Free boundary problems in magnetohydrodynamics

E. Frolova (St. Petersburg State University)

1 Introduction

We consider the free boundary problem governing the motion of a finite mass of a viscous incompressible electrically conducting capillary liquid. The liquid is moving under the action of magnetic field, mass and capillary forces. We assume that the liquid is contained in a bounded variable domain Ω_{1t} whose boundary consists of two disjoint components: the free boundary Γ_t and the fixed surface Σ that is also a boundary of the fixed domain D . The domain $\bar{D} \cup \Omega_{1t}$ is surrounded by a bounded vacuum region Ω_{2t} with the exterior boundary S. The given surfaces Γ_0, S, and Σ are homeomorphic to a sphere, $\Gamma_0 \cap S = \varnothing$, $\Gamma_0 \cap \Sigma = \varnothing$.

The problem consists of determination of the variable domains Ω_{it}, $i = 1, 2$, together with the velocity vector field $\boldsymbol{v}(x,t)$, the pressure $p(x,t)$, $x \in \Omega_{1t}$, and the magnetic field $\boldsymbol{H}(x,t)$, $x \in \Omega_{1t} \cup \Omega_{2t}$. Equations in Ω_{1t} have the form

$$\begin{aligned} &\boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla \cdot T(\boldsymbol{v}, p) - \nabla \cdot T_M(\boldsymbol{H}) = \boldsymbol{f}, \qquad \nabla \cdot \boldsymbol{v}(x,t) = 0, \\ &\mu_1 \boldsymbol{H}_t + \alpha^{-1} rotrot \boldsymbol{H} - \mu_1 rot(\boldsymbol{v} \times \boldsymbol{H}) = 0, \qquad \nabla \cdot \boldsymbol{H}(x,t) = 0, \end{aligned} \tag{1.1}$$

where ν is the kinematic viscosity, α - conductivity, μ_i - magnetic permeability in Ω_{it}. We assume that ν, α, μ_i are positive constants, the density of the fluid is equal to 1.

$T(\boldsymbol{v}, p) = -pI + \nu S(\boldsymbol{v})$ is the viscous stress tensor,

$S(\boldsymbol{v}) = \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)_{i,j=1,2,3}$ is the doubled rate-of-strain tensor,

$T_M(\boldsymbol{H}) = \mu(\boldsymbol{H} \otimes \boldsymbol{H} - \frac{1}{2} I |\boldsymbol{H}|^2)$ is the magnetic stress tensor.

Magnetic field in the vacuum region Ω_{2t} satisfies the equations

$$rot \boldsymbol{H} = 0, \qquad \nabla \cdot \boldsymbol{H}(x,t) = 0. \tag{1.2}$$

Equations (1.1), (1.2) are supplied with the following boundary conditions on the free boundary

$$\begin{aligned} &\big(T(\boldsymbol{v}, p) + [T_M(\boldsymbol{H})]\big)\boldsymbol{n} = \sigma \boldsymbol{n} \mathcal{H}, \\ &\boldsymbol{V}_n = \boldsymbol{v} \cdot \boldsymbol{n}, \\ &[\mu \boldsymbol{H} \cdot \boldsymbol{n}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \quad x \in \Gamma_t, \quad t > 0. \end{aligned} \tag{1.3}$$

Here σ is the coefficient of the surface tension, $\mathcal{H}$ is the doubled mean curvature of Γ_t, $\boldsymbol{V}_n$ is the velocity of evolution of the surface Γ_t in the direction of the exterior normal $\boldsymbol{n}$ to Γ_t, $[u] = u^{(1)} - u^{(2)}$ is the jump of $u(x)$ on Γ_t. The dynamic boundary condition $(1.3)_1$ follows from conservation of momentum under the assumption that the free surface is subject to capillary forces. The kinematic boundary condition $(1.3)_2$ means that the transfer of mass through the surface is excluded and particles of the liquid not leave the free surface.

On the given surfaces S and Σ we set

$$\begin{gathered}\boldsymbol{H}(x,t)\cdot\boldsymbol{n}(x)=0, \quad x\in S, \quad t>0,\\ \boldsymbol{H}(x,t)\cdot\boldsymbol{n}(x)=0, \quad (rot\boldsymbol{H})_\tau=0, \quad \boldsymbol{v}(x,t)=0, \quad x\in\Sigma, \quad t>0,\end{gathered} \tag{1.4}$$

where by $(rot\boldsymbol{H})_\tau$ we denote the tangential part of $rot\mathbf{H}$.

Finally, we add the initial conditions

$$\boldsymbol{v}(x,0)=\boldsymbol{v}_0(x), \quad x\in\Omega_{10}, \qquad \boldsymbol{H}(x,0)=\boldsymbol{H}_0(x), \quad x\in\Omega_{10}\cup\Omega_{20}. \tag{1.5}$$

Problems of magnetohydrodynamics in fixed simply connected domains were studied by O.A. Ladyzhenskaya and V.A. Solonnilov in the classical papers [1], [2]. In 2010 M. Padula and V.A. Solonnikov proved local in time solvability of the problem similar to $(1.1)-(1.5)$ but without a rigid domain D [3]. The solution is obtained in anisotropic Sobolev-Slobodetskii spaces $W_2^{2+l,1+l/2}$, $1/2<l<1$ for a closed surface Γ_0 of arbitrary shape such that Ω_{10} and $\bar{\Omega}_{10}\cup\Omega_{20}$ are simply connected.

In [4] we proved solvability of problem $(1.1)-(1.5)$ with $\boldsymbol{f}\equiv 0$ in an infinite time interval under the additional assumptions that the initial position of the free boundary is close to a sphere and initial data are sufficiently small. We demonstrated that when $t\to+\infty$, then the free boundary tends to a sphere of the same radius. In general, this sphere has a different center, because the barycenter point of the liquid can move. In [5] we extend this result to problem $(1.1)-(1.5)$ under additional smallness assumptions on the force $\boldsymbol{f}$. As the region occupied by the fluid is unknown, we assume that force $\boldsymbol{f}$ is given in the wider domain $\Omega_{10}\cup\Gamma_0\cup\Omega_{20}$. We add the rigid domain D by technical reasons. It helps us to prove the exponential decay for the solution of corresponding homogeneous linear problem.

Sobolev-Slobodetskii space $W_2^{s,s/2}(Q_T)$ in the cylindrical domain $Q_T=\Omega\times(0,T)$ can be defined as $W_2^{s,0}(Q_T)\cap W_2^{0,s/2}(Q_T)$ with the norm

$$\| u \|^2_{W_2^{s,s/2}(Q_T)}=\int\limits_0^T \| u(\cdot,t) \|^2_{W_2^s(\Omega)}\,dt+\int\limits_\Omega \| u(x,\cdot) \|^2_{W_2^{s/2}(0,T)}\,dx. \tag{1.6}$$

The first term in (1.6) is the square of the norm in $W_2^{s,0}(Q_T)=L_2((0,T),W_2^s(\Omega))$, the second is the square of the norm in $W_2^{0,s/2}(Q_T)=L_2(\Omega,W_2^{s/2}(0,T))$. By $W_2^s(\Omega)$ with non-integer $s>0$ we mean the space of functions $u(x)$, $x\in\Omega$ with the finite norm

$$\| u \|^2_{W_2^s(\Omega)}=\| u \|^2_{W_2^{[s]}(\Omega)}+\sum_{|\alpha|=[s]}\int\limits_\Omega\int\limits_\Omega \big|D^\alpha u(x)-D^\alpha u(y)\big|^2\frac{dxdy}{|x-y|^{n+2(s-[s])}},$$

$$\| u \|^2_{W_2^{[s]}(\Omega)}=\sum_{0\le|\alpha|\le[s]}\int\limits_\Omega \big|D^\alpha u(x)\big|^2dx.$$

Spaces of functions defined on the smooth surfaces are introduced in a standard way, with the help of local maps and partition of unity.

2 Main result

In this section we formulate the result of [5].

Imagine the domain D is also filled with a liquid of the density 1, denote by Ω_t the domain $\bar{D} \cup \Omega_{1t}$, and define R_0 by the relation

$$\frac{4}{3}\pi R_0^3 = |\Omega_0|.$$

We assume that the initial position of the free boundary Γ_0 is a small normal perturbation of the sphere S_{R_0}. Precisely,

$$\Gamma_0 = \{x = y + \boldsymbol{N}(y)\rho_0(y), \quad y \in S_{R_0}\},$$

where $\boldsymbol{N}(y) = \frac{y}{|y|}$ is the exterior normal to S_{R_0} and ρ_0 is a given small function. It is clear that

$$\int_{S_{R_0}} ((R_0 + \rho_0)^3 - R_0^3) dS = 0. \tag{2.1}$$

We introduce the function

$$\boldsymbol{\xi}(t) = \frac{1}{|\Omega_0|} \int\limits_{\Omega_t} x dx = \frac{1}{|\Omega_0|} \int\limits_0^t \left(\int\limits_{\Omega_{1\tau}} \boldsymbol{v}(x, \tau) dx \right) d\tau,$$

which is the barycenter point of the domain Ω_t filled with the liquid of the density 1. We assume that at the initial moment of time the barycenter point is located at the origin, it implies

$$\int_{S_{R_0}} y_i((R_0 + \rho_0)^4 - R_0^4) dS = 0, \quad i = 1, 2, 3. \tag{2.2}$$

We are looking for Γ_t in the form

$$\Gamma_t = \{x = y + \boldsymbol{N}(y)\rho(y, t) + \boldsymbol{\xi}(t), \quad y \in S_{R_0}\},$$

where the functions $\rho(y, t)$, $\boldsymbol{\xi}(t)$ are unknown.

