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# COMPUTATION OF QUOTIENT GROUPS OF INVERSE LIMITS OF BURNSIDE RINGS

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**Abstract.** Let $G$ be a finite group. For a subgroup $H$ of $G$, we have the Burnside ring $A(H)$ of $H$. For a set $\mathcal{F}$ of subgroups of $G$, we have the inverse limit $L(G)$ of the category consisting of modules $A(H)$ $(H \in \mathcal{F})$, and the restriction homomorphism from $A(G)$ to $L(G)$. In this article, we discuss computational theory of the cokernel of the homomorphism.

## 1. Introduction

This article is supplementary to the joint work [5] with Masafumi Sugimura.

Let $G$ be a finite group. Let $\mathcal{S}(G)$ denote the set of all subgroups of $G$ and set $\mathcal{F}_G = \mathcal{S}(G) \smallsetminus \{G\}$. Let $\mathcal{G}$ be a subset of $\mathcal{S}(G)$ closed under taking subgroups and conjugations by elements of $G$. For $H \in \mathcal{S}(G)$, let $A(H)$ denote the Burnside ring of $H$ and $A(H, \mathcal{G} \cap \mathcal{S}(H))$ denote the submodule of $A(H)$ generated by $[H/K]$, where $K$ runs over $\mathcal{G} \cap \mathcal{S}(H)$ (see [2], [4]). We have the inverse limit $L_{\mathcal{F}_G}(G, \mathcal{G}) = \varprojlim_{\mathcal{F}_G} \mathcal{A}$ of the functor $\mathcal{A} : H \mapsto A(H, \mathcal{G} \cap \mathcal{S}(H))$, where $H \in \mathcal{F}_G$, which is a submodule of the cartesian product $P_{\mathcal{F}_G}(G, \mathcal{G}) = \prod_{H \in \mathcal{F}_G} A(H, \mathcal{G} \cap \mathcal{S}(H))$, and the restriction homomorphism

$$\mathrm{res}^{G}_{\mathcal{F}_G} : A(G, \mathcal{G}) \longrightarrow L_{\mathcal{F}_G}(G, \mathcal{G})$$

(see [3]). Let $Q_{\mathcal{F}_G}(G, \mathcal{G})$ denote the cokernel of $\mathrm{res}^{G}_{\mathcal{F}_G}$. It is interesting to compute the abelian group $Q_{\mathcal{F}_G}(G, \mathcal{G})$ for a given group $G$. For a natural number $k$, let $C_k$ denote a cyclic group of order $k$. Let $p$ be a prime and let $\mathbb{Z}_p$ denote a module

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consisting of $p$ elements. Let $m$ and $n$ be non-negative integers. The following three computational results have been obtained.

**Theorem** (Y. Hara–M. Morimoto [3]).

(1) *For* $G = C_{p^m}$ $(m \geq 1)$, $Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) \cong \mathbb{Z}_p{}^{m-1}$.

(2) *For* $G = C_{p_1 \cdots p_m}$ *($p_i$ distinct primes),* $Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) = O$.

**Theorem** (M. Sugimura [6]).

(1) *For* $G = C_{p^m} \times C_p$ $(m \geq 1)$, $Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) \cong \mathbb{Z}_p{}^{(m-1)p+1}$.

(2) *For* $G = C_{p^m} \times C_{p^2}$ $(m \geq 2)$, $Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) \cong \mathbb{Z}_p{}^{(m-1)(p^2+1)+p+1}$.

**Theorem** (M. Morimoto–M. Sugimura [5]). *For* $G = C_{p^m} \times C_{p^n}$ $(m \geq n \geq 2)$, $Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) \cong \mathbb{Z}_p{}^r$, *where*

$$r = p^n + 2p^{n-1} + \sum_{k=1}^{n-3} (2k+1)p^{n-k-1} + (2n-4)p$$
$$+ (2n-2) + (m-n)\left(\sum_{k=0}^{n} p^k - p^{n-1}\right).$$

In the present paper, we discuss computational theory to obtain the results above. For example, the next fact follows from Lemma 3.1 and Theorem 6.1.

