COMPUTATION OF QUOTIENT GROUPS OF INVERSE LIMITS OF BURNSIDE RINGS

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Abstract. Let \( G \) be a finite group. For a subgroup \( H \) of \( G \), we have the Burnside ring \( A(H) \) of \( H \). For a set \( \mathcal{F} \) of subgroups of \( G \), we have the inverse limit \( L(G) \) of the category consisting of modules \( A(H) \) \( (H \in \mathcal{F}) \), and the restriction homomorphism from \( A(G) \) to \( L(G) \). In this article, we discuss computational theory of the cokernel of the homomorphism.

1. INTRODUCTION

This article is supplementary to the joint work [5] with Masafumi Sugimura.

Let \( G \) be a finite group. Let \( S(G) \) denote the set of all subgroups of \( G \) and set \( \mathcal{F}_{G} = S(G) \setminus \{G\} \). Let \( \mathcal{G} \) be a subset of \( S(G) \) closed under taking subgroups and conjugations by elements of \( G \). For \( H \in S(G) \), let \( A(H) \) denote the Burnside ring of \( H \) and \( A(H, \mathcal{G} \cap S(H)) \) denote the submodule of \( A(H) \) generated by \([H/K]\) where \( K \) runs over \( \mathcal{G} \cap S(H) \) (see [2], [4]). We have the inverse limit \( L_{\mathcal{F}_{G}}(G, \mathcal{G}) = \lim_{\leftarrow \mathcal{F}_{G}} A \) of the functor \( A : H \mapsto A(H, \mathcal{G} \cap S(H)) \), where \( H \in \mathcal{F}_{G} \), which is a submodule of the cartesian product \( P_{\mathcal{F}_{G}}(G, \mathcal{G}) = \prod_{H \in \mathcal{F}_{G}} A(H, \mathcal{G} \cap S(H)) \), and the restriction homomorphism

\[
\text{res}_{\mathcal{F}_{G}}^{G} : A(G, \mathcal{G}) \longrightarrow L_{\mathcal{F}_{G}}(G, \mathcal{G})
\]

(see [3]). Let \( Q_{\mathcal{F}_{G}}(G, \mathcal{G}) \) denote the cokernel of \( \text{res}_{\mathcal{F}_{G}}^{G} \). It is interesting to compute the abelian group \( Q_{\mathcal{F}_{G}}(G, \mathcal{G}) \) for a given group \( G \). For a natural number \( k \), let \( C_k \) denote a cyclic group of order \( k \). Let \( p \) be a prime and let \( \mathbb{Z}_p \) denote a module

2010 Mathematics Subject Classification. Primary 19A22; Secondary 57S17.

Key words and phrases. Burnside ring, inverse limit, restriction homomorphism, matrix presentation.

This research was partially supported by JSPS KAKENHI Grant Number 26400090.
consisting of $p$ elements. Let $m$ and $n$ be non-negative integers. The following three computational results have been obtained.

**Theorem (Y. Hara–M. Morimoto [3]).**

1. For $G = C_{p^m}$ ($m \geq 1$), $Q_{\mathcal{F}_{G}}(G, S(G)) \cong \mathbb{Z}_{p}^{m-1}$.
2. For $G = C_{p_1 \cdots p_m}$ ($p_i$ distinct primes), $Q_{\mathcal{F}_{G}}(G, S(G)) = O$.

**Theorem (M. Sugimura [6]).**

1. For $G = C_{p^m} \times C_{p}$ ($m \geq 1$), $Q_{\mathcal{F}_{G}}(G, S(G)) \cong \mathbb{Z}_{p}^{(m-1)p+1}$.
2. For $G = C_{p^m} \times C_{p^2}$ ($m \geq 2$), $Q_{\mathcal{F}_{G}}(G, S(G)) \cong \mathbb{Z}_{p}^{(m-1)(p^2+1)+p+1}$.

**Theorem (M. Morimoto–M. Sugimura [5]).** For $G = C_{p^m} \times C_{p^n}$ ($m \geq n \geq 2$), $Q_{\mathcal{F}_{G}}(G, S(G)) \cong \mathbb{Z}_{p}^{r}$, where

$$r = p^n + 2p^{n-1} + \sum_{k=1}^{n-3} (2k + 1)p^{n-k-1} + (2n - 4)p + (2n - 2) + (m - n) \left( \sum_{k=0}^{n} p^k - p^{n-1} \right).$$

In the present paper, we discuss computational theory to obtain the results above. For example, the next fact follows from Lemma 3.1 and Theorem 6.1.

**Theorem 1.1.** Let $G$ be a finite (nontrivial) group and $N$ a normal subgroup of $G$. Suppose each maximal (proper) subgroup $H$ of $G$ contains $N$. Let $\mathcal{K}$ be the set of all subgroups $H$ of $G$ not containing $N$. Then it holds that

$$Q_{\mathcal{F}_{G}}(G, S(G)) \cong Q_{\mathcal{F}_{G}}(\overline{G}, S(\overline{G})) \oplus Q_{\mathcal{F}_{G}}(G, \mathcal{K}),$$

where $\overline{G} = G/N$.