Theorem 1.[5] *Let* $\boldsymbol{v}_0 \in W_2^{1+l}(\Omega_{10})$, $\rho_0 \in W_2^{2+l}(S_{R_0})$, $\boldsymbol{H}_0 \in W_2^{1+l}(\Omega_{io})$, $i = 1, 2$, *with a certain* $l \in (1/2, 1)$, *satisfy natural compatibility conditions and conditions* (2.1), (2.2). *Let* $\boldsymbol{f} \in W_2^{l,l/2}(\Omega \times (0, +\infty))$, $\nabla \boldsymbol{f} \in W_2^{l,l/2}(\Omega \times (0, +\infty))$, $D^2 \boldsymbol{f} \in L_2(\Omega \times (0, +\infty))$, $\Omega = \Omega_{10} \cup \Gamma_0 \cup \Omega_{20}$. *We assume that the following smallness conditions*

$$\|\boldsymbol{v}_0\|_{W_2^{1+l}(\Omega_{10})} + \|\rho_0\|_{W_2^{2+l}(S_{R_0})} + \sum_{i=1,2} \|\boldsymbol{H}_0\|_{W_2^{1+l}(\Omega_{i0})} \leq \varepsilon, \tag{2.3}$$

$$\| e^{bt} \nabla \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega \times (0,+\infty))} + \| e^{bt} \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega \times (0,+\infty))} + \| D^2 \boldsymbol{f} \|_{L_2(\Omega \times (0,+\infty))} \leq \varepsilon, \qquad b > 0$$

are valid. Let at the initial moment of time $dist\{\Gamma_0, \Sigma\} > 3d_0$, $dist\{\Gamma_0, S\} > 3d_0$, $d_0 > (C^* + 1)\varepsilon$ (C^* *is defined in* (5.17)).

There exists a small ε, *such that problem* (1.1) – (1.5) *has a unique solution in an infinite time interval with the following properties: for any* $t > 0$, *the free boundary* Γ_t *is located in the layer* $0 < R_0 - d_0 \leq |y| \leq R_0 + d_0$,

$$\rho(\cdot, t) \in W_2^{2+l}(S_{R_0}), \quad \rho_t(\cdot, t) \in W_2^{1+l}(S_{R_0}), \quad \boldsymbol{v}(\cdot, t) \in W_2^{1+l}(\Omega_{1t}), \qquad \boldsymbol{H}^{(i)}(\cdot, t) \in W_2^{1+l}(\Omega_{it}).$$

The solution is decaying exponentially as t *tends to* $+\infty$.

3 Coordinate transformation

In order to take into account the displacement of the barycenter point, we modify the Hanzawa coordinate transformation used in [3]. We introduce the mapping

$$x = y + \boldsymbol{N}^*(y)\rho^*(y,t) + \chi(y)\boldsymbol{\xi}(t) \equiv e_{\rho,\boldsymbol{\xi}}(y), \quad y \in \Omega, \tag{3.1}$$

where $\chi(y)$ is a smooth non-negative cut-off function, which is equal to 1, when y belongs to the layer $R_0 - d_0 \leq |y| \leq R_0 + d_0$ and vanishing outside the layer $R_0 - 2d_0 \leq |y| \leq R_0 + 2d_0$, $\boldsymbol{N}^*(y)$ and $\rho^*(y,t)$ are sufficiently regular extensions of $\boldsymbol{N}$ and ρ from S_{R_0} into Ω, such that $\rho^*(y,t) = 0$ near S and Σ, C^1-norm of ρ^* is small. We denote by $\mathcal{L}(y, \rho^*, \boldsymbol{\xi})$ the Jacobi matrix of the transform (3.1), $L = \det \mathcal{L}$. Transformation (3.1) maps the domain $\Omega = \Omega_{1t} \cup \Gamma_t \cup \Omega_{2t}$ to $\Omega = \mathcal{F}_1 \cup S_{R_0} \cup \mathcal{F}_2$, where $\mathcal{F}_1$ is the domain bounded by Σ and S_{R_0} and $\mathcal{F}_2 = \Omega \setminus \overline{\mathcal{F}}_1$; $\partial\mathcal{F}_2 = S \cup S_{R_0}$.

With the help of (3.1), we pass from the free boundary problem (1.1) − (1.5) to a nonlinear problem in the fixed domain $\Omega = \mathcal{F}_1 \cup S_{R_0} \cup \mathcal{F}_2$, for the unknown functions $\boldsymbol{u}(y,t) = \boldsymbol{v} \circ e_{\rho,\boldsymbol{\xi}}$, $q(y,t) = p \circ e_{\rho,\boldsymbol{\xi}} - \frac{2\sigma}{R_0}$, $\boldsymbol{h}(y,t) = L\mathcal{L}^{-1}(y, \rho^*, \boldsymbol{\xi})(\boldsymbol{H} \circ e_{\rho,\boldsymbol{\xi}})$. The given function $\boldsymbol{f}$ is transformed to

$$\boldsymbol{f}\left(e_{\rho,\boldsymbol{\xi}}, t\right) = \boldsymbol{f}(y) + \int\limits_0^1 \nabla \boldsymbol{f}(y + s(\boldsymbol{N}^*\rho^* + \chi\boldsymbol{\xi}), t) ds \left(\boldsymbol{N}^*(y)\rho^*(y,t) + \chi(y)\boldsymbol{\xi}(t)\right).$$

We separate linear and nonlinear parts in this problem and obtain

$$\begin{aligned}
&\boldsymbol{u}_t - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{f}(y) + \left(\boldsymbol{f}\left(e_{\rho,\xi}, t\right) - \boldsymbol{f}(y)\right) + \boldsymbol{l}_1(\boldsymbol{u}, q, \boldsymbol{h}, \rho),\\
&\nabla \cdot \boldsymbol{u} = l_2(\boldsymbol{u}, \rho), \quad y \in \mathcal{F}_1,\ t > 0,\\
&\boldsymbol{u}(y,t)\Big|_{y\in\Sigma} = 0,\\
&\nu\Pi_0 S(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u}, \rho),\\
&-q + \nu\boldsymbol{N} \cdot S(\boldsymbol{u})\boldsymbol{N}(y) + \sigma B_0\rho = l_4(\boldsymbol{u}, \boldsymbol{h}, \rho) + l_5(\rho),\\
&\rho_t - \boldsymbol{u} \cdot \boldsymbol{N}(y) + \frac{1}{|\Omega_0|}\int_{\mathcal{F}_1} \boldsymbol{u} dy \cdot \boldsymbol{N}(y) = l_6(\boldsymbol{u}, \rho), \quad y \in S_{R_0}, \quad t > 0,\\
&\mu_1\boldsymbol{h}_t + \alpha^{-1} rot rot \boldsymbol{h} = \boldsymbol{l}_7(\boldsymbol{h}, \boldsymbol{u}, \rho), \qquad \nabla \cdot \boldsymbol{h} = 0, \quad y \in \mathcal{F}_1, \quad t > 0,\\
&rot \boldsymbol{h} = rot \boldsymbol{l}_8(\boldsymbol{h}, \rho), \quad \nabla \cdot \boldsymbol{h} = 0, \quad y \in \mathcal{F}_2,\\
&[\mu\boldsymbol{h} \cdot \boldsymbol{N}] = 0, \quad [\boldsymbol{h}_\tau] = \boldsymbol{l}_9(\boldsymbol{h}, \rho), \quad y \in S_{R_0}, \quad t > 0,\\
&\boldsymbol{h}(y,t) \cdot \boldsymbol{n}(y) = 0, \quad y \in S \cup \Sigma, \quad (rot \boldsymbol{h})_\tau = 0, \quad y \in \Sigma, \quad t > 0,\\
&\boldsymbol{u}(y,0) = \boldsymbol{u}_0(y), \quad y \in \mathcal{F}_1, \quad \boldsymbol{h}(y,0) = \boldsymbol{h}_0(y), \quad y \in \mathcal{F}_1 \cup \mathcal{F}_2,\\
&\rho(y,0) = \rho_0(y), \quad y \in S_{R_0},
\end{aligned} \tag{3.2}$$

here $\Pi_0\boldsymbol{w} = \boldsymbol{w} - \boldsymbol{N}(\boldsymbol{w} \cdot \boldsymbol{N})$, the expression $B_0\rho$ is the first variation of $(\mathcal{H} + \frac{2}{R_0})$ with respect to ρ and has the form

$$B_0\rho = -\frac{1}{R_0^2}(\Delta_{S_1}\rho + 2\rho),$$

Δ_{S_1} is the Laplacean on the unit sphere S_1. By $l_1 - l_9$ we denote the nonlinear terms. Expressions for the nonlinear terms are given in [4].

Theorem 1 follows from the existence result for problem (3.2) in an infinite time interval.

Theorem 2. *Let all the assumptions of Theorem 1 be fulfilled. Then problem* (3.2) *has a unique solution with the following regularity properties:*

$$\boldsymbol{u} \in W_2^{2+l,1+l/2}(Q^1_\infty), \quad \nabla q \in W_2^{l,l/2}(Q^1_\infty), \quad \rho \in W_2^{l/2}(0,+\infty; W_2^{5/2}(S_{R_0})),$$
$$\rho_t \in W_2^{3/2+l,3/4+l/2}(G_\infty), \quad \boldsymbol{h}^{(i)} \in W_2^{2+l,1+l/2}(Q^i_\infty),$$

where $Q^i_\infty = \mathcal{F}_i \times (0,+\infty)$, $G_\infty = S_{R_0} \times (0,+\infty)$, $\boldsymbol{h}^{(i)} = \boldsymbol{h}|_{x\in\mathcal{F}_i}$, $i = 1,2$. *The solution satisfies the inequality*

$$X_{(0,+\infty)}\left(e^{at}\boldsymbol{u}, e^{at}q, e^{at}\rho, e^{at}\boldsymbol{h}\right) \le c\Big(\|\boldsymbol{u}_0\|_{W_2^{1+l}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{2+l}(S_{R_0})} \tag{3.3}$$
$$+ \sum_{i=1}^{2} \|\boldsymbol{h}_0^{(i)}\|_{W_2^{1+l}(\mathcal{F}_i)} + \| e^{at}\boldsymbol{f} \|_{W^{l,l/2}(\Omega\times(0,+\infty))} + \| e^{at}\nabla \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega\times(0,+\infty))}\Big),$$

with a certain small $0 < a < b$. Here we use the notation

$$X_{(t_1,t_2)}(\boldsymbol{u}, q, \rho, \boldsymbol{h}) = \|\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(\mathcal{F}_1\times(t_1,t_2))} + \|\nabla q\|_{W_2^{l,l/2}(\mathcal{F}_1\times(t_1,t_2))} + \|\rho\|_{W_2^{l/2}(t_1,t_2;W_2^{5/2}(S_{R_0}))}$$
$$+\|\rho_t\|_{W_2^{3/2+l,3/4+l/2}(S_{R_0}\times(t_1,t_2))} + \sum_{i=1}^{2} \|\boldsymbol{h}^{(i)}\|_{W_2^{2+l,1+l/2}(\mathcal{F}_\rangle\times(t_1,t_2))}. \tag{3.4}$$

4 Exponential decay for solutions to linear problems

To prove global solvability, we first have to obtain the exponential decay for the corresponding linear problems in Sobolev norms. Omitting all the nonlinear terms in (3.2), we arrive at the linear problem which can be decomposed in two parts: hydrodynamical and magnetic.

