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Some properties of Siegel Eisenstein series for paramodular groups

by
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Abstract

We show that Siegel Eisenstein series for paramodular groups with squarefree polarization matrix have many properties in common with ordinary Eisenstein series for the full Siegel modular group; we discuss among other things Siegel's main theorem, the problem of an explicit pullback formula and the basis problem.

1 Introduction

In the literature, one cannot find so many papers on Siegel modular forms for paramodular groups of general degree. Among the older works for general degree we mention [3, 4, 8]. Most recent papers and the book [7] have their focus on degree 2. We report here on our attempts to solve the basis problem for the paramodular case, following [1]. To do this, we have to introduce appropriate theta series and we have to consider pullback formulas. We get a smooth answer if the polarization matrix has squarefree entries and the weight of the cusp form is sufficiently large. Some problems arise from Hecke operators for primes appearing in the polarization matrix. Details will be given in an article under preparation.

2 Preliminaries

2.1 The matrices considered

We start from an invertible real matrix $\mathcal{P}$ of size m and consider

$$Sp_m(\mathcal{P},\mathbb{R}) := \{g \in GL_{2m}(\mathbb{R}) \mid g^t \cdot \begin{pmatrix} 0 & -\mathcal{P}^t \\ \mathcal{P} & 0 \end{pmatrix} \cdot g = \begin{pmatrix} 0 & -\mathcal{P}^t \\ \mathcal{P} & 0 \end{pmatrix}\}$$

This group is conjugate (inside $GL(2m,\mathbb{R})$) to the usual symplectic group by

$$g \longmapsto \iota_{\mathcal{P}}(g) := \begin{pmatrix} \mathcal{P} & 0 \\ 0 & 1 \end{pmatrix} \cdot g \begin{pmatrix} \mathcal{P}^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

If $\mathcal{P}$ is integral, then the group $Sp_m(\mathcal{P},\mathbb{Z})$ is the "Stufengruppe" in the sense of Siegel [8]. By a further conjugation, we may change $\mathcal{P}$ to $U\cdot\mathcal{P}\cdot V$ with $U,V\in GL_m$, in particular, we may assume that $\mathcal{P}$ (if integral) can be changed into a matrix in elementary divisor form.
The group $Sp_m(\mathcal{P},\mathbb{R})$ then acts on Siegel's upper half space $\mathbb{H}_m$ via the conjugation $\iota_{\mathcal{P}}$. It acts in the same way on functions f on $\mathbb{H}_m$ by

$$(f,g)\longmapsto f\mid_k \iota_{\mathcal{P}}(g),$$

where $|_k$ is the usual stroke operator of weight k.

2.2 Paramodular forms

From now on we assume that $\mathcal{P}$ is integral and diagonal and we write $\widehat{\Gamma}(\mathcal{P})$ for $Sp_m(\mathcal{P},\mathbb{Z})$; the group inside Sp_m corresponding to $\widehat{\Gamma}$ via ι will be denoted by $\Gamma(\mathcal{P})^{\iota}$. The space $\mathcal{M}_k^m(\mathcal{P})$ of paramodular forms for $\mathcal{P}$ and weight k is then the space of all holomorphic functions $f:\mathbb{H}_m\longrightarrow\mathbb{C}$ satisfying $f\mid_k\iota(\gamma)=f$ for all $\gamma\in\widehat{\Gamma}(\mathcal{P})$ (plus some growth condition if $m=1$). The subspace of cusp forms will be denoted by $\mathcal{S}_k^m(\mathcal{P})$.
Note that the paramodular group considered frequently in the literature is not Siegel's Stufengruppe $\widehat{\Gamma}(\mathcal{P})$, but another group conjugate to it, namely

$$\Gamma(\mathcal{P}):=\begin{pmatrix}1&0\\0&\mathcal{P}\end{pmatrix}\widehat{\Gamma}\begin{pmatrix}1&0\\0&\mathcal{P}^{-1}\end{pmatrix}=\begin{pmatrix}\mathcal{P}^{-1}&0\\0&\mathcal{P}\end{pmatrix}\cdot\Gamma(\mathcal{P})^{\iota}\cdot\begin{pmatrix}\mathcal{P}&0\\0&\mathcal{P}^{-1}\end{pmatrix}.$$

3 Siegel Eisenstein series

3.1 The cusps

The inequivalent zero dimensional cusps for a subgroup Γ inside $Sp_m(\mathbb{Q})$ are parametrized by double cosets

$$Sp(m,\mathbb{Q})_\infty\backslash Sp(m,\mathbb{Q})/\Gamma.$$

We call $\mathcal{P}$ squarefree if it is integral and all its elementary divisors are squarefree.

