Note on the analysis of Orr-Sommerfeld equations and application to boundary layer stability

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This note is about the stability of Prandtl boundary layer expansions, in the context of the two-dimensional Navier-Stokes equation. We explain the main arguments behind the recent analysis of [5], which shows an optimal Gevrey stability result for expansions of shear flow type. The core of the reasoning is a resolvent estimate for the fourth order Orr-Sommerfeld equation, which is derived by an iterative process reminiscent of splitting methods in numerical analysis. This iteration is explained here in an abstract setting, that might be helpful to other applications.

1 Introduction

The general concern of the present work is the dynamics of a viscous fluid near a solid boundary, in the regime of high Reynolds number. The starting point of the work is the 2D incompressible Navier-Stokes equation,

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p - \nu \Delta u = f, & \mathbf{x} \in \Omega, \\ \operatorname{div} u = 0, & \mathbf{x} \in \Omega, \\ u|_{\partial\Omega} = 0, & \end{cases} \tag{1.1}$$

set in the half-plane $\Omega = \{\mathbf{x} = (x, y) \in \mathbb{T} \times \mathbb{R}_+\}$. The parameter $0 < \nu \ll 1$ is the inverse Reynolds number. The third line in (1.1) is a no-slip condition on the fluid velocity u, which reflects the experimental observation that viscous fluids stick to solid boundaries, no matter the Reynolds number.

This Dirichet condition, typical of the parabolic nature of Navier-Stokes, is incompatible with the formal limit of the system when $\nu \to 0$, namely the incompressible Euler equation: it is well-known that this equation is well-posed under a simple no-flow condition $u \cdot n|_{\partial\Omega} = 0$, and the tangential velocity is in general non-zero at the boundary. This "discontinuity" of the boundary condition at $\nu = 0$ has a strong consequence on the asymptotics $\nu \to 0$. If u^ν and u^E are respectively the Navier-Stokes and Euler solutions, it tells us that

$$u^\nu \not\to u^E \quad \text{in } H^s(\Omega) \text{ for any } s > \frac{1}{2}.$$

More generally, convergence does not hold in any functional space on which the trace is continuous. In particular, ∇u^ν can not be uniformly bounded in ν in the vicinity of the

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boundary. This is the so-called boundary layer effect: a thin layer of fluid with high velocity gradients develops near the boundary. The boundary layer prevents u^ν of being a regular function of ν, and requires a refined asymptotic description.

A key step of this description was provided by Ludwing Prandtl in 1904 [12], who suggested an asymptotic expansion of the following type:

$$\begin{cases} u_1^\nu \approx U^E(t,x,y) + U^{BL}(t,x,y/\sqrt{\nu}) \\ u_2^\nu \approx V^E(t,x,y) + \sqrt{\nu}\, V^{BL}(t,x,y/\sqrt{\nu}) \end{cases} \tag{1.2}$$

where $u^E = (U^E, V^E)(t,x,y)$ is the Euler solution, and $u^{BL}(t,x,Y) = (U^{BL}, V^{BL})(t,x,Y)$ is a boundary layer corrector, which depends on the variable $Y = y/\sqrt{\nu}$. Thus, in the Prandtl model, the boundary layer has variations of typical length $\sqrt{\nu}$, which is natural considering the heat part of the Navier-Stokes equation. Note that the vertical component is rescaled by a factor $\sqrt{\nu}$ in view of the incompressibility condition. Denoting by

$$U^P(t,x,Y) = U^E(t,x,0) + U^{BL}(t,x,Y), \quad V^P(t,x,Y) = Y\partial_y V^E(t,x,0) + V^{BL}(t,x,Y)$$

the "full" velocity in the boundary layer, and inserting the previous expansion in the Navier-Stokes equation, one derives the famous Prandt system

$$\begin{cases} \partial_t U^P + U^P \partial_x U^P + V^P \partial_Y U^P + \partial_x p^P - \partial_Y^2 P = 0, \\ \partial_Y p^P = 0, \\ \partial_x U^P + \partial_Y V^P = 0, \end{cases} \tag{1.3}$$

endowed with the boundary conditions

$$U^P|_{Y=0} = V^P|_{Y=0} = 0, \quad U^P \to U^E(t,x,0),\ p^P(t,x,Y) \to p^E(t,x,0) \quad \text{as } Y \to +\infty.$$

The Prandtl system is characterized by three features. First, we see from the second line and the boundary condition that the pressure is not an unknown: $p^P(t,x,Y) = p^E(t,x,0)$ for all Y. Second, there is no evolution equation on v^P, which is simply recovered through the divergence-free condition. Third, the tangential diffusion is not kept in the equation for u^P. For these reasons, the description of the boundary layer flow by the Prandtl system (1.3) is very appealing, as it is much easier to simulate than the full 2D Navier-Stokes equation. Unfortunately, the formal expansion (1.2) turns out to have a limited range of validity, due to strong instabilities.

First, instabilities take place at the level of the Prandtl model itself. Briefly, in the case of data that are non-monotonic in y, this model allows for strong exponential growth of high frequencies. This leads to ill-posedness of system (1.3) in the Sobolev setting, while well-posedness results are only available in Gevrey spaces of low enough exponent. The understanding of the stability properties of the Prandtl model has benefited from many recent advances, and we refer to [4, 11, 3, 6, 1, 9] on these aspects.