The result of Morimoto–Sugimura [5] mentioned above is obtained by using Corollary 7.3.

2. **Inverse Limit $L_{\mathcal{F}}(F)$ and Remarks**

Let $\mathcal{S}(G)$ denote the subgroup category of $G$, namely the objects of $\mathcal{S}(G)$ are all subgroups of $G$ and the morphisms of $\mathcal{S}(G)$ are all triples $(H, g, K)$ consisting
of $H \in S(G)$, $g \in G$, $K \in S(G)$ such that $gHg^{-1} \subset K$. Here $(H, g, K)$ is a morphism from $H$ to $K$. Each morphism $(H, g, K)$ induces a (group) homomorphism $\iota_{H,g,K} : H \rightarrow K$ by $\iota(a) = gag^{-1}$ for $a \in H$.

Let $\mathcal{F}$ be a subset of $S(G)$. Let $\mathfrak{G}$ denote the full subcategory of $S(G)$ such that $\text{Obj}(\mathfrak{G}) = \mathcal{F}$ and let $\mathfrak{Ab}$ denote the category of abelian groups of which the objects are all abelian groups and the morphisms are all (group) homomorphisms. Let $F : \mathfrak{G} \rightarrow \mathfrak{Ab}$ be a contravariant functor. Then we have the inverse limit $\lim_{\leftarrow \mathcal{F}} F$ of $F$ with respect to $\mathcal{F}$, where

$$\lim_{\leftarrow \mathcal{F}} F = \left\{ (x_H) \in \prod_{H \in \mathcal{F}} F(H) \mid \forall (H, g, K) \in \text{Mor}(\mathfrak{G}) \, (H, g, K)^* x_K = x_H \right\}$$

(see [1]). This inverse limit will be denoted by $L_{\mathcal{F}}(F)$.

We subsequently use the next fact without specifically mentioning.

**Proposition 2.1.** If $F(H)$ is a free $\mathbb{Z}$-module for each $H \in \mathcal{F}$ then $\lim_{\leftarrow \mathcal{F}} F$ is a direct summand of $\prod_{H \in \mathcal{F}} F(H)$.

Let $e$ denote the identity element of $G$. For a morphism $(H, e, K)$ of $S(G)$, let $\text{res}^K_H$ denote the homomorphism $(H, e, G)^* : F(K) \rightarrow F(H)$. Let $P_{\mathcal{F}}(F)$ denote the cartesian product $\prod_{H \in \mathcal{F}} F(H)$. The image of the restriction homomorphism $\text{res}^G = \prod_{H \in \mathcal{F}} \text{res}^G_H : F(G) \rightarrow P_{\mathcal{F}}(F)$ is contained in $L_{\mathcal{F}}(F)$ and therefore $\text{res}^G$ induces $\text{res}_{\mathcal{F}}^G : F(G) \rightarrow L_{\mathcal{F}}(F)$. Let $B_{\mathcal{F}}(F)$ denote the image of $\text{res}_{\mathcal{F}}^G$ and $Q_{\mathcal{F}}(F)$ the quotient group $L_{\mathcal{F}}(F)/B_{\mathcal{F}}(F)$, namely the cokernel of $\text{res}_{\mathcal{F}}^G$.

Let $\mathcal{F}^*$ denote a complete set of representatives of $G$-conjugacy classes of subgroups belonging to $\mathcal{F}$ such that $\mathcal{F}^* \subset \mathcal{F}$. Let $\text{proj} : \prod_{H \in \mathcal{F}} F(H) \rightarrow \prod_{H \in \mathcal{F}^*} F(H)$ be the canonical projection. This gives the homomorphisms $\text{proj}_L : L_{\mathcal{F}}(F) \rightarrow L_{\mathcal{F}^*}(F)$ and $\text{proj}_B : B_{\mathcal{F}}(F) \rightarrow B_{\mathcal{F}^*}(F)$ as well as the homomorphism $\rho_Q : Q_{\mathcal{F}}(F) \rightarrow Q_{\mathcal{F}^*}(F)$. Clearly, the diagram

$$\begin{array}{ccc}
B_{\mathcal{F}}(F) & \rightarrow & L_{\mathcal{F}}(F) \\
\downarrow \text{proj}_B & & \downarrow \text{proj}_L \\
B_{\mathcal{F}^*}(F) & \rightarrow & L_{\mathcal{F}^*}(F) \\
\end{array}$$

commutes.
Proposition 2.2. In the diagram above, \( \text{proj}_B \), \( \text{proj}_L \), and \( \rho_Q \) all are isomorphisms.

Proof. By definition of inverse limit, \( \text{proj}_L \) is an isomorphism. Since \( \text{proj}_B \) is surjective and \( \text{proj}_L \) is injective, \( \text{proj}_B \) is bijective. As \( \text{proj}_L \) and \( \text{proj}_B \) are isomorphisms, \( \rho_Q \) is an isomorphism.