Proposition *Assume that $\mathcal{P}$ is squarefree. Then the number of inequivalent cusps for $\Gamma(\mathcal{P})^{\iota}$ is one.*

Remarks: The situation is different for higher-dimensional cusps, in particular, there will be several inequivalent ϕ-operators (and Klingen-Eisenstein series) even in the squarefree case.

3.2 Siegel's Eisenstein series

The Eisenstein series considered by Siegel [8] can be viewed as the one attached to the cusp "infinity" in the case of general integral $\mathcal{P}$ and for squarefree $\mathcal{P}$ it is then the unique Eisenstein series for $\widehat{\Gamma}(\mathcal{P})$:

$$E(\mathcal{P})(Z) := \sum_{\gamma} 1 \mid_k \gamma = \sum_{C,D} \det(C\mathcal{P}^{-1}Z + D)^{-k}.$$

Here γ runs over $\Gamma(\mathcal{P})^{\iota}_{\infty}\backslash\Gamma(\mathcal{P})^{\iota}$ or -equivalently - (C, D) runs over non-associated second rows of matrices in $\widehat{\Gamma}(\mathcal{P})$. Siegel [8] computed the (rational) Fourier coefficients of these Eisenstein series ($k > m + 1$).

4 The doubling method

4.1 The integral in general

We try to follow the techniques from [1]: We start from elementary divisor matrices S and T of size n' and n We put $\mathcal{P} = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$. For any $F \in \mathcal{M}_k(\mathcal{P})$ we define an element of $\mathcal{M}_k(S) \otimes \mathcal{M}_k(T)$ by

$$(z, w) \longmapsto F(\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}) \qquad (z \in \mathbb{H}_{n'}, w \in \mathbb{H}_n).$$

In particular, we can study the map

$$\Lambda_n^{n'} : \left\{ \begin{array}{ccc} \mathcal{S}_k^n(T) & \longrightarrow & \mathcal{M}_k^{n'}(S) \\ f & \longmapsto & z \longmapsto < f, E(\mathcal{P})(\begin{pmatrix} -\bar{z} & 0 \\ 0 & * \end{pmatrix}) > \end{array} \right. ,$$

where $<,>$ is the usual Petersson inner product on $\mathcal{S}_k^n(T)$.
We mention that in this general context we always have $\Lambda_n^{n'} = 0$ if $n' < n$ and Λ_n^n maps cusp forms to cusp forms. For the application we have in mind we have to investigate the injectivity of Λ_n^n.

4.2 Injectivity

We focus on the case $n = n'$ and $S = T$ squarefree. The map $\Lambda := \Lambda_n^{n'}$ is hermitian w.r.t. $<,>$, we may therefore assume that f is an eigenform of Λ. Furthermore, generalizing the techniques from [1] and relying heavily on arguments from [8], we may unfold the integral to arrive at an expression

$$\Lambda(f) \sim \sum_{A,B,D} f \mid_k \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mu(A,B,D)^{-k}, \tag{1}$$

where A, B, D run over explictly given rational matrices and $\mu(A,B,D)$ is given essentially given by elementary divisors.
In [1, 2] we showed (for the trivial polarization $T = 1_n$) how this infinite sum - for a Hecke eigenform f- can be expressed by a value of the standard L-function attached to f. In the more general case at hand, we do not understand the Hecke algebra well enough to get a similar result. Naturally, this is a problem of local nature, i.e. for the primes p occuring in the polarization matrix T.
Without such knowledge of the Hecke algebra, we can only show a weak result at the moment:

Proposition: *For k large enough, the map Λ is bijective.*
We only have to show that Λ is injective. For the proof (inspired by [5]), we observe that the Petersson product $< f, f \mid_k \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} >$ can be estimated by $< f, f >$ itself. Then we obtain from (1)

$$< \Lambda(f), f >= (1 + X) \cdot < f, f >$$

where X can be estimated by

$$\left(\sum{}^{*} \mu(A,B,D)^{-k}\right)$$

and where $*$ indicates that we omit the summand $A = D = 1_n, B = 0$. All we have to show is that the absolute value of this sum is smaller than 1; we are left with a counting problem for the number of A, B, D occuring in (1).

Remark: One can make the bound for k more explicit by a refinement of the consideration above (i.e. split off the contribution of primes away from T); such an explicit bound however will then depend on the primes occuring in T).