The hydrodynamical problem has the form

$$\begin{aligned}
&\boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(y,t), \quad \nabla\cdot\boldsymbol{v} = 0, \quad y \in \mathcal{F}_1,\\
&\Pi_0 S(\boldsymbol{v})\boldsymbol{N} = 0,\\
&-p + \nu\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N} + \sigma B_0\rho = 0,\\
&\rho_t - \left(\boldsymbol{v} - |\Omega_0|^{-1}\int_{\mathcal{F}_1} \boldsymbol{v}(y,t)dy\right)\cdot\boldsymbol{N} = 0, \quad y \in S_{R_0},\\
&\boldsymbol{v}(y,t) = 0, \quad y \in \Sigma,\\
&\boldsymbol{v}(y,0) = \boldsymbol{v}_0(y), \quad y \in \mathcal{F}_1, \quad \rho(x,0) = \rho_0(y), \quad y \in S_{R_0}.
\end{aligned} \tag{4.1}$$

Linearization of (2.1) (2.2) leads to the following orthogonality conditions

$$\int_{S_{R_0}} \rho_0(y)dS = 0, \quad \int_{S_{R_0}} y_i\rho_0(y)dS = 0, \quad i = 1,2,3. \tag{4.2}$$

Theorem 3. *Let* $\boldsymbol{v}_0 \in W_2^{1+l}(\mathcal{F}_1)$, $\rho_0 \in W_2^{2+l}(S_{R_0})$, $\boldsymbol{f} \in W_2^{l,l/2}(\mathcal{F}_1 \times (0,T))$, $T \in (0,+\infty]$, *conditions* (4.2) *and natural compatibility conditions be satisfied. The given function* $\boldsymbol{f}$ *is decaying exponentially as* $t \to +\infty$ *and*

$$\|e^{a_1 t}\boldsymbol{f}\|_{W_2^{l,l/2}(\mathcal{F}_1\times(0,T))} < +\infty, \qquad a_1 > 0. \tag{4.3}$$

Then problem (4.1) *has a unique solution:* $\boldsymbol{v} \in W_2^{2+l,1+l/2}(Q_T^1)$, $\nabla p \in W_2^{l,l/2}(Q_T^1)$, $\rho \in W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))$, $\rho_t \in W_2^{3/2+l,3/4+l/2}(G_T)$, $Q_T^1 = \mathcal{F}_1 \times (0,T)$, $G_T = S_{R_0} \times (0,T)$, *and the estimate*

$$\begin{aligned} &\|e^{at}\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_T^1)} + \|e^{at}\nabla p\|_{W_2^{l,l/2}(Q_T^1)} + \|e^{at}\rho\|_{W_2^{l/2}(0,T;W_2^{5/2}(S_{R_0}))} + \\ &\|e^{at}\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + \sup_{t<T}\|e^{at}\boldsymbol{v}(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_1)} + \sup_{t<T}\|e^{at}\rho(\cdot,t)\|_{W_2^{2+l}(S_{R_0})} \\ &\leq c\big(\|\boldsymbol{v}_0\|_{W_2^{1+l}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{2+l}(S_{R_0})} + \|e^{at}\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^1)}\big) \end{aligned} \tag{4.4}$$

holds with a certain constant $0 < a < a_1$.

Proof. Existence of a solution to the hydrodynamical linear problem with such regularity properties is proved in [3], [4]. Here we explain the proof of estimate (4.4). To deduce the energy estimate, we multiply the first equation in (4.1) by $\boldsymbol{v}$, integrate over $\mathcal{F}_1$, and integrate by parts. We arrive at the relation

$$\frac{1}{2}\frac{d}{dt} \parallel \boldsymbol{v}(\cdot,t) \parallel^2_{L_2(\mathcal{F}_1)} + \frac{\nu}{2} \parallel S(\boldsymbol{v}) \parallel^2_{L_2(\mathcal{F}_1)} + \int\limits_{\partial\mathcal{F}_1} \big(-\nu S(\boldsymbol{v})\boldsymbol{N}\cdot\boldsymbol{v} + p\boldsymbol{v}\cdot\boldsymbol{N}\big)ds = \int\limits_{\mathcal{F}_1} \boldsymbol{f}\cdot\boldsymbol{v}dy. \tag{4.5}$$

Due to the boundary conditions, the surface integral equals

$$\int\limits_{S_{R_0}} \sigma B_0\rho\big(\rho_t + \frac{1}{|\Omega_0|}\int\limits_{\mathcal{F}_1}\boldsymbol{v}(y,t)dy\cdot\boldsymbol{N}\big)ds = \int\limits_{S_{R_0}} \sigma\rho_t B_0\rho ds + \sigma\int\limits_{S_{R_0}} B_0\rho\boldsymbol{\xi}'(t)\cdot\boldsymbol{N}ds. \tag{4.6}$$

The first term at the right-hand side of (4.6) can be written in the form

$$-\frac{\sigma}{R_0^2}\int_{S_1}\big(\Delta_{S_1}\rho + 2\rho\big)\rho_t ds = \frac{\sigma}{2R_0^2}\frac{d}{dt}\int_{S_1}\big(|\nabla_\omega\rho|^2 - 2\rho^2\big)ds = \frac{1}{2}\frac{d}{dt}M(t),$$

where

$$M(t) = \frac{\sigma}{R_0^2}\int_{S_1}\big(|\nabla_\omega\rho|^2 - 2\rho^2\big)ds.$$

It can be easily demonstrated (see [4]) that if the orthogonality conditions (4.2) are fulfilled at the initial moment of time, then the same conditions are fulfilled for the solution $\rho(y,t)$ of the problem (4.1) at any time $t > 0$. It means that ρ is orthogonal to the first and the second eigenfunctions of Laplace-Beltarmi operator Δ_{S_1}. It implies that $M(t)$ is positively defined:

$$M(t) \geq C \parallel \rho(\cdot,t) \parallel^2_{W_2^1(S_{R_0})}. \tag{4.7}$$

The second term at the right-hand side of (4.6) is equal to zero due to the condition $B_0N_i = 0$. Consequently, (4.5) takes the form

$$\frac{1}{2}\frac{d}{dt}\Big(\parallel \boldsymbol{v}(\cdot,t) \parallel^2_{L_2(\mathcal{F}_1)} + M(t)\Big) + \frac{\nu}{2} \parallel S(\boldsymbol{v}) \parallel^2_{L_2(\mathcal{F}_1)} = \int\limits_{\mathcal{F}_1} \boldsymbol{f}\cdot\boldsymbol{v}dy. \tag{4.8}$$

To add the dissipative term for ρ, we use the so-called "free energy" method, introduced by M.Padula.

Lemma1([7],[8]) *Assume that* $\rho \in W_2^{1/2,0}(S_{R_0} \times (0,T))$, *has the time derivative* $\rho_t \in L_2(S_{R_0} \times (0,T))$, *and satisfies the orthogonality condition* $\int\limits_{S_{R_0}} \rho(y,t)ds = 0$. *There exists a vector field* $\boldsymbol{w}(\cdot,t) \in W_2^1(\mathcal{F}_1)$, *such that* $\boldsymbol{w}_t(\cdot,t) \in L_2(\mathcal{F}_1)$, *and*

$$\nabla \cdot \boldsymbol{w} = 0, \qquad y \in \mathcal{F}_1, \qquad t > 0, \qquad \boldsymbol{w}\Big|_{\Sigma} = 0, \qquad \boldsymbol{w} \cdot \boldsymbol{N}\Big|_{S_{R_0}} = \rho.$$

This vector field satisfies the estimates

$$\| \boldsymbol{w}(\cdot,t) \|_{W_2^1(\mathcal{F}_1)} \leq c \, \| \rho(\cdot,t) \|_{W_2^{1/2}(S_{R_0})}, \qquad \| \boldsymbol{w}(\cdot,t) \|_{L_2(\mathcal{F}_1)} \leq c \, \| \rho(\cdot,t) \|_{L_2(S_{R_0})},$$
$$\| \boldsymbol{w}_t(\cdot,t) \|_{L_2(\mathcal{F}_1)} \leq c\big(\| \rho_t(\cdot,t) \|_{L_2(S_{R_0})} + \| \rho(\cdot,t) \|_{W_2^{1/2}(S_{R_0})} \big).$$

We multiply the first equation in (4.1) by the auxiliary vector field $\boldsymbol{w}$, integrate over $\mathcal{F}_1$, and integrate by parts. Taking into account boundary conditions, we arrive at

$$\frac{d}{dt} \int\limits_{\mathcal{F}_1} \boldsymbol{v} \cdot \boldsymbol{w} dx + \frac{\nu}{2} \int\limits_{\mathcal{F}_1} S(\boldsymbol{v}) : S(\boldsymbol{w}) dx - \int\limits_{\mathcal{F}_1} \boldsymbol{v} \cdot \boldsymbol{w}_t dx + M(t) = \int\limits_{\mathcal{F}_1} \boldsymbol{f} \cdot \boldsymbol{w} dy. \tag{4.9}$$