Still, even when the Prandtl expansion can be constructed and yields a Navier-Stokes solution with an arbitrarily small source term (say $O(\nu^N)$ for any N), this solution can be very unstable at the level of the Navier-Stokes equation. To discuss this issue of boundary layer flow stability, we restrict in this note to the special case of steady shear flows. Namely, we will consider velocity fields of the type

$$\overline{u}^\nu(t,x,y) = \left(U^E(y) + U^{BL}(\frac{y}{\sqrt{\nu}}), 0\right). \tag{1.4}$$

For simplicity, we will assume that the source term f in (1.1a) is such that u^ν is an exact solution of the Navier-Stokes equation. A perturbation u to u^ν will then obey the following system:

$$\begin{cases} \partial_t u + (U^E + U^{BL}(\frac{\cdot}{\sqrt{\nu}}))\partial_x u + u_2 \frac{d}{dy}(U_E + U^{BL}(\frac{\cdot}{\sqrt{\nu}}))e_1 + u\cdot\nabla u + \nabla p - \nu\Delta u = 0, \quad \mathbf{x}\in\Omega, \\ \qquad \text{div } u = 0, \quad \mathbf{x}\in\Omega, \\ \qquad u|_{\partial\Omega} = 0. \end{cases} \tag{1.5}$$

The main stability question is then: if $u_0 = u|_{t=0}$ is small, does u remain small over a reasonable period of time, namely independent of ν ? The difficulty raised by this question can be seen from a standard L^2 estimate: it involves the integral term $\int_\Omega u_2 \partial_y (U(\cdot/\sqrt{\nu}))u_1$, which through a naive estimate yields

$$\partial_t \|u\|^2_{L^2} \le \frac{C}{\sqrt{\nu}}\|u\|^2_{L^2},$$

as $\partial_y U = O(\frac{1}{\sqrt{\nu}})$. It gives the poor control $\|u(t)\|^2_{L^2} \le \|u_0\|_{L^2} e^{Ct\sqrt{\nu}}$. In particular, if $u_0 = O(\nu^N)$ for large N, this yields a stability estimate only over times of order $\sqrt{\nu}|\ln(\nu)|$.

It turns out that this estimate is, in full generality, optimal. Indeed, it was shown by Grenier in [7] that for some shear flow profiles U^{BL} with inflexion points (and zero U^E), there exist perturbations that have an approximate behaviour of the form

$$u(t,x,y) \approx e^{\lambda\frac{t}{\sqrt{\nu}}} e^{i\frac{x}{\sqrt{\nu}}} \hat{u}^\nu(y) \tag{1.6}$$

with $\text{Re}\,\lambda > 0$. Let us stress that the tangential frequency $k = \frac{1}{\sqrt{\nu}} \gg 1$ experiences a growth at rate $\sigma = \frac{\text{Re}\,\lambda}{\sqrt{\nu}} = \text{Re}\,\lambda k$, that is proportional to the frequency. This suggests that stability over times of order $O(1)$ can only be achieved for analytic data u_0, in coherence with the stability results in [14, 10]. The instability mechanism exhibited by Grenier comes from a linear instability for the Euler equation

$$\partial_\tau u + U^{BL}\partial_X u + u_2 \frac{d}{dY}U^{BL} e_1 + \nabla_{X,Y} p = 0,$$

after proper rescaling. The necessary condition that U^{BL} has an inflexion point comes from the celebrated Rayleigh criterion. But in the context of boundary layer problems, it is associated to the so-called region of separation, or reverse flow, for which the Prandtl model itself is known to badly behave in many ways [2, 4].

Hence, from this perspective, it is more natural to consider concave increasing boundary layer profiles U^{BL}, which are linearly stable for Euler. Still, as known from the celebrated works of Tollmien and Schlichting, and recently revisited in [8], viscosity has a destabilizing effect.

Roughly, when properly rescaled, this Tollmien-Schlichting instability provides approximate solutions of (1.5) of the form

$$u(t,x,y) \approx e^{\lambda_\nu t} e^{i\nu^{-3/8}x} \hat{u}^\nu(y) \tag{1.7}$$

with $\operatorname{Re}\lambda_\nu = O(\nu^{-1/4})$. Let us note that in this case, the high-frequency instability is weaker than in the case with inflexion points: frequency $k = \nu^{-3/8} \gg 1$ is amplified at a rate $\sigma_\nu = O(\nu^{-1/4}) = O(k^{2/3})$. In view of this remark, one may hope to show stability of boundary layer flows (1.4) for a more general class of initial data, namely of Gevrey class with exponent less than $\frac{3}{2}$. This is exactly the kind of result that is obtained in [5] and that we will now explain.

2 An optimal Gevrey stability result

In this note, we restrict to steady shear flow solutions of (1.1) of boundary layer type:

$$\bar{u}^\nu(t,x,y) = \big(U^E(y) - U^E(0) + U(y/\sqrt{\nu})\big)\, e_1$$

with U satisfying a strict concavity condition. We will remind the Gevrey stability estimate established in [5] in this special context. Let us emphasize that more general stability results are available in [5] (for time dependent, weakly concave shear flows of boundary layer type). Still, the steady and strictly concave case contains key ideas of the more global results. More precisely, we assume that the functions U^E and U^{BL} that appear in (1.4) satisfy

(a) $U^E, U \in BC^2(\mathbb{R}_+)$, $U|_{Y=0} = 0$, $\lim_{Y\to\infty} U = U^E|_{y=0} > 0$, and

$$\sum_{k=0,1,2} \sup_{Y\geq 0} (1+Y)^k |U^{(k)}(Y)| < \infty\,. \tag{2.1}$$

(b) There exists $M > 0$ such that $-MU''(Y) \geq (U'(Y))^2$ for $Y \geq 0$.