\( \square \)

3. Definition of \( Q_{\mathcal{F}}(G, \mathcal{G}) \) and Interpretation of \( L_{\mathcal{F}}(G, \mathcal{G}) \)

Let \( \mathcal{G} \) be a subset of \( \mathcal{S}(G) \) closed under taking conjugations by elements in \( G \). For a subgroup \( H \) of \( G \), let \( \mathcal{G}(\cap H) \) denote the set consisting of all \( K \cap H \), where \( K \) ranges over \( \mathcal{G} \). We have the restriction homomorphism \( \text{res}^{\mathcal{G}}_H : A(G, \mathcal{G}) \to A(H, \mathcal{G}(\cap H)) \). Let \( P_{\mathcal{F}}(G, \mathcal{G}) \) denote the cartesian product \( \prod_{H \in \mathcal{F}} A(H, \mathcal{G}(\cap H)) \). We have the restriction homomorphism \( \text{res}^G : A(G, \mathcal{G}) \to P_{\mathcal{F}}(G, \mathcal{G}) \). Let \( L_{\mathcal{F}}(G, \mathcal{G}) \) denote the inverse limit \( \lim_{\leftarrow \mathcal{F}} A \) of the Burnside ring functor \( A : H \mapsto A(H, \mathcal{G}(\cap H)) \), where \( H \in \mathcal{S}(G) \). Since the image of \( \text{res}^G \) above is contained in \( L_{\mathcal{F}}(G, \mathcal{G}) \), we obtain the restriction homomorphism \( \text{res}^{\mathcal{G}}_F : A(G, \mathcal{G}) \to L_{\mathcal{F}}(G, \mathcal{G}) \). Let \( B_{\mathcal{F}}(G, \mathcal{G}) \) denote the image of \( \text{res}^{\mathcal{G}}_F \) and let \( Q_{\mathcal{F}}(G, \mathcal{G}) \) denote the cokernel of \( \text{res}^{\mathcal{G}}_F \), i.e. \( Q_{\mathcal{F}}(G, \mathcal{G}) = L_{\mathcal{F}}(G, \mathcal{G}) / B_{\mathcal{F}}(G, \mathcal{G}) \).

Lemma 3.1. Let \( \mathcal{F} \) be a subset of \( \mathcal{S}(G) \) closed under taking subgroups, and let \( \mathcal{G} \) be a subset of \( \mathcal{S}(G) \) closed under taking subgroups and conjugations by elements of \( G \). Then \( L_{\mathcal{F}}(G, \mathcal{G}) \) coincides with \( \overline{B_{\mathcal{F}}(G, \mathcal{G})} \), where

\[
\overline{B_{\mathcal{F}}(G, \mathcal{G})} = \{ x \in P_{\mathcal{F}}(G, \mathcal{G}) \mid n x \in B_{\mathcal{F}}(G, \mathcal{G}) \text{ for some } n \in \mathbb{N} \}.
\]

Proof. First recall that \( L_{\mathcal{F}}(G, \mathcal{G}) \) is a direct summand of the \( \mathbb{Z} \)-free module \( P_{\mathcal{F}}(G, \mathcal{G}) \). The lemma above follows from the facts that \( B_{\mathcal{F}}(G, \mathcal{G}) \subset L_{\mathcal{F}}(G, \mathcal{G}) \) and that \( \text{rank}_{\mathbb{Z}} B_{\mathcal{F}}(G, \mathcal{G}) = \text{rank}_{\mathbb{Z}} L_{\mathcal{F}}(G, \mathcal{G}) \).

\( \square \)

4. Definition of Groups \( R(-) \) and Remarks

Let \( W \) be a finitely generated free \( \mathbb{Z} \)-module. For a submodule \( U \) of \( W \), we define the submodule \( \overline{U} \) of \( W \) as \( \{ x \in W \mid n x \in U \text{ for some } n \in \mathbb{N} \} \). Therefore \( \overline{U} \) is the smallest direct summand of \( W \) containing \( U \). We define a finite module \( R_W(U) \) by

\[
R_W(U) = \overline{U} / U.
\]
Clearly, $R_W(U)$ coincides with $R_{\overline{U}}(U)$. We readily see

**Proposition 4.1.** Let $U = U_1 \oplus U_2$ be a submodule of $W$.

(1) $\overline{U}$ contains $\overline{U_1} + \overline{U_2}$.

(2) If $\overline{U} = \overline{U_1} + \overline{U_2}$ holds then $R_W(U) = R_W(U_1) \oplus R_W(U_2)$.

Let $f : V \rightarrow W$ be a homomorphism between finitely generated free $\mathbb{Z}$-modules $V$ and $W$. We define a finite module $R(f)$ by

\[(4.2) \quad R(f) = R_W(f(V)) \quad (= \overline{f(V)}/f(V)).\]

We immediately obtain

**Proposition 4.2.** Let $V$, $W$, $V'$, and $W'$ be finitely generated free $\mathbb{Z}$-modules, and let $f : V \rightarrow W$ be a homomorphism.