We multiply (4.9) by a small positive number γ and add it to (4.8), it gives

$$\frac{1}{2} \frac{d}{dt}\Big(E(t)\Big) + D(t) = \int\limits_{\mathcal{F}_1} \boldsymbol{f} \cdot \boldsymbol{v} dy + \gamma \int\limits_{\mathcal{F}_1} \boldsymbol{f} \cdot \boldsymbol{w} dy, \tag{4.10}$$

where

$$E(t) = \| \boldsymbol{v}(\cdot,t) \|^2_{L_2(\mathcal{F}_1)} + 2\gamma \int\limits_{\mathcal{F}_1} \boldsymbol{v} \cdot \boldsymbol{w} dx + M(t),$$
$$D(t) = \tfrac{\nu}{2} \| S(\boldsymbol{v}) \|^2_{L_2(\mathcal{F}_1)} + \gamma \tfrac{\nu}{2} \int\limits_{\mathcal{F}_1} S(\boldsymbol{v}) : S(\boldsymbol{w}) dx - \gamma \int\limits_{\mathcal{F}_1} \boldsymbol{v} \cdot \boldsymbol{w}_t dx + \gamma M(t).$$

Due to the condition $\mathbf{v} = \mathbf{0}$ on the surface Σ, we can use the Korn inequality. For the sufficiently small γ, it helps us to demonstrate that (see details in [4])

$$1/2\big(\| \boldsymbol{v}(\cdot,t) \|^2_{L_2(\mathcal{F}_1)} + M(t)\big) \leq E(t) \leq 3/2\big(\| \boldsymbol{v}(\cdot,t) \|^2_{L_2(\mathcal{F}_1)} + M(t)\big),$$
$$D(t) \geq \alpha\big(\| \boldsymbol{v}(\cdot,t) \|^2_{W_2^1(\mathcal{F}_1)} + M(t)\big), \quad \alpha > 0. \tag{4.11}$$

We multiply (4.10) by e^{ct} with a certain $0 < c \leq 2a_1$, and obtain

$$\frac{d}{dt}\Big(\frac{1}{2} e^{ct} E(t)\Big) - \frac{c}{2} e^{ct} E(t) + e^{ct} D(t) = \int\limits_{\mathcal{F}_1} e^{ct} \boldsymbol{f} \cdot (\boldsymbol{v} + \gamma \boldsymbol{w}) dy. \tag{4.12}$$

At first, we fix γ in such a way that (4.11) hold. Then, we choose so small c that

$$D(t) - \frac{c}{2} E(t) \geq \alpha_1 \left(\| \boldsymbol{v}(\cdot,t) \|^2_{L_2(\mathcal{F}_1)} + M(t)\right), \qquad \alpha_1 > 0. \tag{4.13}$$

We introduce the notations

$$e^{ct} E(t) = \mathcal{U}^2(t), \qquad e^{ct}\left(D(t) - \frac{c}{2} E(t)\right) = \mathcal{R}^2(t).$$

Identity (4.12) reads

$$\frac{1}{2}\frac{d}{dt}\big(\mathcal{U}^2(t)\big)+\mathcal{R}^2(t)=\int\limits_{\partial\mathcal{F}_1} e^{ct}\boldsymbol{f}\cdot(\boldsymbol{v}+\gamma\boldsymbol{w})dy. \tag{4.14}$$

We estimate the right-hand side of (4.14) by the Hölder inequality, making use of Lemma 1 and (4.7)

$$\begin{aligned}\int\limits_{\mathcal{F}_1} e^{ct}\big|\boldsymbol{f}\cdot(\boldsymbol{v}+\gamma\boldsymbol{w})\big|dy &\le e^{ct}\parallel \boldsymbol{f}\parallel_{L_2(\mathcal{F}_1)}\big(\parallel \boldsymbol{v}\parallel_{L_2(\mathcal{F}_1)}+\gamma\parallel \boldsymbol{w}\parallel_{L_2(\mathcal{F}_1)}\big)\\ &\le C_1e^{\frac{c}{2}t}\parallel \boldsymbol{f}(\cdot,t)\parallel_{L_2(\mathcal{F}_1)}\mathcal{U}(t).\end{aligned}$$

Consequently, (4.14) gives

$$\frac{d}{dt}\big(\mathcal{U}(t)\big)\le C_1e^{\frac{c}{2}t}\parallel \boldsymbol{f}\parallel_{L_2(\mathcal{F}_1)}.$$

It follows that

$$\mathcal{U}(t)\le C_1\int\limits_0^t e^{\frac{c}{2}\tau}\parallel \boldsymbol{f}(\cdot,\tau)\parallel_{L_2(\mathcal{F}_1)}d\tau+\mathcal{U}(0). \tag{4.15}$$

Estimate (4.15), implies the exponential decay for the solution in L_2 norms. Multiplying (4.15) by $e^{-\frac{1}{2}(c-\beta)t}$, where $c-\beta>0$, we have

$$\mathcal{U}(t)e^{-\frac{1}{2}(c-\beta)t}\le C_1\int\limits_0^t e^{-\frac{1}{2}(c-\beta)(t-\tau)}e^{\frac{\beta}{2}\tau}\parallel \boldsymbol{f}(\cdot,\tau)\parallel_{L_2(\mathcal{F}_1)}d\tau+e^{-\frac{1}{2}(c-\beta)t}\mathcal{U}(0). \tag{4.16}$$

From inequality (4.16) it follows that the expression

$$\int\limits_0^T\left(e^{-\frac{1}{2}(c-\beta)t}\mathcal{U}(t)\right)^2dt=\int\limits_0^T e^{\beta t}E(t)dt$$

is controlled by

$$\int\limits_0^T\|e^{\frac{\beta}{2}t}\boldsymbol{f}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)}dt+\mathcal{U}^2(0).$$

As a result we obtain

$$\begin{aligned}&\int\limits_0^T e^{\beta t}\left(\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)}+\|\rho(\cdot,t)\|^2_{W_2^1(S_{R_0})}\right)dt\\ &\le c\left(\|\boldsymbol{v}_0\|^2_{L_2(\mathcal{F}_1)}+\|\rho_0\|^2_{W_2^1(S_{R_0})}+\int\limits_0^T\|e^{\frac{\beta}{2}t}\boldsymbol{f}(\cdot,t)\|^2_{L_2(\mathcal{F}_1)}dt\right),\end{aligned} \tag{4.17}$$

with a certain positive $\beta<c\le 2a_1$.

We introduce the functions:

$$\tilde{\boldsymbol{v}}=e^{at}\boldsymbol{v},\qquad \tilde{p}=e^{at}p,\qquad \tilde{\rho}=e^{at}\rho,\qquad \tilde{\boldsymbol{f}}=e^{at}\boldsymbol{f},\quad 0<a\le\frac{\beta}{2}<a_1,$$

These functions satisfy the relations

$$\begin{aligned}
&\tilde{\boldsymbol{v}}_t - \nu\nabla^2\tilde{\boldsymbol{v}} + \nabla\tilde{p} = a\tilde{\boldsymbol{v}} + \tilde{\boldsymbol{f}}, \quad \nabla\cdot\tilde{\boldsymbol{v}} = 0, \quad y \in \mathcal{F}_1,\\
&\Pi_0 S(\tilde{\boldsymbol{v}})\boldsymbol{N} = 0,\\
&-\tilde{p} + \nu\boldsymbol{N}\cdot S(\tilde{\boldsymbol{v}})\boldsymbol{N} + \sigma B_0\tilde{\rho} = 0,\\
&\tilde{\rho}_t = \big(\tilde{\boldsymbol{v}} - |\Omega_0|^{-1}\int_{\mathcal{F}_1}\tilde{\boldsymbol{v}}(y,t)dy\big)\cdot\boldsymbol{N} + a\tilde{\rho}, \quad y \in S_{R_0},\\
&\tilde{\boldsymbol{v}}(y,t) = 0, \quad y \in \Sigma,\\
&\tilde{\boldsymbol{v}}(y,0) = \boldsymbol{v}_0(y), \quad y \in \mathcal{F}_1, \quad \tilde{\rho}(x,0) = \rho_0(y), \quad y \in S_{R_0}.
\end{aligned} \tag{4.18}$$

We use the estimate of a solution to the hydrodynamical linear problem [4] and apply interpolation inequalities for the terms $\|\tilde{\boldsymbol{v}}\|_{W_2^{l,l/2}(Q_T^1)}$, $\|\tilde{\rho}\|_{W_2^{l+3/2,l/2+3/4}(G_T)}$. To estimate $\| \tilde{\boldsymbol{v}} \|_{L_2(Q_T^1)}$, $\| \tilde{\rho} \|^2_{W_2^1(G_T)}$, we use (4.17). As a result, we obtain (4.4) with a certain $a < a_1$. $\square$

The homogeneous magnetic problem has the form

$$\begin{aligned}
&\mu_1\boldsymbol{H}_t + \alpha^{-1}rotrot\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{H} = 0, \quad x \in \mathcal{F}_1,\\
&rot\boldsymbol{H} = 0, \quad \nabla\cdot\boldsymbol{H} = 0, \quad x \in \mathcal{F}_2,\\
&[\mu\boldsymbol{H}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{H}_\tau] = 0, \quad y \in S_{R_0},\\
&\boldsymbol{H}\cdot\boldsymbol{n} = 0, \quad y \in S\cup\Sigma, \quad (rot\boldsymbol{H})_\tau = 0, \quad y \in \Sigma,\\
&\boldsymbol{H}(y,0) = \boldsymbol{H}_0(y), \quad y \in \mathcal{F}_1\cup\mathcal{F}_2.
\end{aligned} \tag{4.19}$$

Theorem 4. *For arbitrary $\boldsymbol{H}_0 \in W_2^{1+l}(\mathcal{F}_i)$, $i = 1, 2$, satisfying the natural compatibility conditions, problem* (4.19) *has a unique solution $\mathbf{H^{(i)}} \in \mathbf{W_2^{2+l,1+l/2}(Q_T^i)}$. The inequality*

$$\sum_{i=1}^{2}\big(\|e^{at}\boldsymbol{H}^{(i)}\|_{W_2^{2+l,1+l/2}(Q_T^i)} + \sup_{t<T}\|e^{at}\boldsymbol{H}^{(i)}(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_i)}\big) \le c\sum_{i=1}^{2}\|\boldsymbol{H}_0^{(i)}\|_{W_2^{1+l}(\mathcal{F}_i)} \tag{4.20}$$

holds with a certain $a > 0$ and with the constant c independent of T.