Conditions (a)-(b) imply that $U'(Y) > 0$ and $U''(Y) < 0$ for all $Y \geq 0$. Although the requirement $U''|_{Y=0} \neq 0$ is a bit restrictive, it simplifies the whole stability argument and allows to recover the optimal Gevrey exponent, which is far from trivial even under (a)-(b).

We then write the Navier-Stokes equation (1.1) in perturbative form: $u^\nu = \bar{u}^\nu + u$, with

$$\begin{cases} \partial_t u + \mathbb{A}_\nu u \;=\; -\mathbb{P}\big(u\cdot\nabla u\big)\,, \qquad t>0\,,\\ \qquad\quad u|_{t=0} \;=\; a\,. \end{cases} \tag{2.2}$$

Here, $u = (u_1(t,x,y), u_2(t,x,y))$, $(x,y) \in \Omega = \mathbb{T}\times\mathbb{R}_+$ (2π-periodic in x), and $\mathbb{P} : L^2(\Omega)^2 \to L^2_\sigma(\Omega)$ is the Helmholtz-Leray projection. The linear operator $\mathbb{A}_\nu$ is defined as

$$\mathbb{A}_\nu u \;=\; -\nu\mathbb{P}\Delta u + \mathbb{P}\bigg(\big(U^E + U^{BL}\big)\partial_x u + u_2 \partial_y \big(U^E + U^{BL}\big)\mathbf{e}_1\bigg)\,, \tag{2.3}$$

with the domain $D(\mathbb{A}_\nu) = W^{2,2}(\Omega)^2 \cap W^{1,2}_0(\Omega)^2 \cap L^2_\sigma(\Omega)$. To state the main result in [5], we need to introduce a few more notations. Let

$$\big(\mathcal{P}_n f\big)(y) \;=\; f_n(y) e^{inx}\,, \qquad f_n(y) \;=\; \frac{1}{2\pi}\int_0^{2\pi} f(x,y) e^{-inx}\mathrm{d}x\,, \qquad n\in\mathbb{Z}\,, \tag{2.4}$$

be the projection on the Fourier mode n in x, $n \in \mathbb{Z}$. We then introduce, for $\gamma \in (0,1]$, $d \geq 0$, and $K > 0$ the Banach space $X_{d,\gamma,K}$ as

$$X_{d,\gamma,K} = \{f \in L^2_\sigma(\Omega) \mid \|f\|_{X_{d,\gamma,K}} = \sup_{n\in\mathbb{Z}} (1+|n|^d) e^{K|n|^\gamma} \|\mathcal{P}_n f\|_{L^2(\Omega)} < \infty\}\,. \tag{2.5}$$

Fields in this space have L^2 regularity in y, and Gevrey regularity in x, of class s for any $s \geq \frac{1}{\gamma}$. We are now ready to state our main result.

Theorem 1 ([5]). *Suppose that assumptions (a)-(b) hold. For any $\gamma \in [\frac{2}{3}, 1)$, $d \geq 3$, and $K > 0$, there exist $C, T,' K' > 0$ such that the following statement holds for all sufficiently small $\nu > 0$. If $\|a\|_{X_{d,\gamma,K}} \leq \nu^{\frac{1}{2}+\beta}$ with $\beta = \frac{2(1-\gamma)}{\gamma}$ then the system (2.2) admits a unique solution $u \in C([0,T']; L^2_\sigma(\Omega)) \cap L^2(0,T'; W^{1,2}_0(\Omega)^2)$ satisfying the estimate*

$$\sup_{0<t\leq T'} \left(\|u(t)\|_{X_{d,\gamma,K'}} + (\nu t)^{\frac{1}{4}} \|u(t)\|_{L^\infty(\Omega)} + (\nu t)^{\frac{1}{2}} \|\nabla u(t)\|_{L^2(\Omega)} \right) \leq C \|a\|_{X_{d,\gamma,K}}\,. \tag{2.6}$$

Our result can be seen as improving the celebrated result of Sammartino and Caflisch, dedicated to the stability of Prandtl expansions in analytic regularity, see [13, 14]. Note nevertheless that article [13] treats general x-dependent boundary layer expansions, while ours restricts to the case of shear flows. Extension of our result to arbitrary (meaning x-dependent) expansions is a very interesting open problem.