(1) If $\alpha : V' \rightarrow V$ is an epimorphism then $R(f) = R(f \circ \alpha)$.

(2) If $\beta : W \rightarrow W'$ is a homomorphism such that $\beta|_{\overline{f(V)}} : \overline{f(V)} \rightarrow W'$ is split injective then $R(f) \cong R(\beta \circ f)$.

Let $M = [u_{ij}]$ be an $m \times n$-matrix with entries in $\mathbb{Z}$, i.e. $M \in M_{m,n}(\mathbb{Z})$. Then we have the homomorphism $f_M : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$ defined by $f_M(\mathbf{x}) = \mathbf{x}M$, where $\mathbf{x} = [x_1, \ldots, x_m] \in \mathbb{Z}^m$. We define the finite module $R(M)$ by

\[(4.3) \quad R(M) = R(f_M).\]

Therefore $R(M)$ coincides with $R_{\mathbb{Z}_n}(U)$, where $U = \langle u_1, \ldots, u_m \rangle_\mathbb{Z}$ with $u_i = [u_{i1}, \ldots, u_{in}]$ ($i = 1, \ldots, m$).

**Example 4.3.** For natural numbers $p_1, \ldots, p_n$ and the matrix

\[
M = \begin{bmatrix}
p_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & p_n & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

$R(M)$ is isomorphic to $\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_n}$. 
We remark that in general, the module \( R \left( \begin{array}{cc}
M_{11} & M_{12} \\
O & M_{22}
\end{array} \right) \) is not isomorphic to 
\( R(M_{11}) \oplus R(M_{22}) \), nor to 
\( R(\begin{array}{cc}M_{11} & M_{12}\end{array}) \oplus R(M_{22}) \).

Let \( \mathcal{A} = \{a_1, \ldots, a_m\} \) and \( \mathcal{B} = \{b_1, \ldots, b_n\} \) be \( \mathbb{Z} \)-bases of \( V \) and \( W \), respectively. For a homomorphism \( f : V \to W \), the matrix \( M_f = [u_{ij}] \in \mathrm{M}_{m,n}(\mathbb{Z}) \) is defined by
\[
f(a_i) = \sum_{j=1}^{n} u_{ij} b_j \quad (i = 1, \ldots, m).
\]
The matrix \( M_f \) is called the \textit{matrix presentation} of \( f \) with respect to \( \mathcal{A} \) and \( \mathcal{B} \). We readily obtain

**Proposition 4.4.** Let \( f : V \to W \) be a homomorphism and let \( M_f = [u_{ij}] \in \mathrm{M}_{m,n}(\mathbb{Z}) \) be the matrix presentation of \( f \). Then the module \( R(f) \) is isomorphic to \( R(M_f) \), and hence to \( R_{\mathbb{Z}^n}(U) \), where \( U = \langle u_1, \ldots, u_n \rangle_{\mathbb{Z}} \) and \( u_i = [u_{i1}, \ldots, u_{im}] \) (\( i = 1, \ldots, m \)).

For matrices \( M, N \in \mathrm{M}_{m,n}(\mathbb{Z}) \), we say that \( M \) is \textit{similar} to \( N \), and write \( M \sim N \), if there exist \( X \in \mathrm{GL}_m(\mathbb{Z}) \) and \( Y \in \mathrm{GL}_n(\mathbb{Z}) \) such that 
\[
N = X \cdot M \cdot Y.
\]
By Proposition 4.2, we get

**Proposition 4.5.** Let \( M \) and \( N \) be matrices in \( \mathrm{M}_{m,n}(\mathbb{Z}) \). If \( M \) is similar to \( N \) then 
\( R(M) \) is isomorphic to \( R(N) \).

**Corollary 4.6.** Let \( X \in \mathrm{M}_{p,q}(\mathbb{Z}) \), \( Y \in \mathrm{M}_{s,t}(\mathbb{Z}) \), and \( Z \in \mathrm{M}_{q,t}(\mathbb{Z}) \). Then
\[
R \left( \begin{array}{cc}
X & X \cdot Z \\
O & Y
\end{array} \right) \cong R \left( \begin{array}{cc}
X & O \\
O & Y
\end{array} \right) \cong R(X) \oplus R(Y).
\]

**Proof.** First note 
\[
\begin{bmatrix}
I & Z \\
O & I
\end{bmatrix} \in \mathrm{GL}_{q+t}(\mathbb{Z}).
\]
The corollary above follows from the equality
\[
\begin{bmatrix}
X & X \cdot Z \\
O & Y
\end{bmatrix} = \begin{bmatrix}
X & O \\
O & Y
\end{bmatrix} \begin{bmatrix}
I & Z \\
O & I
\end{bmatrix}
\]
and Proposition 4.5. \( \square \)

We give a computational example of Proposition 4.5.
Example 4.7. Let $p$ be a natural number and $M$ the $(p + 3) \times (2p + 2)$-matrix

$$
\begin{bmatrix}
1 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\
p & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & 1 \\
0 & p & 0 & 0 & 1 & 0 & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & p & 0 & \vdots & \ddots & 0 & 1 & \vdots & \vdots \\
0 & \cdots & 0 & p & 1 & \cdots & 1 & 0 \\
0 & \cdots & 0 & 0 & p & \cdots & p & p
\end{bmatrix}
$$

We readily check that $M$ is similar to the matrix

$$
N = \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \ddots & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & p & 0 & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 0
\end{bmatrix}
$$

Therefore we get $R(M) \cong R(N) \cong \mathbb{Z}_p$.