Theorem 4 is proved in [3], [4]. To obtain (4.20), problem (4.19) is rewritten in the form of the Cauchy problem

$$\boldsymbol{H}_t + \mathcal{A}\boldsymbol{H} = \boldsymbol{0}, \quad \boldsymbol{H}|_{t=0} = \boldsymbol{H}_0,$$

where the operator $\mathcal{A}$ is defined on the space $\mathcal{H}^2(\Omega)$ (space of solenoidal vector fields from $W_2^2(\Omega)$, satisfying boundary conditions (4.19)). The characteristic property of $\mathcal{A}$ is

$$\int_\Omega \mu\mathcal{A}\boldsymbol{H}\cdot\boldsymbol{h}dx = \alpha^{-1}\int_{\mathcal{F}_1} rot\boldsymbol{H}\cdot rot\boldsymbol{h}dx, \quad \forall\boldsymbol{h},\boldsymbol{H} \in \mathcal{H}^2.$$

$\mathcal{A}$ is a positive defined self-adjoint operator. The spectrum of $-\mathcal{A}$ consists of a countable number of real negative eigenvalues with the accumulation point at $-\infty$. This guarantees the weighted estimate (4.20) (see details in [9]).

5 Nonlinear problem

In this section we outline the main ideas of the proof of Theorem 2. We start with the existence result on the finite time interval $[0,T]$. We separate initial conditions in (3.2) in two parts

$$\boldsymbol{u}_0=\boldsymbol{u}_0^{''}+\boldsymbol{u}_0^{'},\quad \rho_0=\rho_0^{''}+\rho_0^{'},\quad \boldsymbol{h}_0=\boldsymbol{h}_0^{''}+\boldsymbol{h}_0^{'},$$

where the functions $\boldsymbol{u}_0^{''}$, $\rho_0^{''}$, $\boldsymbol{h}_0^{''}$ satisfy the same compatibility conditions as $\boldsymbol{u}_0$, ρ_0, $\boldsymbol{h}_0$ in nonlinear problem (3.2):

$$\int\limits_{S_1}\rho_0^{''}(R_0y)dS=-\frac{1}{R_0}\int\limits_{S_1}\rho_0^2(R_0y)dS-\frac{1}{3R_0^2}\int\limits_{S_1}\rho_0^3(R_0y)dS,$$

$$\int\limits_{S_1}y_i\rho_0^{''}(R_0y)dS=-\frac{3}{2R_0}\int\limits_{S_1}y_i\rho_0^2(R_0y)dS-\frac{1}{R_0^2}\int\limits_{S_1}y_i\rho_0^3(R_0y)dS-\frac{1}{4R_0^3}\int\limits_{S_1}y_i\rho_0^4(R_0y)dS,\quad i=1,2,3,$$

$$\nabla\cdot\boldsymbol{u}_0^{''}=l_2(\boldsymbol{u}_0,\rho_0),\quad y\in\mathcal{F}_1,$$

$$\nu\Pi_{S_{R_0}}S(\boldsymbol{u}_0^{''})\boldsymbol{N}(y)=\boldsymbol{l}_3(\boldsymbol{u}_0,\rho_0),\quad y\in S_{R_0},\quad \boldsymbol{u}_0^{''}=0,\quad y\in\Sigma,$$

$$rot\boldsymbol{h}_0^{''}=rot\boldsymbol{l}_8(\boldsymbol{h}_0^{(2)},\rho_0),\quad y\in\mathcal{F}_2,\quad \nabla\cdot\boldsymbol{h}_0^{''}=0,\quad y\in\mathcal{F}_1\cup\mathcal{F}_2,$$

$$[\boldsymbol{h}_{0\tau}^{''}]=\boldsymbol{l}_9(\boldsymbol{h}_0,\rho_0),\quad y\in S_{R_0},\quad [\mu\boldsymbol{h}_0^{''}\cdot\boldsymbol{N}]=0,\quad y\in S_{R_0},$$

$$\boldsymbol{h}_0^{''}\cdot\boldsymbol{N}=0,\quad y\in\Sigma\cup S,\quad (rot\boldsymbol{h}_0^{''})_\tau=0,\quad y\in\Sigma,$$

and have the order ε^2:

$$\|\rho_0^{''}\|_{W_2^{2+l}(S_{R_0})}+\|\boldsymbol{u}_0^{''}\|_{W_2^{1+l}(\mathcal{F}_1)}\le c\big(\|\rho_0\|_{W_2^{2+l}(S_{R_0})}+\|\boldsymbol{u}_0\|_{W_2^{l+l}(\mathcal{F}_1)}\big)^2. \tag{5.1}$$

$$\sum_{i=1}^{2}\|\boldsymbol{h}_0^{''}\|_{W_2^{1+l}(\mathcal{F}_i)}\quad\le c\big(\sum_{i=1}^{2}\|\boldsymbol{h}_0^{(i)}\|_{W_2^{1+l}(\mathcal{F}_i)}+\|\rho_0\|_{W_2^{2+l}(S_{R_0})}\big)^2. \tag{5.2}$$

Possibility of constructing such functions follows from inverse trace theorems and proved in [3], [6].

To simplify the presentation, we introduce the notation

$$Y(t)=\|\boldsymbol{u}(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_1)}+\|\rho(\cdot,t)\|_{W_2^{2+l}(S_{R_0})}+\sum_{i=1}^{2}\|\boldsymbol{h}(\cdot,t)\|_{W_2^{1+l}(\mathcal{F}_i)},$$

and denote by $Y^{'}(t)$, $Y^{''}(t)$ the same expression for the functions $\boldsymbol{u}^{'}$, $\rho^{'}$, $\boldsymbol{h}^{'}$ or $\boldsymbol{u}^{''}$, $\rho^{''}$, $\boldsymbol{h}^{''}$. Henceforth, we also use the notation $X_{(t_1,t_2)}(\boldsymbol{u},q,\rho,\boldsymbol{h})$ introduced in (3.4).

The functions $\boldsymbol{u}_0^{'},\rho_0^{'},\boldsymbol{h}_0^{'}$ evidently satisfy compatibility conditions in linear problem (4.1), (4.19). By Theorems 3,4, this problem has a unique solution $\boldsymbol{u}^{'},q^{'},\rho^{'},\boldsymbol{h}^{'}$. In accordance with (4.4), (4.20), we have

$$X_{(0,T)}\left(e^{at}\boldsymbol{u}^{'},e^{at}q^{'},e^{at}\rho^{'},e^{at}\boldsymbol{h}^{'}\right)\le c\left(Y^{'}(0)+\parallel e^{at}\boldsymbol{f}\parallel_{W_2^{l,l/2}(Q_T^1)}\right), \tag{5.3}$$

$$Y^{'}(t) \le c_1 e^{-at} \left(Y^{'}(0) + \left(\int\limits_0^t \parallel e^{a\tau} \boldsymbol{f}(\cdot,\tau) \parallel^2_{L_2(\mathcal{F}_1)} d\tau \right)^{1/2} \right), \tag{5.4}$$

with a certain $0 < a < b$.

The functions $\boldsymbol{u}^{''}$, $q^{''}$, $\rho^{''}$, $\boldsymbol{h}^{''}$ we find from the following nonlinear system

$$\begin{aligned}
&\boldsymbol{u}^{''}_t - \nu\nabla^2\boldsymbol{u}^{''} + \nabla q^{''} = \int\limits_0^1 \nabla \boldsymbol{f}(y + s(\boldsymbol{N}^*(\rho^{'}+\rho^{''})^* + \chi\boldsymbol{\xi}), t) ds \left(\boldsymbol{N}^*(\rho^{'}+\rho^{''})^* + \chi\boldsymbol{\xi}\right)\\
&+ \boldsymbol{l}_1(\boldsymbol{u}' + \boldsymbol{u}^{''}, q' + q^{''}, \boldsymbol{h}' + \boldsymbol{h}^{''}, \rho' + \rho^{''}),\\
&\nabla\cdot\boldsymbol{u}^{''} = l_2(\boldsymbol{u}' + \boldsymbol{u}^{''}, \rho' + \rho^{''}), \quad in \quad \mathcal{F}_1, \qquad \boldsymbol{u}^{''}(y,t)\big|_{y\in\Sigma} = 0,\\
&\nu\Pi_0 S(\boldsymbol{u}^{''})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u}' + \boldsymbol{u}^{''}, \rho' + \rho^{''}),\\
&- q^{''} + \nu\boldsymbol{N}\cdot S(\boldsymbol{u}^{''})\boldsymbol{N}(y) + \sigma B_0\rho^{''} = l_4(\boldsymbol{u}' + \boldsymbol{u}^{''}, \boldsymbol{h}^{'} + \boldsymbol{h}^{''}, \rho' + \rho^{''}) + l_5(\rho' + \rho^{''}),\\
&\rho^{''}_t - \boldsymbol{u}^{''}\cdot\boldsymbol{N}(y) + |\Omega_0|^{-1}\int\limits_{\mathcal{F}_1} \boldsymbol{u}^{''} dz\cdot\boldsymbol{N}(y) = l_6(\boldsymbol{u}' + \boldsymbol{u}^{''}, \rho' + \rho^{''}), \quad on \quad S_{R_0}, \qquad (5.5)\\
&\mu_1\boldsymbol{h}^{''}_t + \alpha^{-1} rot rot \boldsymbol{h}^{''} = \boldsymbol{l}_7(\boldsymbol{h}' + \boldsymbol{h}^{''}, \boldsymbol{u}' + \boldsymbol{u}^{''}, \rho' + \rho^{''}), \quad \nabla\cdot\boldsymbol{h}^{''} = 0, \quad in \quad \mathcal{F}_1,\\
&rot\boldsymbol{h}^{''} = rot\boldsymbol{l}_8(\boldsymbol{h}' + \boldsymbol{h}^{''}, \rho' + \rho^{''}), \quad \nabla\cdot\boldsymbol{h}^{''} = 0, \quad in \quad \mathcal{F}_2,\\
&[\mu\boldsymbol{h}^{''}\cdot\boldsymbol{N}] = 0, \quad [\boldsymbol{h}^{''}_\tau] = \boldsymbol{l}_9(\boldsymbol{h}' + \boldsymbol{h}^{''}, \rho' + \rho^{''}), \quad on \quad S_{R_0},\\
&\boldsymbol{h}^{''}(y,t)\cdot\boldsymbol{n}(y) = 0, \quad on \quad S\cup\Sigma, \quad (rot\boldsymbol{h}^{''})_\tau = 0, \quad on \quad \Sigma,\\
&\boldsymbol{u}^{''}(y,0) = \boldsymbol{u}^{''}_0(y), \quad y\in\mathcal{F}_1, \quad \boldsymbol{h}^{''}(y,0) = \boldsymbol{h}^{''}_0(y), \qquad y\in\mathcal{F}_1\cup\mathcal{F}_2,\\
&\rho^{''}(y,0) = \rho^{''}_0(y), \quad y\in S_{R_0}.
\end{aligned}$$

We choose T so big that $c_1 e^{-aT} < \frac{1}{4}$ (c_1 is the constant in (5.4)). Problem (5.5) can be solved for $t\in[0,T]$, provided ε is sufficiently small.