The heart of the proof is the derivation of a growth bound for the semigroup generated by the operator $\mathbb{A}_\nu$ in (2.3). Once the estimate for the semigroup is obtained, the proof of Theorem 1 follows from Duhamel's formula; for details, see [5, Section 7]. As the operator $\mathbb{A}_\nu$ has constant coefficients in x, one can proceed independently for each Fourier mode n. The key estimates are contained in the following

Theorem 2 ([5]). *Suppose that (a)-(b) hold. There exist $C, C_0, C_1, K_0 > 0$ such that the following estimates hold for any $\gamma \in [\frac{2}{3}, 1]$ and for all sufficiently small $\nu > 0$.*

$$\|\mathcal{P}_n e^{-t\mathbb{A}_\nu}\|_{L^2\to L^2} \leq \left\{ \begin{array}{ll} Ce^{C_1 t}\,, & |n| \leq C_0\,, \\ Ce^{-\frac{1}{4}n^2\nu t}\,, & |n| \geq C_0 \nu^{-\frac{3}{4}}\,, \\ C|n|^{2(1-\gamma)} e^{K_0|n|^\gamma t}\,, & C_0 \leq |n| \leq C_0\nu^{-\frac{3}{4}}\,. \end{array} \right. \tag{2.7}$$

The hard part of this theorem is the last estimate, in the regime $\mathcal{O}(1) \leq |n| \leq \mathcal{O}(\nu^{-\frac{3}{4}})$. It is deduced from a bound on the resolvent equation:

$$\left((\mu + in(U^E + U^{BL}(\frac{\cdot}{\sqrt{\nu}})) \right) \hat{u} + \hat{u}_2 \frac{d}{dy}(U_E + U^{BL}(\frac{\cdot}{\sqrt{\nu}}))e_1 + \begin{pmatrix} in \\ \partial_y \end{pmatrix} \hat{p} - \nu(\partial_y^2 - n^2)u = \hat{f}$$

After introduction of the stream function and use of the rescaled variable $Y = \frac{y}{\sqrt{\nu}}$, this resolvent equation can be expressed in terms of an Orr-Sommerfeld equation:

$$-\varepsilon(\partial_Y^2 - \alpha^2)^2\Phi + (U - c)(\partial_Y^2 - \alpha^2)\Phi - U''\Phi = H, \tag{2.8}$$

where $\alpha = n\sqrt{\nu}$, $\varepsilon = -\frac{i}{n}$, $c = \frac{\mu}{i\alpha}$. See [5] for details. This fourth-order equation is completed by both a Dirichlet and Neumann condition on the stream function φ. To show the second estimate of Theorem 2, we must be able to control Φ in terms of H for any c with $\mathcal{I}m\, c$

large compared to $n^{\gamma-1}$. In particular, for $\gamma \in [\frac{2}{3}, 1[$, $\mathcal{I}m\, c$ can take small values, while the standard estimates only provide a control for $\mathcal{I}m\, c \gg 1$, that is for $\gamma = 1$. Hence, the case $\gamma < 1$ requires new ideas, and as we discussed in the introduction, it should rely on the concavity of U.

A core idea in [5] is to solve the Orr-Sommerfeld by an iterative process, based on the alternate resolution of the inviscid part (Rayleigh equation) and viscous part (Airy type equation) of the equation. This Rayleigh-Airy iteration is reminiscent of splitting methods in numerical analysis, and finds its origin in the work of [8], about the linear instability of monotonic shear flows in the Navier-Stokes equation. However, we stress that the approach in [8] is dedicated to a specific regime of spectral parameter and Fourier frequency, for which the fundamental solutions of the Rayleigh and Airy equations have quite explicit expressions. The specific regime of [8] is enough to construct an unstable eigenfunction, but is far from sufficient here.

The next section is dedicated to the presentation of the iteration argument from [5]. We will describe it in an abstract setting, to illustrate the main steps and the robustness of the method. We hope that it will be useful in this way to other problems.

3 Abstract iteration argument

In this section we consider the abstract form of the Rayleigh-Airy iteration used in [5]. This method in [5] is simple and robust, and in fact, it can be modeled by the analysis of a suitable combination of self-adjoint operators in a general Hilbert space. The keypoint is that, after multiplication by $\pm i$ (depending on the sign of n), the Orr-Sommerfeld equation can be written in the form:

$$L_{\lambda,\epsilon}\Phi = h\,, \qquad h \in H \tag{3.1}$$

where $\lambda \in \mathbb{C}$, $\epsilon > 0$ and

$$L_{\lambda,\epsilon} = \epsilon A^2 - \lambda A + i\big(-B_1 A + B_2\big) \tag{3.2}$$

for suitable self-adjoint operators A, B_1, B_2. Namely, A is the Laplace operator $(\partial_y^2 - \alpha^2)$, while each B_j is a multiplication by a real-valued function. One important remark here is that this iteration method is useful in constructing a solution to (2.8) but subject to only one boundary condition, e.g., the Dirichlet boundary condition when $A = (\partial_y^2 - \alpha^2)$ is realized with the Dirichlet boundary condition. Recovering the second boundary condition (the Neumann boundary condition) in the case of (2.8) is another problem and requires an additional (hard) work. The latter problem is more specific to the actual boundary problem and is not handled in the abstract framework. Nevertheless, to construct solutions with one boundary layer condition using the iteration method is certainly one of the core parts in [5] and can be explained in the abstract setting. With this remark in mind we shall make the following abstract assumption:

Assumption 1. *Let H be a Hilbert space.*
(i) $-A$ *is self-adjoint in H and positive, i.e., there exists $\alpha > 0$ such that $\langle -Au, u\rangle_H \geq \alpha^2 \|u\|_H^2$ for all $u \in D(A)$.*

(ii) B_1 *and* B_2 *are bounded self-adjoint in H, $B_1 B_2 = B_2 B_1$, and B_2 is injective and nonnegative. Moreover, $D(A) \subset D(AB_1)$ and there exists $C > 0$ such that* $\|B_2^{\frac{1}{2}} u\|_H +$

$\|[B_1,-A]u\|_H \le C\|(-A)^{\frac{1}{2}}u\|_H$ *for all* $u \in D(A)$. *Here* $[B_1,-A]$ *is the formal commutator* $[B_1,-A] = B_1(-A) - (-A)B_1$.