5. Coincidence of $R(\text{res}_F^G)$ and $R(\text{res}_{F_{\max}}^G)$

For $\mathcal{F} \subset S(G)$, let $\mathcal{F}_{\max}$ be the set of all maximal elements of $\mathcal{F}$ and let $\mathcal{F}^*$ be a complete set of representatives of conjugacy classes of subgroups belonging to $\mathcal{F}$ such that $\mathcal{F}^* \subset \mathcal{F}$. If $\mathcal{F}_{\max}$ is closed under taking conjugations by elements of $G$, the equality $\mathcal{F}_{\max} = \bigcup_{H \in \mathcal{F}_{\max}^*} (H)$ holds, where $(H) = \{gHg^{-1} \mid g \in G\}$. Let $\mathcal{G}$ be a subset of $S(G)$ closed under taking conjugations by elements in $G$. We have the commutative diagram

$$
\begin{array}{ccc}
A(G, \mathcal{G}) & \xrightarrow{\text{res}_F^G} & L_\mathcal{F}(G, \mathcal{G}) \\
& & \downarrow \eta \\
& & P_\mathcal{F}(G, \mathcal{G})
\end{array}
\begin{array}{ccc}
& \xrightarrow{j_1} & \\
& \downarrow \text{proj} & \\
& \xrightarrow{j_2} & P_{\mathcal{F}_{\max}}^*(G, \mathcal{G})
\end{array}
\begin{array}{ccc}
& & \xrightarrow{\text{res}_{\mathcal{F}_{\max}^*}} \\
L_{\mathcal{F}_{\max}^*}(G, \mathcal{G}) & \xrightarrow{\eta} & P_{\mathcal{F}_{\max}^*}(G, \mathcal{G})
\end{array}
$$

consisting of canonical homomorphisms.

**Proposition 5.1.** For the diagram above, the following holds.

1. The homomorphisms $j_1$ and $j_2$ are split injective.
2. The composition $\text{proj} \circ j_1 : L_\mathcal{F}(G, \mathcal{G}) \to P_{\mathcal{F}_{\max}}^*(G, \mathcal{G})$ is split injective.
(3) \( R(\text{res}_G^F) \cong R(\text{res}_G^F_{\max}) \cong R(j_2 \circ \text{res}_G^F_{\max}). \)

Proof. Since \( A(H) \) is a free \( \mathbb{Z} \)-module for each \( H \in \mathcal{F}_{\max}, \) \( j_1 \) and \( j_2 \) are split injective, which implies \( R(\text{res}_G^F_{\max}) \cong R(j_2 \circ \text{res}_G^F_{\max}). \) Since \( j_2 \) and \( \eta \) both are split injective, \( \text{proj} \circ j_1 \) is split injective and hence \( R(\text{res}_G^F) \cong R(\text{proj} \circ j_1 \circ \text{res}_G^F). \) Thus we conclude \( R(\text{res}_G^F) \cong R(\text{res}_G^F_{\max}). \)

The next corollary immediately follows from the proposition above.

**Corollary 5.2.** For the commutative diagram

\[
\begin{array}{ccc}
L_\mathcal{F}(G, \mathcal{G}) & \xrightarrow{\eta_1} & A(G, \mathcal{G}) \\
\downarrow & & \downarrow \\
L_{\mathcal{F}_{\max}}(G, \mathcal{G}) & \xrightarrow{j} & P_{\mathcal{F}_{\max}}(G, \mathcal{G})
\end{array}
\]

consisting of canonical homomorphisms, it holds that

\( R(\text{res}_G^F) \cong R(\text{res}_G^F_{\max}) \cong R(j \circ \text{res}_G^F_{\max}) \cong R(M) \cong \mathbb{Z}_p. \)

**Example 5.3** (cf. [3, Proposition 2.2]). Let \( p \) be a prime, \( C_p \) a cyclic group of order \( p, \) \( G = C_p \times C_p, \) \( \mathcal{F} = \mathcal{F}_G, \) and \( M \) the matrix given in Example 4.7. Then it holds that

(5.1) \( Q_\mathcal{F}(G, S(G)) \cong R(\text{res}_G^F) \cong R(j \circ \text{res}_G^F_{\max}) \cong R(M) \cong \mathbb{Z}_p. \)