Theorem 5. *Let all the assumptions of Theorem 1 be fulfilled. The functions* $\boldsymbol{u}^{'}$, $q^{'}$, $\rho^{'}$, $\boldsymbol{h}^{'}$ *are subject to* (5.3), (5.4). *For a given* $T > 0$, *there exists such* $\varepsilon > 0$ *that if the given functions satisfy smallness conditions* (2.3) *with this* ε, *then problem* (5.5) *is uniquely solvable on the time interval* $[0,T]$ *and the solution satisfies the estimate*

$$\begin{aligned}
&X_{(0,T)}\left(\boldsymbol{u}^{''}, q^{''}, \rho^{''}, \boldsymbol{h}^{''}\right) + \sup_{t<T} Y^{''}(t)\\
&\le c_2(T)\varepsilon\left(Y(0) + \parallel \boldsymbol{f} \parallel_{W_2^{l,l/2}(Q_T^1)} + \parallel \nabla\boldsymbol{f} \parallel_{W_2^{l,l/2}(\Omega\times(0,T))}\right).
\end{aligned} \tag{5.6}$$

Theorem 5 is proved in [5] by the successive approximations method. Estimates of the nonlinear terms are given in [3], [4], [6]. The functions

$$\boldsymbol{u} = \boldsymbol{u}^{'} + \boldsymbol{u}^{''}, \quad q = q^{'} + q^{''}, \quad \rho = \rho^{'} + \rho^{''}, \quad \mathbf{h} = \mathbf{h}^{'} + \mathbf{h}^{''}$$

is a solution to problem (3.2) on time interval $[0,T]$. Now we choose such ε that $c_2(T)\varepsilon$ in (5.6) is not grater then $\frac{1}{4}$. In consequence of (5.4), (5.6), solution to problem (3.2) satisfies the estimate

$$Y(T) \le \tfrac{1}{2}Y(0) + \tfrac{1}{4}\left(\| \boldsymbol{f} \|_{W_2^{l,l/2}(Q_T^1)} + \| \nabla \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega\times(0,T))}\right)$$
$$+\tfrac{1}{4}\left(\int\limits_0^T \| e^{a\tau}\boldsymbol{f}(\cdot,\tau) \|^2_{L_2(\mathcal{F}_1)}\, d\tau\right)^{1/2}. \tag{5.7}$$

The existence result in an infinite time interval is proved step by step. Let us have proved existence of a solution to problem (3.2) on time interval $[0, kT]$. Let $|\boldsymbol{\xi}(t)|$ be uniformly bounded for $t \in [0, kT]$, and the estimate

$$Y(iT) \le \frac{1}{2}Y((i-1)T) + \frac{1}{4}\left(F[i] + \left(\int\limits_{(i-1)T}^{iT} \| e^{a(\tau-(i-1)T)}\boldsymbol{f}(\cdot,\tau) \|^2_{L_2(\mathcal{F}_1)}\, d\tau\right)^{1/2}\right), \tag{5.8}$$

where

$$F[i] = \| \boldsymbol{f} \|_{W_2^{l,l/2}(\mathcal{F}_1\times((i-1)T,iT))} + \| \nabla \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega\times((i-1)T,iT))}$$

holds for $i = 1, ..., k$. On time interval $[(i-1)T, iT]$, the solution can be decomposed in two parts: $\boldsymbol{u} = \boldsymbol{u}' + \boldsymbol{u}''$, $q = q' + q''$, $\rho = \rho' + \rho''$, $\boldsymbol{h} = \boldsymbol{h}' + \boldsymbol{h}''$, satisfying the following estimates

$$X_{[(i-1)T,iT]}\left(\boldsymbol{u}'', q'', \rho'', \boldsymbol{h}''\right) \le \frac{1}{4}\left(Y((i-1)T) + F[i]\right), \tag{5.9}$$

$$X_{[(i-1)T,iT]}\left(e^{a(t-(i-1)T)}\boldsymbol{u}', e^{a(t-(i-1)T)}q', e^{a(t-(i-1)T)}\rho', e^{a(t-(i-1)T)}\boldsymbol{h}'\right)$$
$$\le c\left(Y((i-1)T) + \| e^{a(t-(i-1)T)}\boldsymbol{f} \|_{W_2^{l,l/2}(\mathcal{F}_1\times((i-1)T,iT))}\right), \quad a < b. \tag{5.10}$$

We consider $\boldsymbol{u}_{kT} = \boldsymbol{u}(\cdot, kT)$, $\rho_{kT} = \rho(\cdot, kT)$, $\boldsymbol{h}_{kT} = \boldsymbol{h}(\cdot, kT)$ as initial data at $t = kT$ and repeat the above scheme on $[kT, (k+1)T]$. Due to the conservation of volume, condition (2.1) holds for ρ_{kT}. The barycenter is located at the point $\xi(kT)$, which not necessarily coincides with the origin. We have

$$\int\limits_{\Omega_{kT}} x_i dx = \xi_i(kT)\frac{4}{3}\pi R_0^3 = \xi_i(kT)\int\limits_{\Omega_{kT}} dx, \quad i = 1, 2, 3.$$

We pass to the spherical coordinates with the center at the point $\xi(kT)$, and see that the linear part of (2.2) for ρ_{kT} has the same form as for ρ_0, precisely, $\int\limits_{S_1} y_i\rho(R_0 y, kT)dS = 0$. Consequently, we can use all the results of section 4.

We again separate the data at $t = kT$ in two parts

$$\boldsymbol{u}_{kT} = \boldsymbol{u}''_{kT} + \boldsymbol{u}'_{kT}, \quad \rho_{kT} = \rho''_{kT} + \rho'_{kT}, \quad \boldsymbol{h}_{kT} = \boldsymbol{h}''_{kT} + \boldsymbol{h}'_{kT},$$

where the functions $\boldsymbol{u}''_{kT}$, ρ''_{kT}, $\boldsymbol{h}''_{kT}$ satisfy the same compatibility conditions in (3.2) as $\boldsymbol{u}_{kT}$, ρ_{kT}, $\boldsymbol{h}_{kT}$ and have the order ε^2. The solution $\boldsymbol{u}', q', \rho', \boldsymbol{h}'$ to linear problem (4.1), (4.19) with initial data $\boldsymbol{u}'_{kT}, \rho'_{kT}, \boldsymbol{h}'_{kT}$ satisfies (4.4), (4.20) on time interval $[kT, (k+1)T]$. It gives

$$Y'((k+1)T) \le \frac{1}{4}\left(Y'(kT) + \left(\int\limits_{kT}^{(k+1)T} \| e^{a(\tau-kT)}\boldsymbol{f}(\cdot,\tau) \|^2_{L_2(\mathcal{F}_1)}\, d\tau\right)^{1/2}\right) \tag{5.11}$$

and (5.10) for $i = k + 1$.

To apply Theorem 5 on time interval $[kT, (k+1)T]$, we have to take care of the term

$$\sup_{kT<t<(k+1)T} |\boldsymbol{\xi}(t)|.$$

It is clear that $\boldsymbol{\xi}(t) - \boldsymbol{\xi}(kT)$ is estimated by $\| \boldsymbol{u} \|_{L_2(\mathcal{F}_1 \times (kT,(k+1)T))}$, and it remains to estimate $|\xi(kT)|$. We use (5.8) for $i = 1, ..., k.$, and deduce

$$Y(kT) \le \frac{1}{2^k} Y(0) + \sum_{i=1}^{k} \frac{1}{2^{k-i+2}} \left(F[i] + \left(\int\limits_{(i-1)T}^{iT} \| e^{a(\tau-(i-1)T)} \boldsymbol{f}(\cdot, \tau) \|^2_{L_2(\mathcal{F}_1)} \right)^{1/2} \right). \quad (5.12)$$

Under our assumptions on $\mathbf{f}$, (5.12) gives

$$Y(kT) \le \frac{1}{(\min\{2, e^{aT}\})^k} \left(Y(0) + \| e^{at} \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega\times(0,+\infty))} + \| e^{at} \nabla \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega\times(0,+\infty))} \right). \quad (5.13)$$

This implies the exponential decay for $Y(t)$. In particular,

$$\|\boldsymbol{u}(\cdot, t)\|_{L_2(\mathcal{F}_1)} \le ce^{-\alpha t} \left(Y(0) + \| e^{at} \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega\times[0,+\infty))} + \| e^{at} \nabla \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega\times[0,+\infty))} \right) \le 3ce^{-\alpha t} \varepsilon,$$

with a certain $\alpha > 0$. In consequence of (5.9), (5.10), Jacobian L is uniformly bounded for $t \in [0, kT]$. Using this fact and the Holder inequality, we obtain

$$|\boldsymbol{\xi}(kT)| = \left| \int\limits_0^{kT} dt \int\limits_{\Omega_{1t}} \boldsymbol{v}(x, t) dx \right| \le \int_0^{kT} dt \int_{\mathcal{F}_1} |\boldsymbol{u}(y, t)||L| dy$$

$$\le c \int\limits_0^{kT} \|\boldsymbol{u}(\cdot, t)\|_{L_2(\mathcal{F}_1)} dt \le c_1 \int\limits_0^{+\infty} \varepsilon e^{-\alpha t} dt \le C\varepsilon, \quad (5.14)$$

with the constant C independent of kT and ε.

