series $\mathcal{F}_L(X)$:*

(1) If $l = n-1$, then

$$\mathcal{F}_L(X) = f^{n+1}\cdot X^2 \cdot \mathcal{F}_{L^{(n)}}(X) + (1-X)(1+fX)\cdot \mathcal{F}_{L_0^{(n)}}(fX).$$

(2) If $l < n-1$, then

$$\mathcal{F}_L(X) = f^{n+1}\cdot X^2 \cdot \sum_{L'\in\mathcal{G}_{L,l,1}} \mathcal{F}_{L'}(X) - f^{2n+3}\cdot X^4 \cdot \sum_{L''\in\mathcal{G}_{L,l,2}} \mathcal{F}_{L''}(X) +$$
$$(1-X)(1-f^{n-l}X)^{-1}\cdot\left(\prod_{i=1}^{n-l}(1-f^{2i}X^2)\right)\cdot \mathcal{F}_{L_0^{(l+1,n)}}(f^{n-l}X).$$

(3) Let (L, q_L) be an anisotropic quadratic lattice over $\mathbb{Z}_p$ of rank n. Then

$$\mathcal{F}_L(X) = \begin{cases} p^{n+1}\cdot X^2\cdot \mathcal{F}_{L^{(n)}}(X) + (1-X)(1+pX)\cdot\mathcal{F}_{L_0^{(n)}}(pX) & \textit{if } 2\le n\le 4; \\ p^2\cdot X^2\cdot\mathcal{F}_{L^{(1)}}(X) + (1-X)(1+pX) & \textit{if } n = 1. \end{cases}$$

As mentioned at the end of Section 2, the local intersection multiplicity $\lg \hat{\mathcal{O}}_{\mathcal{X},x}$ is described in terms of the Gross-Keating invariant by Proposition 5.4 of [GK93]. Their formula is based on an inductive formula which will be stated below. Let (L, q_L) be an anisotropic ternary quadratic lattice over $\mathbb{Z}_p$ and let $\mathrm{GK}(L) = (a_1, a_2, a_3)$ with a reduced basis (e_1, e_2, e_3). Let $L^{(3)}$ be the sublattice spanned by $(e_1, e_2, 1/p\cdot e_3)$ and let M be the sublattice spanned by (e_1, e_2).

Let $\alpha_p(a_1, a_2, a_3)$ be the local intersection multiplicity associated to a quadratic lattice (L, q_L) in the setting of Gross-Keating. Then it satisfies the following inductive formula (cf. Lemma 5.6 of [GK93] or Proposition 1.6 of [Rap07]):

$$\alpha_p(a_1,a_2,a_3) = \alpha_p(a_1,a_2,a_3-2) + \mathcal{T}_{a_1,a_2}. \tag{4.1}$$

Here, $\mathcal{T}_{a_1,a_2}$ is the local intersection multiplicity of two cycles at the special fiber, associated to an anisotropic binary quadratic lattice M with $\mathrm{GK}(M) = (a_1, a_2)$.

On the other hand, by differentiating the formula of Theorem 4.1.(3), we have

$$\mathcal{F}_L'(1/p^2) = \mathcal{F}_{L^{(3)}}'(1/p^2) + (1-\frac{1}{p^2})(p+1)\cdot\mathcal{F}_M'(1/p). \tag{4.2}$$

We can now compare two inductive formulas Equation (4.1) and Equation (4.2). Remarkably, these two are equal. More precisely, we obtain the following theorem.

Theorem 4.2 (Theorems 7.7 and 7.8 in [CY]). *The two inductive formulas Equations 4.1 and 4.2 do match each other. As a direct consequence, we have*

$$\begin{cases} \alpha_p(a_1, a_2, a_3) = c_1 \cdot \mathcal{F}'_L(1/p^2); \\ \mathcal{T}_{a_1,a_2} = c_2 \cdot \mathcal{F}'_M(1/p), \end{cases}$$

for explicitly computable constants c_1 and c_2.

Thus, in addition to matching two values of the intersection multiplicity and the Siegel series, we further find an analogy of inherent structures by matching two inductive formulas of both sides.

4.2. **Applications.** Theorem 4.2 yields two major applications. The first application is to compute the intersection number of two modular correspondences on the special fiber.

In the Gross-Keating setting, since we computed the intersection multiplicity of the intersection of two cycles on the special fiber, it is natural to consider the intersection number of two cycles, which is defined by

$$(\mathcal{T}_{m_1,p}, \mathcal{T}_{m_2,p}) := \mathrm{length}_{\mathbb{F}_p}\mathbb{F}_p[x,y]/(\phi_{m_1}, \phi_{m_2}). \tag{4.3}$$

Then we have the following result on the intersection number:

Theorem 4.3 (Theorem 8.2 or Theorem 1.3 in [CY]). *We assume that p is odd. Then for any two positive integers m_1, m_2 such that m_1m_2 is not a square, we have that*

$$(\mathcal{T}_{m_1,p}, \mathcal{T}_{m_2,p}) = \frac{1}{288} \sum_{\substack{T \in \mathrm{Sym}_2(\mathbb{Z})_{>0} \\ \mathrm{diag}(T)=(m_1,m_2)}} c(T) = (\mathcal{T}_{m_1,\mathbb{C}}, \mathcal{T}_{m_2,\mathbb{C}}).$$

Here, $c(T)$ is the Fourier coefficient of the Siegel-Eisenstein series for $\mathrm{Sp}_4(\mathbb{Q})$ of weight 2 with respect to the (2×2)- half-integral symmetric matrix T.

Here, the intersection number on the special fiber is independent of an odd prime p.

Since we obtain the interpretation of the local intersection multiplicites on the special fiber of the Gross-Keating's setting in terms of the Siegel series, we can also describe the local intersection multiplicities on the special fiber of $\mathrm{GSpin}(n, 2)$ Shimura varieties with $n \leq 3$ in terms of the Seigel series. The second application is the following theorem:

Theorem 4.4 (Theorem 1.4 or Theorem 2.3 in [CY]). *The local intersection multiplicities on the special fiber of $\mathrm{GSpin}(n, 2)$ Shimura varieties with $n \leq 3$ can be described in terms of a suitable (derivative of) Siegel series.*

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