6. **Decomposition of \( R(\text{res}_G^F) \)**

In this section, let \( N \) be a normal subgroup of \( G \) and let \( \mathcal{F} \) and \( \mathcal{G} \) be subsets of \( S(G) \) closed under taking conjugations by elements of \( G. \) We use the notation

\[
\begin{align*}
\mathcal{F}(\geq N) & = \{ H \in \mathcal{F} \mid H \supset N \}, \\
\mathcal{F}(\geq N)/N & = \{ H/N \mid H \in \mathcal{F}(\geq N) \}, \\
\mathcal{F}(\geq N)' & = \mathcal{F} \setminus \mathcal{F}(\geq N).
\end{align*}
\]
Let
\[ \text{res}_{\mathcal{F}}^{G} : A(G, \mathcal{G}) \to L_{\mathcal{F}}(G, \mathcal{G}), \]
\[ \text{res}_{1} : A(G, \mathcal{G}(\geq N)) \to L_{\mathcal{F}(\geq N)}(G, \mathcal{G}(\geq N)), \]
\[ \text{res}_{2} : A(G, \mathcal{G}(\geq N')) \to L_{\mathcal{F}}(G, \mathcal{G}(\geq N')), \] and
\[ \overline{\text{res}}_{1} : A(G/N, \mathcal{G}(\geq N)/N) \to L_{\mathcal{F}(\geq N)/N}(G/N, \mathcal{G}(\geq N)/N) \]
denote the restriction homomorphisms, respectively.

**Theorem 6.1.** If the condition \( N \subset \bigcap_{L \in \mathcal{F}_{\max}} L \) is satisfied then it holds that
\[ R(\text{res}_{\mathcal{F}}^{G}) \cong R(\text{res}_{1}) \oplus R(\text{res}_{2}) \cong R(\overline{\text{res}}_{1}) \oplus R(\text{res}_{2}). \]

**Proof.** We can readily see \( R(\text{res}_{1}) \cong R(\overline{\text{res}}_{1}) \). Observe the commutative diagram
\[
\begin{array}{ccc}
A(G, \mathcal{G}(\geq N)) & \xrightarrow{\text{res}_{\mathcal{F}}^{G}} & L_{\mathcal{F}}(G, \mathcal{G}(\geq N)) \\
\downarrow j_{1} & & \downarrow \eta_{1} \\
A(G, \mathcal{G}) & \xrightarrow{\text{res}_{1}} & L_{\mathcal{F}}(G, \mathcal{G}) \\
\downarrow j_{2} & & \downarrow \eta_{2} \\
A(G, \mathcal{G}(\geq N')) & \xrightarrow{\text{res}_{2}} & L_{\mathcal{F}}(G, \mathcal{G}(\geq N')) \\
\end{array}
\]
consisting of canonical homomorphisms. By Proposition 5.1, we have \( R(\text{res}_{\mathcal{F}}^{G}) \cong R(\rho \circ \text{res}_{\mathcal{F}}^{G}), R(\text{res}_{1}) \cong R(\rho_{1} \circ \text{res}_{1}), \) and \( R(\text{res}_{2}) \cong R(\rho_{2} \circ \text{res}_{2}). \) There are canonical direct sum decompositions
\[ A(G, \mathcal{G}) = A(G, \mathcal{G}(\geq N)) \oplus A(G, \mathcal{G}(\geq N')), \] and
\[ P_{\mathcal{F}_{\max}}(G, \mathcal{G}) = P_{\mathcal{F}_{\max}}(G, \mathcal{G}(\geq N)) \oplus P_{\mathcal{F}_{\max}}(G, \mathcal{G}(\geq N')). \]

With respect to these direct sums, \( \rho \circ \text{res}_{\mathcal{F}}^{G} \) coincides with \( (\rho_{1} \circ \text{res}_{1}) \oplus (\rho_{2} \circ \text{res}_{2}). \) Thus we get \( R(\rho \circ \text{res}_{\mathcal{F}}^{G}) \cong R(\rho_{1} \circ \text{res}_{1}) \oplus R(\rho_{2} \circ \text{res}_{2}). \) \( \square \)