Now we can repeat the proof of Theorem 5 on time interval $[kT, (k+1)T]$, replacing everywhere $Y(0)$ by $Y(kT)$. The constant $c_2(T)$ in (5.6) and, as a consequence, the value of ε can be chosen independent of k beginning with $k = 2$. Taking a sum of solutions to problem (5.5) with initial data $\boldsymbol{u}''_{kT}$, ρ''_{kT}, $\boldsymbol{h}''_{kT}$ and to linear problem (4.1), (4.19) with initial data $\boldsymbol{u}'_{kT}$, ρ'_{kT}, $\boldsymbol{h}'_{kT}$, we obtain a solution to problem (3.2) on time interval $[kT, (k+1)T]$. We repeat the above scheme for any $k \in \boldsymbol{N}$ and step by step obtain a solution to problem (3.2) on an infinite time interval $[0, +\infty)$.

By (5.9), (5.10), (5.13), we have

$$X_{[(i-1)T,iT]} \left(e^{a(t-(i-1)T)} \boldsymbol{u}', e^{a(t-(i-1)T)} q', e^{a(t-(i-1)T)} \rho', e^{a(t-(i-1)T)} \boldsymbol{h}' \right) \quad (5.15)$$

$$\le c \frac{1}{(\min\{2, e^{aT}\})^{i-1}} \left(Y(0) + 2 \| e^{at} \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega\times(0,+\infty))} + \| e^{at} \nabla \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega\times(0,+\infty))} \right),$$

where the constant c is independent of i, and

$$X_{[(i-1)T,iT]} \left(\boldsymbol{u}'', q'', \rho'', \boldsymbol{h}'' \right) \quad (5.16)$$

$$\le \frac{1}{\min\{2, e^{aT}\}^i} \left(Y(0) + \| e^{at} \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega\times(0,+\infty))} + \| e^{at} \nabla \boldsymbol{f} \|_{W_2^{l,l/2}(\Omega\times(0,+\infty))} \right).$$

Estimates (5.15), (5.16) imply (3.3), provided that $e^{aT} < 2$.

Theorem 1 follows from Theorem 2. We find the position of the free boundary for any $t > 0$ by the formula

$$\Gamma_t = \{x = y + \boldsymbol{N}(y)\rho(y,t) + \xi(t), \quad y \in S_{R_0}\},$$

make coordinate transform, and obtain a solution $\boldsymbol{v}$, p, $\boldsymbol{H}$ to the free boundary problem (1.1) − (1.5).

In accordance with (3.3), we can conclude that Jacobian of mapping (3.1) is uniformly bounded for any $t > 0$, and exponential decay in Sobolev norms takes place for $t \to +\infty$. By the same reasonings as in (5.14), we have

$$|\boldsymbol{\xi}(+\infty)| \leq \int_0^{+\infty} dt \int_{\Omega_{1t}} |\boldsymbol{v}(x,t)|dx \leq c \int\limits_0^{+\infty} \|\boldsymbol{u}(\cdot,t)\|_{L_2(\mathcal{F}_1)} dt \leq c \int\limits_0^{+\infty} \varepsilon e^{-\alpha t} dt \leq C^* \varepsilon. \quad (5.17)$$

It means that $|\boldsymbol{\xi}(t)|$ is uniformly bounded for any $t > 0$. To be sure that the free boundary do not intersect the fixed parts of the boundary, we have to assume that at the initial moment of time $\mathrm{dist}\{\Gamma_0, \Sigma\} > 3d_0$ $\mathrm{dist}\{\Gamma_0, S\} > 3d_0$, $d_0 > (C^* + 1)\varepsilon$ (see assumptions of Theorem 1).

The same scheme can be applied to the free boundary problem describing the motion of a finite mass of a viscous incompressible fluid when the external force is acting on the fluid, but there is no magnetic field (see [10], [11]).

6 Free boundary problem of magnetohydrodynamics for two liquids

The next step is to consider the motion of a finite mass of viscous incompressible electrically conducting capillary liquid inside the other viscous incompressible liquid under the action of magnetic field. In this case the domain Ω_{2t} is also filled with a liquid. The interface between the liquids is unknown. Let the bounded variable domain Ω_{1t} be filled by the liquid of density d_1 and viscosity ν_1. The domain Ω_{1t} is surrounded by the bounded domain Ω_{2t}, filled by the liquid of density d_2 and viscosity ν_2. The boundary of Ω_{2t} consists of two disjoint components: the free boundary Γ_t and the fixed boundary S. We assume that both Γ_0 and S are homeomorphic to a sphere, $dist\{\Gamma_0, S\} \geq \delta > 0$.

The problem consists of determination for $t > 0$ the variable domains Ω_{it}, $i = 1, 2$ together with the velocity vector field $\mathbf{v^{(i)}}$, the pressure $p^{(i)}$, and the magnetic field $\mathbf{H^{(i)}}$. Equations in Ω_{it} have the form

$$\begin{gathered} \mathbf{v^{(i)}}_t + (\mathbf{v^{(i)}} \cdot \nabla)\mathbf{v^{(i)}} - \nabla \cdot T(\mathbf{v^{(i)}}, p^{(i)}) - \nabla \cdot T_M(\mathbf{H^{(i)}}) = 0, \\ \mu_i \mathbf{H^{(i)}}_t + \alpha_i^{-1} rotrot \mathbf{H^{(i)}} - \mu_i rot(\mathbf{v^{(i)}} \times \mathbf{H^{(i)}}) = 0, \\ \nabla \cdot \mathbf{v^{(i)}} = 0, \qquad \nabla \cdot \mathbf{H^{(i)}} = 0, \qquad x \in \Omega_{it}, \end{gathered} \quad (6.1)$$

where μ_i, - magnetic permeability, ν_i - kinematic viscosity, α_i - conductivity , d_i - density. We assume that $\nu_i, \alpha_i, d_i, \mu_i$ are positive constants.

On the free surface Γ_t, we have the following boundary conditions

$$\begin{gathered}\big([T(\mathbf{v},p)]+[T_M(\mathbf{H})]\big)\mathbf{n}=\sigma\mathbf{n}\mathcal{H},\\ \mathbf{V}_n=\mathbf{v}\cdot\mathbf{n},\qquad [\mathbf{v}]=0,\\ [\tfrac{1}{\alpha}(rot\mathbf{H})_\tau]=[\mu(\mathbf{v}\times\mathbf{H})_\tau],\\ [\mu\mathbf{H}\cdot\mathbf{n}]=0,\quad [\mathbf{H}_\tau]=0,\quad x\in\Gamma_t,\end{gathered}\tag{6.2}$$

where σ - coefficient of the surface tension, $\mathcal{H}$ - is the doubled mean curvature of Γ_t, $\mathbf{V}_n$ is the velocity of evolution of the surface Γ_t in the direction of the normal $\mathbf{n}$ to Γ_t, which is exterior with respect to the domain Ω_{1t}. Condition $(6.2)_3$ on the jump of the tangential part of $rot\mathbf{H}$ follows from the fact that on the interface tangential part of electric field is continuous and Maxwell equations.

We assume that the fixed boundary S is a perfectly conducting bounded closed surface. Boundary conditions on S have the form

$$\mathbf{H}\cdot\mathbf{n}=0,\quad (rot\mathbf{H})_\tau=0,\quad \mathbf{v}=0,\quad x\in S.\tag{6.3}$$

We add the initial conditions

$$\mathbf{v}(x,0)=\mathbf{v}_0(x),\qquad \mathbf{H}(x,0)=\mathbf{H}_0(x),\quad x\in\Omega_{10}\cup\Omega_{20}.\tag{6.4}$$

We assume that the initial position of the free boundary Γ_0 can be regarded as a small normal perturbation of the given smooth closed surface G

$$\Gamma_0=\{x=y+\mathbf{N}(y)\rho_0(y),\quad y\in G\},$$

where $N(y)$ is the external normal to the surface G, $\rho_0\in W_2^{2+l}(G)$ is a given function, and $|\rho_0|\le\frac{\delta}{4}$. We are looking for the free boundary in a similar form

$$\Gamma_t=\{x=y+\mathbf{N}(y)\rho(y,t),\quad y\in G\},$$

where the function $\rho(y,t)$ is unknown.