(iii) *There exists* $C > 0$ *such that* $\|B_2^{-\frac{1}{2}}[B_1,-A]u\|_H \le C\|(-A)^{\frac{1}{2}}u\|_H$ *for all* $u \in D(A)$.

Since B_2 is nonnegative and injective its square root and its inverse are defined by the spectral resolution of B_2. Note that $B_2^{-\frac{1}{2}}$ is not necessarily bounded. In the actual application to the boundary layer problem, we take $H = L^2(\mathbb{R}_+)$, $A = \partial_Y^2 - \alpha^2$. The condition $\|B_2^{\frac{1}{2}}u\|_H \le C\|(-A)^{\frac{1}{2}}u\|_H$ corresponds to the Hardy inequality. The non-negativity of B_2 stated in (ii) is nothing but the non-negativity of the multiplier function, which corresponds to the concavity of the boundary layer profile: $-U'' \ge 0$. The last condition (iii) describes the strong concavity condition assumed in [5], i.e., $-MU'' \ge (U')^2$ for some constant $M > 0$ (*cf* the previous section).

In the search for good estimates of equation (3.1), we shall consider first the invertibility of $L_{\lambda,0}$ under Assumption 1 (i), (ii).

Lemma 1. *Suppose that the conditions (i) and (ii) of Assumption 1 hold. Let* $\mathrm{Re}\,\lambda \neq 0$. *Then* $L_{\lambda,0} : D(A) \to H$ *is invertible. Moreover,*

$$\|AL_{\lambda,0}^{-1}h\|_H \le \frac{C}{\alpha|\mathrm{Re}\,\lambda|}\big(\frac{1+|\mathcal{I}m\,\lambda|}{|\mathrm{Re}\,\lambda|}+1\big)\|h\|_H\,, \qquad h \in H\,, \tag{3.3}$$

$$\|AL_{\lambda,0}^{-1}h\|_H \le \frac{C}{|\mathrm{Re}\,\lambda|}\|B_2^{-\frac{1}{2}}h\|_H\,, \qquad \text{if } B_2^{-\frac{1}{2}}h \in H\,. \tag{3.4}$$

Here C *depends only on* $\|B_j\|_{H\to H}$, $j = 1,2$.

Proof. Since $L_{\lambda,0} = i\big\{(B_1 - i\lambda)(-A) + B_2\big\}$ it suffices to consider

$$(B_1 - i\lambda)(-A)\varphi + B_2\varphi = h\,,$$

which is equivalent with

$$-A\varphi + (B_1 - i\lambda)^{-1}B_2\varphi = (B_1 - i\lambda)^{-1}h \tag{3.5}$$

since B_1 is self-adjoint and $\mathrm{Re}\,\lambda \neq 0$. It suffices to show the a priori estimate of the solution to (3.5). For the actual construction of the solution, we replace $-A$ by $-A+\mu$ with $\mu \gg 1$ first, in which the unique solvability falls into the standard Neumann series argument, and then we take $\mu \to 0$ together with the a priori estimates of solutions which we will show below. Note that the argument below holds even if we replace $-A$ by $-A+\mu$ with $\mu \ge 0$, and the estimates are independent of $\mu \ge 0$. The details on this process are omitted here.

Taking the inner product with φ in (3.5), we have

$$\|(-A)^{\frac{1}{2}}\varphi\|_H^2 + \langle B_2\varphi, (B_1 + i\bar{\lambda})^{-1}\varphi\rangle_H = \langle h, (B_1 + i\bar{\lambda})^{-1}\varphi\rangle_H\,. \tag{3.6}$$

Set $f = (B_1 + i\bar{\lambda})^{-1}\varphi$. Then

$$\langle B_2\varphi, f\rangle_H = \langle B_2(B_1 + i\bar{\lambda})f, f\rangle_H = \langle B_2B_1f, f\rangle_H + i\bar{\lambda}\langle B_2f, f\rangle_H\,.$$

Hence we have from $B_2B_1 = B_1B_2$,

$$\mathrm{Re}\,\langle B_2\varphi, f\rangle_H = \langle (B_1 + \mathcal{I}m\,\lambda)B_2f, f\rangle_H$$

and

$$\mathcal{I}m\,\langle B_2\varphi, f\rangle_H = \mathrm{Re}\,\lambda \|B_2^{\frac{1}{2}} f\|_H^2\,.$$

Taking the real part and the imaginary part of (3.6), we obtain for $f = (B_1 + i\bar{\lambda})^{-1}\varphi$,

$$\|(-A)^{\frac{1}{2}}\varphi\|_H^2 + \langle (B_1 + \mathcal{I}m\,\lambda)B_2 f, f\rangle_H = \mathrm{Re}\,\langle h, f\rangle_H\,, \tag{3.7}$$

and

$$\|B_2^{\frac{1}{2}} f\|_H^2 = \frac{1}{\mathrm{Re}\,\lambda}\mathcal{I}m\,\langle h, f\rangle_H\,. \tag{3.8}$$