**Example 6.2.** Let \( p \) be a prime and \( m \) a natural number \( \geq 2 \). Let \( a \) and \( b \) be generators of \( C_{p^{m}} \) and \( C_{p} \), respectively, and let \( G = C_{p^{m}} \times C_{p} \) be the group generated by \( a \) and \( b \). Let \( N \) be the subgroup of \( G \) generated by \( a^{p^{m-1}} \). We regard \( N = C_{p} \times E \) as the subgroup of \( C_{p^{m}} \times C_{p} \). Let \( \mathcal{F} = \mathcal{F}_{G}, K = G/N, \) and \( \mathcal{H} = \mathcal{F}_{K}. \) We remark
that $K \cong C_{p^{m-1}} \times C_{p}$. Let $M$ be the $(p+1) \times (2p+1)$-matrix
\[
\begin{bmatrix}
p & 0 & \cdots & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\
0 & p & \ddots & & \cdots & 0 & 1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & p & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 0 & p & p & \cdots & p
\end{bmatrix}.
\]
We readily show that $M$ is similar to the matrix
\[
\begin{bmatrix}
p & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & p & \ddots & & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & p & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]
Thus we get $R(M) \cong \mathbb{Z}_{p^{p}}$. It follows that
\[
Q_{\mathcal{F}}(G, S(G)) \cong R(\text{res}_{\mathcal{F}}^{G}) \cong R(\text{res}_{\mathcal{F}_{\text{max}}^{*}}^{G}) \cong R(\text{res}_{\mathcal{H}_{\text{max}}^{*}}^{K}) \oplus R(M).
\]
We readily show that $M$ is similar to the matrix
\[
\begin{bmatrix}
p & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & p & \ddots & & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & p & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]
Thus we get $R(M) \cong \mathbb{Z}_{p^{p}}$. It follows that
\[
Q_{\mathcal{F}}(G, S(G)) \cong R(\text{res}_{\mathcal{F}}^{G}) \cong R(\text{res}_{\mathcal{F}_{\text{max}}^{*}}^{G}) \cong R(\text{res}_{\mathcal{H}_{\text{max}}^{*}}^{K}) \oplus R(M).
\]
We readily show that $M$ is similar to the matrix
\[
\begin{bmatrix}
p & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & p & \ddots & & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 0 & p & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]
Thus we get $R(M) \cong \mathbb{Z}_{p^{p}}$. It follows that
\[
Q_{\mathcal{F}}(G, S(G)) \cong R(\text{res}_{\mathcal{F}}^{G}) \cong R(\text{res}_{\mathcal{F}_{\text{max}}^{*}}^{G}) \cong R(\text{res}_{\mathcal{H}_{\text{max}}^{*}}^{K}) \oplus R(M).
\]

7. Decomposition of $R(\text{res}_{\mathcal{F}}^{G})$ for $G = C_{p^{m}} \times C_{p^{n}}$

Throughout this section, let $G = C_{p^{m}} \times C_{p^{n}}$ with $m \geq n \geq 2$ for a prime $p$. Let $a$ and $b$ be generators of the cyclic groups $C_{p^{m}}$ and $C_{p^{n}}$, respectively. Let $N$ denote the subgroup generated by $a^{p^{m-1}}$ and $b^{p^{n-1}}$. Thus $N$ is isomorphic to $C_{p} \times C_{p}$. Let $H_{0}$ denote the subgroup generated by $a^{p^{m-1}}$. We can regard $H_{0}$ as the subgroup $C_{p} \times E$ of $C_{p^{m}} \times C_{p^{n}}$, where $E$ is the trivial group. The group $G$ also contains subgroups $H_{i}$ ($i = 1, \ldots, p$) of order $p$ generated by $a^{ip^{m-1}}b^{p^{n-1}}$. We can regard $H_{p}$ as the subgroup $E \times C_{p}$ of $C_{p^{m}} \times C_{p^{n}}$. Let
\[
\mathcal{G} = S(G),
\]
\[
\mathcal{G}_{1} = \{H \in S(G) \mid H \supset N\},
\]
\[
\mathcal{G}_{2,i} = \{H \in S(G) \setminus \mathcal{G}_{1} \mid H \supset H_{i}\},
\]
where $i = 0, \ldots, p$. Then the equality
\[
\mathcal{G} = \mathcal{G}_{1} \amalg \mathcal{G}_{2,0} \amalg \left( \coprod_{i=1}^{p} \mathcal{G}_{2,i} \right) \amalg \{E\}
\]
holds, which gives the decomposition formula

\[(7.2) \quad A(G, \mathcal{G}) = A(G, \mathcal{G}_1) \oplus A(G, \mathcal{G}_{2,0}) \oplus \bigoplus_{i=1}^{p} A(G, \mathcal{G}_{2,i}) \oplus A(G, \{E\})\]

of the Burnside module. In addition, we have the canonical identifications

\[\mathcal{G}_1 = S(G/N), \]
\[\mathcal{G}_1 \cup \mathcal{G}_{2,0} = S(G/H_0), \]

\[(7.3) \quad G/N = C_{p^{m-1}} \times C_{p^{n-1}}, \text{ and} \]
\[G/H_0 = C_{p^{m-1}} \times C_{p^{n}}. \]

Set \(X_{m,n} = A(G, \mathcal{G}_1),\ Y_{m,n,i} = A(G, \mathcal{G}_{2,i})\ (i = 0, \ldots, p),\) and \(Z_{m,n} = A(G, \{E\}).\) By (7.2) we get

\[(7.4) \quad A(C_{p^m} \times C_{p^n}) = X_{m,n} \oplus \bigoplus_{i=0}^{p} Y_{m,n,i} \oplus Z_{m,n}. \]

It follows from (7.3) that

\[(7.5) \quad A(C_{p^{m-1}} \times C_{p^{n-1}}) \cong X_{m,n}, \quad A(C_{p^{m-1}} \times C_{p^{n}}) \cong X_{m,n} \oplus Y_{m,n,0} \]