We denote by $\mathcal{F}_1$ the domain bounded by G, by $\mathcal{F}_2$ the domain bounded by G and S. We construct the mapping which transforms $\Omega=\mathcal{F}_1\cup G\cup\mathcal{F}_2$ to $\Omega=\Omega_{1t}\cup\Gamma_t\cup\Omega_{2t}$. To this end, we extend N and ρ into Ω. By N^* we mean a smooth non-vanishing vector field in Ω which coincides with N on G. By $\rho^*(y,t)$ we denote an extension of unknown function $\rho(y,t)$ from G into Ω with preservation of class, which vanishes in a $\frac{\delta_0}{4}$ neighborhood of the surface S and satisfies the condition $\frac{\partial\rho^*(y,t)}{\partial N}\Big|_G=0$. We introduce this mapping by the relation

$$x=y+\mathbf{N}^*(y)\rho^*(y,t)=e_\rho(y).\tag{6.5}$$

When ρ is sufficiently small (which is certainly the case for small t), transform (6.5) establishes one-to-one correspondence between $\mathcal{F}_i$ and Ω_{it}, $i=1,2$. We denote by $\mathcal{L}(y,\rho^*)$ the Jacobi matrix of the transformation (6.5), $L=det\mathcal{L}$, $\widehat{\mathcal{L}}=L\mathcal{L}^{-1}$ is the cofactor matrix. The normal $\mathbf{n}$ to the free boundary is connected with N by the formula

$$\mathbf{n}(e_\rho(y))=\frac{\widehat{\mathcal{L}}\mathbf{N}(y)}{|\widehat{\mathcal{L}}\mathbf{N}(y)|}.\tag{6.6}$$

Let

$$\mathbf{v}(e_\rho, t) = \mathbf{u}(y,t), \qquad p(e_\rho, t) = q(y,t).$$

To simplify the calculations, we introduce the new unknown function

$$\mathbf{h} = \widehat{\mathcal{L}}\mathbf{H}(e_\rho, t).$$

As it is demonstrated in [3], $\mathbf{h}$ is a solenoidal vector field and satisfies the homogeneous condition $[\mu\mathbf{h}\cdot\mathbf{N}] = 0$, $y \in G$. Transformation (6.5) converts the problem (6.1) – (6.4) to a nonlinear problem in the fixed domain $\Omega = \mathcal{F}_1 \cup G \cup \mathcal{F}_2$. We separate linear and nonlinear parts in this problem and write the boundary condition $(6.2)_1$ for the tangential and normal parts separately, then it can be written in the following form:

$$\begin{aligned}
&\mathbf{u}_t^{(i)} - \nu_i\nabla^2\mathbf{u}^{(i)} + \frac{1}{d_i}\nabla q^{(i)} = \mathbf{l}_1^{(i)}(\mathbf{u}^{(i)}, q^{(i)}, \mathbf{h}^{(i)}, \rho), \quad y \in \mathcal{F}_i\\
&\nabla\cdot\mathbf{u}^{(i)} = l_2^{(i)}(\mathbf{u}^{(i)}, \rho), \quad y \in \mathcal{F}_i,\\
&[\nu\Pi_0 S(\mathbf{u})\mathbf{N}] = \mathbf{l}_3^{(i)}(\mathbf{u}, \rho), \quad y \in G,\\
&-[\frac{1}{d}q] + [\nu\mathbf{N}\cdot S(\mathbf{u})\mathbf{N}(y)] + \sigma B\rho = l_4(\mathbf{u}, \mathbf{h}, \rho), \quad y \in G,\\
&\rho_t - \mathbf{u}\cdot\mathbf{N} = l_5(\mathbf{u}, \rho), \quad [\mathbf{u}] = 0, \quad y \in G,\\
&\mu_i\mathbf{h}_t^{(i)} + \alpha_i^{-1} rot rot\mathbf{h}^{(i)} = \mathbf{l}_6^{(i)}(\mathbf{h}^{(i)}, \mathbf{u}^{(i)}, \rho), \quad y \in \mathcal{F}_i,\\
&\nabla\cdot\mathbf{h}^{(i)} = 0, \quad y \in \mathcal{F}_i,\\
&[\mu\mathbf{h}\cdot\mathbf{N}] = 0, \quad [\mathbf{h}_\tau] = \mathbf{l}_7(\mathbf{h}, \rho), \quad [\frac{1}{\alpha}(rot\mathbf{h})_\tau] = \mathbf{l}_8(\mathbf{h}, \mathbf{u}, \rho) \quad y \in G,\\
&\mathbf{h}^{(2)}\cdot\mathbf{n} = 0, \quad (rot\mathbf{h}^{(2)})_\tau = 0, \quad \mathbf{u}^{(2)} = 0 \quad y \in S,\\
&\mathbf{u}^{(i)}(y,0) = \mathbf{u}_0^{(i)}(y), \quad \mathbf{h}^{(i)}(y,0) = \mathbf{h}_0^{(i)}(y), \quad y \in \mathcal{F}_i, \quad \rho(y,0) = \rho_0(y), \quad y \in G.
\end{aligned} \tag{6.7}$$

Here $\Pi_0\mathbf{u} = \mathbf{u} - \mathbf{N}(\mathbf{u}\cdot\mathbf{N})$ is the tangential part of the vector field $\mathbf{u}$, $-B\rho$ is the first variation of $\mathcal{H}$ with respect to ρ. The nonlinear terms $\mathbf{l}_1^{(i)} - \mathbf{l}_7$ are similar to the nonlinear terms calculated in [3], [4] The nonlinear term $\mathbf{l}_8$ has the form

$$\begin{aligned}
\mathbf{l}_8 = [\frac{1}{\alpha}(rot\mathbf{h})_\tau] &= [\frac{1}{\alpha}\,(rot\mathbf{h} - (rot\mathbf{h}\cdot\mathbf{N})\mathbf{N})]\\
&= [\frac{1}{\alpha}\left(rot\mathbf{h} - \frac{1}{L}\mathcal{L}rot\mathcal{L}^T\frac{1}{L}\mathcal{L}\mathbf{h}\right)]\\
&+ [\frac{1}{\alpha}\left((\frac{1}{L}\mathcal{L}rot\mathcal{L}^T\frac{1}{L}\mathcal{L}\cdot\mathbf{n}(e_\rho)\mathbf{n})(e_\rho) - (rot\mathbf{h}\cdot\mathbf{N})\mathbf{N}\right)]\\
&+ [\mu\left(\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h} - ((\mathcal{L}^{-1}\mathbf{u}\times\mathbf{h})\cdot\mathbf{n}(e_\rho)\mathbf{n}(e_\rho))\right)],
\end{aligned}$$

where $\mathbf{n}(e_\rho)$ is given in (6.6).

Here we formulate the local solvability result for problem (6.7). The proof will be given in subsequent publications.

Theorem 6. *Let* $\mathbf{u}_{0i} \in W_2^{1+l}(\mathcal{F}_i)$, $\mathbf{H_{0i}} \in W_2^{1+l}(\mathcal{F}_i)$, $i = 1, 2$, $\rho_0 \in W_2^{2+l}(G)$ *with a certain*

$l \in (1/2, 1)$ *and the following compatibility conditions*

$$\nabla \cdot \mathbf{u}_0^{(i)} = l_2^{(i)}(\mathbf{u}_0^{(i)}, \rho_0), \quad y \in \mathcal{F}_i,$$
$$[\nu \Pi_0 S(\mathbf{u}_0)\mathbf{N}] = \mathbf{l}_3(\mathbf{u}_0, \rho_0), \quad y \in G,$$
$$\nabla \cdot \mathbf{h}_0^{(i)} = 0, \quad y \in \mathcal{F}_i,$$
$$[\mu \mathbf{h}_0 \cdot \mathbf{N}] = 0, \quad [(\mathbf{h}_0)\tau] = \mathbf{l}_7(\mathbf{h}_0, \rho_0), \quad [\frac{1}{\alpha}(rot\mathbf{h}_0)_\tau] = \mathbf{l}_8(\mathbf{h}_0, \mathbf{u}_0, \rho_0), \quad [\mathbf{u}_0] = 0 \quad y \in G,$$
$$\mathbf{h}_0^{(2)} \cdot \mathbf{n} = 0, \quad (rot\mathbf{h}_0^{(2)})_\tau = 0, \quad \mathbf{u}_0^{(2)} = 0 \quad y \in S$$

hold. We assume that the smallness condition

$$\|\rho_0\|_{W_2^{2+l}(G)} \leq \varepsilon$$

is satisfied. Then problem (6.7) *has a unique solution on a certain small time interval* $(0, T)$ *with the following regularity properties*

$$\rho \in W_2^{5/2+l,0}(G_T) \cap W_2^{l/2}((0,T), W_2^{5/2}(G)), \quad \rho_t \in W_2^{3/2+l,3/4+l/2}(G_T),$$
$$\mathbf{u}^{(i)} \in W_2^{2+l,1+l/2}(\mathcal{F}_i \times (0,T)), \qquad \mathbf{h}^{(i)} \in W_2^{2+l,1+l/2}(\mathcal{F}_i \times (0,T)),$$
$$q \in W_2^{1/2+l,0}(G_T) \cap W_2^{l/2}((0,T); W_2^{1/2}(G)), \qquad \nabla q \in W_2^{l,l/2}(\mathcal{F}_i \times (0,T)).$$

References

[1] O. A. Ladyzhenskaya and V. A. Solonnikov, *Solution of some non-stationary problems of magnetohydrodynamics for a viscous incompressible fluid.* - Trudy mat. Inst. Steklov **59** (1960), 115-173.

[2] O. A. Ladyzhenskaya and V. A. Solonnikov, *The linearization principle and invariant manifolds for problems of magnetohydrodynamics.* - Zap. Nauchn. Sem. POMI **38**, (1973), 46-93.

[3] M. Padula and V. A. Solonnikov, *On the free boundary problem of magnethydrodynamics.*- Zap. Nauchn. Sem. POMI **385**, (2010), 135-186.

[4] V. A. Solonnikov and E. V. Frolova *Solvability of a free boundary problem of magnetohydrodynamics in an infinite time interval.* - Zap. Nauchn. Sem. POMI **410**, (2013), 131-167.

[5] E. V. Frolova *Free boundary problem of magnetohydrodynamics.* - Zap. Nauchn. Sem. POMI **425**, (2014), 149-178.

[6] V.A. Solonnikov, *On the stability of uniformly rotating liquid in a weak magnetic field.*- Problemy Mat. Anal. **57**, (2011), 165-191.

[7] M. Padula and V. A. Solonnikov, *On stability of equlilbrium figures of a uniformly rotating liquid drop in n-dimentional space.* - RIMS Kokyuroku Bessatsu, B1, (2007).

[8] M. Padula, *On the exponential stability of the rest state of a viscous compressible fluid.* - J. Math. Fluid Mech. **1**, (1999), 62-77.

[9] S. Mosconi and V.A. Solonnikov, On a problem of magnetohydrodinamics in a multi-connected domain. Nonlinear analysis, **74** N2 (2010), 462-478.

[10] I.V.Denisova, Global classical solvability of an interface problem on the motion of two fluids, RIMS Kokyuroku Series, Kyoto University **1875**, (2014), 84-108.

[11] I.V.Denisova, Global L_2-solvability of a problem governing two-phase fluid motion without surface tension, Port. Math. **71**, (2014), 1-24.