Since $B_1B_2 = B_2B_1$ we have $B_1B_2f = B_2^{\frac{1}{2}}B_1B_2^{\frac{1}{2}}f$, which yields

$$|\langle B_1B_2f, f\rangle_H| = |\langle B_1B_2^{\frac{1}{2}}f, B_2^{\frac{1}{2}}f\rangle_H| \le \|B_1\|_{H\to H}\|B_2^{\frac{1}{2}}f\|_H^2\,,$$

where we have used the boundedness of B_1. Thus we have from (3.7) and (3.8),

$$\begin{aligned}\|(-A)^{\frac{1}{2}}\varphi\|_H^2 &\le \big(\|B_1\|_{H\to H} + |\mathcal{I}m\,\lambda)\|B_2^{\frac{1}{2}}f\|_H^2 + \mathrm{Re}\,\langle h, f\rangle_H \\ &= \frac{\|B_1\|_{H\to H} + |\mathcal{I}m\,\lambda|}{\mathrm{Re}\,\lambda}\mathcal{I}m\,\langle h, f\rangle_H + \mathrm{Re}\,\langle h, f\rangle_H\end{aligned}$$

Since

$$\|f\|_H = \|(B_1 + i\bar{\lambda})^{-1}\varphi\|_H \le \frac{1}{|\mathrm{Re}\,\lambda|}\|\varphi\|_H \tag{3.9}$$

we obtain also from $\|\varphi\|_H \le \alpha^{-1}\|(-A)^{\frac{1}{2}}\varphi\|_H$ by (i) of Assumption 1,

$$\|(-A)^{\frac{1}{2}}\varphi\|_H \le \frac{1}{\alpha|\mathrm{Re}\,\lambda|}\left(\frac{\|B_1\|_{H\to H} + |\mathcal{I}m\,\lambda|}{|\mathrm{Re}\,\lambda|} + 1\right)\|h\|_H\,. \tag{3.10}$$

Then

$$\begin{aligned}\|B_2^{\frac{1}{2}}f\|_H^2 &\le \frac{1}{|\mathrm{Re}\,\lambda|^2}\|h\|_H\|\varphi\|_H \\ &\le \frac{1}{\alpha^2|\mathrm{Re}\,\lambda|^3}\left(\frac{\|B_1\|_{H\to H} + |\mathcal{I}m\,\lambda|}{|\mathrm{Re}\,\lambda|} + 1\right)\|h\|_H^2\,.\end{aligned} \tag{3.11}$$

Therefore,

$$\begin{aligned}\|(-A)\varphi\|_H &= \|(B_1 - i\lambda)^{-1}h - (B_1 - i\lambda)^{-1}B_2\varphi\|_H \\ &\le \frac{1}{|\mathrm{Re}\,\lambda|}\|h\|_H + \|(B_1 - i\lambda)^{-1}(B_1 + i\bar{\lambda})B_2f\|_H \\ &\le \frac{1}{|\mathrm{Re}\,\lambda|}\|h\|_H + 3\|B_2f\|_H \\ &\le \frac{1}{|\mathrm{Re}\,\lambda|}\|h\|_H + 3\|B_2\|_{H\to H}\|B_2^{\frac{1}{2}}f\|_H\,.\end{aligned} \tag{3.12}$$

Combining (3.11) with (3.12), we obtain (3.3). If $B_2^{-\frac{1}{2}}h \in H$ in addition, then

$$\|B_2^{\frac{1}{2}}f\|_H^2 = \frac{1}{\operatorname{Re}\lambda}\mathcal{I}m\,\langle h, f\rangle_H \leq \frac{1}{|\operatorname{Re}\lambda|}\|B_2^{-\frac{1}{2}}h\|_H\|B_2^{\frac{1}{2}}f\|_H$$

which gives $\|B_2^{\frac{1}{2}}f\|_H \leq \frac{1}{|\operatorname{Re}\lambda|}\|B_2^{-\frac{1}{2}}h\|_H$. Then (3.12) yields (3.4). The proof is complete.

We now turn to the existence of the solution to (3.1) for sufficiently small $\epsilon > 0$ when $|\lambda| \leq 1$ with $\operatorname{Re}\lambda > 0$. Due to the boundedness of B_1 and B_2 the solvability for the case $|\lambda| \gg 1$ is in fact not difficult. This is the reason why we focus only on the case $|\lambda| \leq 1$ as a typical case of $|\lambda| \leq \mathcal{O}(1)$. The basic strategy is to construct the solution around the solution to $L_{\lambda,0}\varphi_0 = h$, which is already solved in Lemma 1 since $\operatorname{Re}\lambda \neq 0$. Clearly we have

$$L_{\lambda,\epsilon}\varphi_0 = h + \epsilon A^2\varphi_0\,, \tag{3.13}$$

where the equation should be considered in the weak sense due to the lack of the regularity of φ_0, i.e.,

$$\epsilon\langle A\varphi_0, A\chi\rangle_H - \lambda\langle A\varphi_0, \chi\rangle_H + i\langle(-B_1A + B_2)\varphi_0, \chi\rangle_H = \langle h, \chi\rangle_H + \epsilon\langle A\varphi_0, A\chi\rangle_H\,, \qquad \chi \in D(A)\,. \tag{3.14}$$

Now we observe that

$$L_{\lambda,\epsilon} = -A\Big(- \epsilon A + i(B_1 - i\lambda)\Big) + i[B_1, -A] + iB_2 \tag{3.15}$$

where $[B_1, -A] = B_1(-A) - (-A)B_1$ without taking into account the relation on the domain of the operators. To correct the error $\epsilon A^2\varphi_0$ in (3.13), we consider the problem

$$-\epsilon A\psi + i(B_1 - i\lambda)\psi = \epsilon h \tag{3.16}$$

for a given $h \in H$. That is, we have dropped the lower order terms $i[B_1, -A]$ and iB_2, and we make use of the presence of the common factor A. The solvability of (3.16) is not difficult, as stated below.