4.3 Variants

In the expression (1) we may plug in the Fourier expasion of f to get a formula for the Fourier coefficients of $\Lambda(f)$. This can be generalized for $\Lambda_n^{n'}$ with $n' > n$ and one gets in this way explict formulas for the Fourier coefficients of Klingen-Eisenstein series in the paramodular (squarefree) context, if f is assumed to be an eigenform of Λ.

5 Theta series and the basis problem

5.1 Theta series for chains of lattices

Examples of modular forms for paramodular groups can be constructed by theta series as follows: We start from an integral matrix $\mathcal{P}$ of size m with $\mathcal{P} = diag(t_1, \ldots, t_m)$ and $t_i \mid t_{i+1}$. Furthermore let L_j be a t_j-modular even integral lattice (i.e. $L_j^\sharp = t_j^{-1}L_j$) of full rank in an $2k$-dimensional positive definite quadratic space V over $\mathbb{Q}$ $(1 \leq j \leq m)$; a chain

$$L_1 \supset L_2 \supset \cdots \supset L_m$$

with the properties above will be called "paramodular of type $\mathcal{T}$".
Then we define a theta series of degree n attached to a chain as above by

$$\vartheta^n(L_1, \ldots, L_n; Z) = \sum_{x_1 \in L_1, \ldots, x_n \in L_n} e^{\pi i tr(Q(x_1, \ldots, x_n) \cdot Z)};$$

here $Q(x_1, \ldots, x_n)$ denotes the Gram matrix for $(x_1, \ldots, x_n) \in V^n$.

Proposition *For a chain $(L_1 \supset \ldots L_n)$, paramodular of type $\mathcal{P}$ with $\mathcal{P} = diag(t_1, t_2, \ldots, t_n)$ the theta series $\theta^n(L_1, \ldots, L_n)$ is a modular form of weight k for the group $\Gamma(\mathcal{P})$.*

There are refinements/generalizations of this statement for lattices with level and also for theta series with harmonic polynomials.

5.2 Siegel's Theorem

We say that two chains $(K_1, \ldots, K_n)$ and $(L_1, \ldots, L_n)$ of lattices in V are in the same class if there is an isometry $\phi \in \mathbf{O}(V)$ with $\phi(K_j) = L_j$ for all j. The notion of a genus of such chains is explained in a similar way (by local conditions).

Proposition : *Let $\mathcal{P}$ be an elementary divisor matrix of squarefree level and $k \equiv 0 \bmod 4$. Then there exists exactly one genus of n-tuples $(L_1, \ldots, L_n)$ of lattices of rank $m = 2k$ which are paramodular of type $\mathcal{P}$.*

Proposition (= Siegel's main theorem)
Let $\mathcal{P}$ be a squarefree elementary divisior matrix and let $gen(L_1, \ldots L_n)$ be the unique genus of n-tupels of positive definite lattices of rank $2k$ with $4 \mid k$ and $k > n+1$, paramodular of type $\mathcal{P}$. Let $\mathcal{E}(\mathcal{P})$ denote the (unique) Siegel-Eisenstein series of degree n and weight k for $\Gamma(\mathcal{P})$. Then

$$\mathcal{E}(\mathcal{P})(Z) \sim \sum \frac{1}{\mid O(M_1, \ldots, M_n) \mid} \vartheta^n(M_1, \ldots, M_n),$$

where the sum goes over representatives of the classes in the genus $gen(L_1, \ldots, L_n)$ and $O(M_1, \ldots, M_n)$ is the group of isometries of the chain $(M_1, \ldots, M_n)$.

5.3 The basis problem

Combining Siegel's theorem with the the bijectivity of the map Λ and with the decomposition property

$$\vartheta^n(L_1, \ldots . L_{2n})(\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}) = \vartheta^n(L_1, \ldots, L_n)(z) \cdot \vartheta^n(L_{n+1}, \ldots, L_{2n})(w) \qquad (z, w \in \mathbb{H}_n)$$

we obtain (in the same way as in [1]) - after switching from $\Gamma(T)$ to $\widehat{\Gamma}(T)$-

Theorem: *Let T be an elementary divisor matrix of size n and of squarefree level. Let k be a positive integer divisble by 4 and sufficiently large. Then all cusp forms in $\mathcal{S}_k^n(T)$ are linear combinations of theta series $\vartheta^n(L_1^\sharp, \ldots, L_n^\sharp)$.*

Remark: There are versions of the theorem above for noncuspidal modular forms (by using maps $\Lambda_n^{n'}$ with $n' > n$) and also for theta series with harmonic polynomials (including vector-valued cases) by using diffential operators.

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