In addition, we have

\[(7.6) \quad Y_{m,n,i} \cong Y_{n,n,i} \cong Y_{n,n,0} \ (i = 1, \ldots, p). \]

Let \(\mathcal{F} = S(G) \setminus \{G\},\ \mathcal{F}_1 = \{H \in \mathcal{F} \mid H \supset N\},\ \mathcal{F}_{2,i} = \{H \in \mathcal{F} \mid H \supset H_i\} \ (i = 0, \ldots, p),\) and let \(f_{m,n}, g_{m,n}, h_{m,n,i}, k_{m,n}\) be the restriction homomorphisms

[\text{res}_{\mathcal{F}}^G : A(G) \to L_{\mathcal{F}}(G, S(G)),\]
[\text{res}_{\mathcal{F}_1}^G : X_{m,n} \to L_{\mathcal{F}_1}(G, \mathcal{G}_1),\]
[\text{res}_{\mathcal{F}_{2,i}}^G : Y_{m,n,i} \to L_{\mathcal{F}_{2,i}}(G, \mathcal{G}_{2,i}),\]
[\text{res}_{\mathcal{F}}^G : Z_{m,n} \to L_{\mathcal{F}}(G, \{E\}),\]

respectively.

**Theorem 7.1.** Under the situation above, the direct sum decomposition formula

\[(7.7) \quad R(f_{m,n}) \cong R(g_{m,n}) \oplus R(h_{m,n,0}) \oplus \bigoplus_{i=1}^{p} R(h_{m,n,i}) \oplus R(k_{m,n})\]

holds.
Proof. Similarly to the diagram (6.1), we have diagrams

\[
\begin{array}{c}
X_{m,n} \xrightarrow{g_{m,n}} L_{\mathcal{F}_{1}}(G, \mathcal{G}_{1}) \xrightarrow{f_{m,n}} P_{\mathcal{F}_{\text{max}}}(G, \mathcal{G}) \quad A(G) \xrightarrow{f_{m,n}} L_{\mathcal{F}}(G, \mathcal{G}) \xrightarrow{g_{m,n}} P_{\mathcal{F}_{\text{max}}}(G, \mathcal{G}) \quad A(G, \mathcal{G}_{1}') \xrightarrow{g_{m,n}} L_{\mathcal{F}}(G, \mathcal{G}_{1}') \xrightarrow{f_{m,n}} P_{\mathcal{F}_{\text{max}}}(G, \mathcal{G}_{1}') \quad Y_{m,n,0} \xrightarrow{h_{m,n,0}} L_{\mathcal{F}_{2,0}}(G, \mathcal{G}_{2,0}) \xrightarrow{g_{m,n,0}} P_{\mathcal{F}_{\text{max}}}(G, \mathcal{G}_{2,0}) \quad A(G, \mathcal{G}_{2,0}') \xrightarrow{h_{m,n,0}} L_{\mathcal{F}}(G, \mathcal{G}_{2,0}') \xrightarrow{g_{m,n,0}} P_{\mathcal{F}_{\text{max}}}(G, \mathcal{G}_{2,0}')
\end{array}
\]

and so on, where \( \mathcal{G}_{1}' = \mathcal{G} \setminus \mathcal{G}_{1} \) and \( \mathcal{G}_{2,0}' = \mathcal{G}_{1}' \setminus \mathcal{G}_{2,0} \). The theorem above is obtained by iteration of use of Theorem 6.1.

Concerning with Theorem 7.1, we remark

**Proposition 7.2.**

1. \( R(f_{m-1,n-1}) \cong R(g_{m,n}) \).
2. \( R(f_{m-1,n}) \cong R(g_{m,n}) \oplus R(h_{m,n,0}) \).
3. \( R(f_{m,n}) \cong R(g_{m,n}) \oplus R(h_{m,n,0}) \oplus R(h_{n,n,0})^\oplus \mathbb{Z}_p \).

**Proof.** The claims (1) and (2) follow from (7.5). Since \( R(k_{m,n}) \cong \mathbb{Z}_p \), the claim (3) follows from Theorem 7.1 and (7.6).\qed

The next fact immediately follows from the proposition above.

**Corollary 7.3.**

1. \( R(f_{n-1,n}) \cong R(f_{n-1,n-1}) \oplus R(h_{n,n,0}) \).
2. \( R(f_{m,n}) \cong R(f_{m-1,n}) \oplus R(h_{n,n,0})^\oplus \mathbb{Z}_p \).

Now recall Lemma 3.1 and Sugimura’s theorem described in the introduction. By induction arguments on \( m \) and \( n \) with the corollary above, we can readily prove...
Proposition 7.4. Any element $x \in Q_{\mathcal{F}_{G}}(G, S(G))$ has exponent $p$, i.e. $px = 0$, and hence $Q_{\mathcal{F}_{G}}(G, S(G))$ is isomorphic to a direct sum of copies of $\mathbb{Z}_p$.

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