Lemma 2. *Suppose that the conditions (i) and (ii) of Assumption 1 hold. Let* $\operatorname{Re}\lambda > 0$*. Then for any* $h \in H$ *there exists a unique solution* $\psi \in D(A)$ *to* (3.16) *and satisfies*

$$\|\psi\|_H \leq \frac{\epsilon}{\operatorname{Re}\lambda}\|h\|_H\,, \tag{3.17}$$

$$\|(-A)^{\frac{1}{2}}\psi\|_H \leq \big(\frac{\epsilon}{\operatorname{Re}\lambda}\big)^{\frac{1}{2}}\|h\|_H\,, \tag{3.18}$$

$$\|A\psi\|_H \leq C\Big(\big(\frac{\epsilon}{(\operatorname{Re}\lambda)^3}\big)^{\frac{1}{2}} + 1\Big)\|h\|_H\,. \tag{3.19}$$

Here C depends only on the constant appearing in the condition (ii) of Assumption 1.

Proof. Again it suffices to show the a priori estimate. Taking the inner product with ψ in (3.16), we have

$$\epsilon\|(-A)^{\frac{1}{2}}\psi\|_H^2 + i\langle(B_1 - i\lambda)\psi, \psi\rangle_H = \epsilon\langle h, \psi\rangle_H\,. \tag{3.20}$$

The real part of (3.20) yields from the symmetry of B_1,

$$\epsilon\|(-A)^{\frac{1}{2}}\psi\|_H^2 + \mathrm{Re}\,\lambda\,\|\psi\|_H^2 = \epsilon\mathrm{Re}\,\langle h,\psi\rangle_H\,. \tag{3.21}$$

This gives the bounds (3.17) and (3.18). We will show (3.19). By taking the inner product with $-A\psi$ in (3.16) we have

$$\epsilon\|A\psi\|_H^2 + i\langle(B_1-i\lambda)\psi,-A\psi\rangle_H = \epsilon\langle h,-A\psi\rangle_H\,.$$

The real part of this identity gives

$$\epsilon\|A\psi\|_H^2 + \mathrm{Re}\,\lambda\,\|(-A)^{\frac{1}{2}}\psi\|_H^2 = \mathcal{I}m\,\langle B_1\psi,-A\psi\rangle_H + \epsilon\mathrm{Re}\,\langle h,-A\psi\rangle_H\,. \tag{3.22}$$

We observe that

$$\begin{aligned}\langle B_1\psi,-A\psi\rangle_H &= \langle B_1\psi,-A\psi\rangle_H - \langle -A\psi,B_1\psi\rangle_H + \langle -A\psi,B_1\psi\rangle_H\\ &= \langle\psi,[B_1,-A]\psi\rangle_H + \overline{\langle B_1\psi,-A\psi\rangle_H}\,,\end{aligned}$$

and thus,

$$\begin{aligned}|\mathcal{I}m\,\langle B_1\psi,-A\psi\rangle_H| = \frac{1}{2}|\langle\psi,[B_1,-A]\psi\rangle_H| &\le \frac{1}{2}\|\psi\|_H\|[B_1,-A]\psi\|_H\\ &\le C\|\psi\|_H\|(-A)^{\frac{1}{2}}\psi\|_H\\ &\le \frac{C}{\mathrm{Re}\,\lambda}\|\psi\|_H^2 + \frac{\mathrm{Re}\,\lambda}{2}\|(-A)^{\frac{1}{2}}\psi\|_H^2\,.\end{aligned} \tag{3.23}$$

Here we have used the condition (ii) of Assumption 1. Collecting (3.22) and (3.23) with (3.17), we obtain (3.19). The proof is complete.

We define $\{\varphi_k\}_{k=0}^\infty$ and $\{\psi_k\}_{k=0}^\infty$ iteratively as follows. As already stated, $\varphi_0\in D(A)$ is the solution to $L_{\lambda,0}\varphi_0 = h$. Then $\psi_0\in D(A)$ is defined as the solution to (3.16) with $h = A\varphi_0$. Then $\varphi_0+\psi_0$ solves $L_{\lambda,\epsilon}(\varphi_0+\psi_0) = h + i[B_1,-A]\psi_0 + iB_2\psi_0$ in the weak form,

$$\begin{aligned}&\langle\epsilon A(\varphi_0+\psi_0),A\chi\rangle_H - \lambda\langle A(\varphi_0+\psi_0),\chi\rangle_H + i\langle(-B_1A+B_2)(\varphi_0+\psi_0),\chi\rangle_H\\ &\quad = \langle h + i[B_1,-A]\psi_0 + iB_2\psi_0,\chi\rangle_H\,,\qquad \chi\in D(A)\,,\end{aligned} \tag{3.24}$$

and $i[B_1,-A]\psi_0 + iB_2\psi_0$ is the new error term. Note that the term $[B_1,-A]\psi_0$ makes sense by Assumption 1 and $\psi_0\in D(A)$. We iterate this process. That is, set φ_{k+1} is the solution to

$$L_{\lambda,0}\varphi_{k+1} = -i[B_1,-A]\psi_k - iB_2\psi_k\,, \tag{3.25}$$

while ψ_{k+1} is set as the solution to (3.16) with $h = A\varphi_{k+1}$. Then $\Phi_N = \sum_{k=0}^N\varphi_k + \sum_{k=0}^N\psi_k$ solves $L_{\lambda,\epsilon}\Phi_N = h + i[B_1,-A]\psi_N + iB_2\psi_N$ in the weak form,

$$\begin{aligned}&\langle\epsilon A\Phi_N,A\chi\rangle_H - \lambda\langle A\Phi_N,\chi\rangle_H + i\langle(-B_1A+B_2)\Phi_N,\chi\rangle_H\\ &\quad = \langle h + i[B_1,-A]\psi_N + iB_2\psi_N,\chi\rangle_H\,,\qquad \chi\in D(A)\,.\end{aligned} \tag{3.26}$$

Then our aim is to show that the series $\sum_{k=0}^{N}\varphi_k$ and $\sum_{k=0}^{N}\psi_k$ converge in $D(A)$ when $N\to\infty$. Indeed, once we have shown this convergence, the limit $\Phi=\lim_{N\to\infty}\Phi_N$ belongs to $D(A)$ and solves $L_{\lambda,\epsilon}\Phi=h$ in the weak sense:

$$\langle\epsilon A\Phi, A\chi\rangle_H-\lambda\langle A\Phi,\chi\rangle_H+i\langle(-B_1A+B_2)\Phi,\chi\rangle_H=\langle h,\chi\rangle_H\,,\qquad \chi\in D(A)\,. \tag{3.27}$$

By the fact that $-\lambda A\Phi+i\big(-B_1A+B_2)\Phi\in H$, the identity (3.27) implies that $A\Phi\in D(A)$, that is, $\Phi\in D(A^2)$ and Φ solves $L_{\lambda,\epsilon}\Phi=h$ in the classical sense.

It remains to check the convergence of $\sum_{k=0}^{\infty}\varphi_k$ and $\sum_{k=0}^{\infty}\psi_k$. Let us recall that $|\lambda|\le 1$ and $\operatorname{Re}\lambda>0$. We will assume all conditions of Assumption 1. For $k\ge 0$ we see from (3.25) and Lemma 1,

$$\begin{aligned}\|A\varphi_{k+1}\|_H&\le\frac{C}{|\operatorname{Re}\lambda|}\big(\|B_2^{-\frac12}[B_1,-A]\psi_k\|_H+\|B_2^{\frac12}\psi_k\|_H\big)\\&\le\frac{C}{|\operatorname{Re}\lambda|}\|(-A)^{\frac12}\psi_k\|_H\,.\end{aligned}$$

Here we have used the key condition (iii) as well as (ii) of Assumption 1. Since ψ_k is the solution to (3.16) with $h=A\varphi_k$, Lemma 2 implies $\|(-A)^{\frac12}\psi_k\|_H\le(\frac{\epsilon}{\operatorname{Re}\lambda})^{\frac12}\|A\varphi_k\|_H$. Thus we obtain

$$\|A\varphi_{k+1}\|_H\le C\big(\frac{\epsilon}{(\operatorname{Re}\lambda)^3}\big)^{\frac12}\|A\varphi_k\|_H\,,\qquad k\ge 0\,. \tag{3.28}$$

Lemma 2 also implies

$$\|A\psi_k\|_H\le C\bigg(\big(\frac{\epsilon}{(\operatorname{Re}\lambda)^3}\big)^{\frac12}+1\bigg)\|A\varphi_k\|_H\,,\qquad k\ge 0\,. \tag{3.29}$$

Therefore, we conclude from (3.28) and (3.29) that the series $\sum_{k=0}^{\infty}\varphi_k$ and $\sum_{k=0}^{\infty}\psi_k$ converges in $D(A)$ if $|\lambda|\le 1$ and $\frac{\epsilon}{(\operatorname{Re}\lambda)^3}$ is sufficiently small. In the end, this results in the following theorem.

Theorem 3. *Suppose that Assumption 1 holds. Let $|\lambda|\le 1$ and $\operatorname{Re}\lambda>0$. If $\dfrac{\epsilon}{(\operatorname{Re}\lambda)^3}$ is sufficiently small then for any $h\in H$ there exists a solution $\Phi\in D(A^2)$ to the problem $L_{\lambda,\epsilon}\Phi=h$ satisfying the estimate*

$$\|A\Phi\|_H\le C\|AL_{\lambda,0}^{-1}h\|_H\,. \tag{3.30}$$

Here C is independent of λ, ϵ, α, and h.

When applied in the context of the Orr-Sommerfeld equation, this theorem allows to solve the resolvent estimate for $\mathcal{P}_n\mathbb{A}_\nu$, in a range of spectral parameters appropriate to the semigroup estimates of Theorem 2, as expected.

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