**Theorem 1.1.** *Let $G$ be a finite (nontrivial) group and $N$ a normal subgroup of $G$. Suppose each maximal (proper) subgroup $H$ of $G$ contains $N$. Let $\mathcal{K}$ be the set of all subgroups $H$ of $G$ not containing $N$. Then it holds that*

$$Q_{\mathcal{F}_G}(G, \mathcal{S}(G)) \cong Q_{\mathcal{F}_{\overline{G}}}(\overline{G}, \mathcal{S}(\overline{G})) \oplus Q_{\mathcal{F}_G}(G, \mathcal{K}),$$

*where* $\overline{G} = G/N$.

The result of Morimoto–Sugimura [5] mentioned above is obtained by using Corollary 7.3.

## 2. Inverse limit $L_{\mathcal{F}}(F)$ and remarks

Let $\mathfrak{S}(G)$ denote the subgroup category of $G$, namely the objects of $\mathfrak{S}(G)$ are all subgroups of $G$ and the morphisms of $\mathfrak{S}(G)$ are all triples $(H, g, K)$ consisting of $H \in \mathcal{S}(G)$, $g \in G$, $K \in \mathcal{S}(G)$ such that $gHg^{-1} \subset K$. Here $(H, g, K)$ is a morphism from $H$ to $K$. Each morphism $(H, g, K)$ induces a (group) homomorphism $\iota_{H,g,K} : H \to K$ by $\iota(a) = gag^{-1}$ for $a \in H$.

Let $\mathcal{F}$ be a subset of $\mathcal{S}(G)$. Let $\mathfrak{F}$ denote the full subcategory of $\mathfrak{S}(G)$ such that $\mathrm{Obj}(\mathfrak{F}) = \mathcal{F}$ and let $\mathfrak{Ab}$ denote the category of abelian groups of which the objects are all abelian groups and the morphisms are all (group) homomorphisms. Let $F : \mathfrak{F} \to \mathfrak{Ab}$ be a contravariant functor. Then we have the inverse limit $\varprojlim_{\mathcal{F}} F$ of $F$ with respect to $\mathcal{F}$, where

$$\varprojlim_{\mathcal{F}} F = \left\{ (x_H) \in \prod_{H \in \mathcal{F}} F(H) \;\middle|\; \begin{array}{c} (H, g, K)^* x_K = x_H \\ \forall\, (H, g, K) \in \mathrm{Mor}(\mathfrak{F}) \end{array} \right\}$$

(see [1]). This inverse limit will be denoted by $L_{\mathcal{F}}(F)$.

We subsequently use the next fact without specifically mentioning.

**Proposition 2.1.** *If $F(H)$ is a free $\mathbb{Z}$-module for each $H \in \mathcal{F}$ then $\varprojlim_{\mathcal{F}} F$ is a direct summand of $\prod_{H \in \mathcal{F}} F(H)$.*

Let $e$ denote the identity element of $G$. For a morphism $(H, e, K)$ of $\mathfrak{S}(G)$, let $\mathrm{res}^K_H$ denote the homomorphism $(H, e, G)^* : F(K) \to F(H)$. Let $P_{\mathcal{F}}(F)$ denote the cartesian product $\prod_{H \in \mathcal{F}} F(H)$. The image of the restriction homomorphism $\mathrm{res}^G = \prod_{H \in \mathcal{F}} \mathrm{res}^G_H : F(G) \to P_{\mathcal{F}}(F)$ is contained in $L_{\mathcal{F}}(F)$ and therefore $\mathrm{res}^G$ induces $\mathrm{res}^G_{\mathcal{F}} : F(G) \to L_{\mathcal{F}}(F)$. Let $B_{\mathcal{F}}(F)$ denote the image of $\mathrm{res}^G_{\mathcal{F}}$ and $Q_{\mathcal{F}}(F)$ the quotient group $L_{\mathcal{F}}(F)/B_{\mathcal{F}}(F)$, namely the cokernel of $\mathrm{res}^G_{\mathcal{F}}$.

Let $\mathcal{F}^*$ denote a complete set of representatives of $G$-conjugacy classes of subgroups belonging to $\mathcal{F}$ such that $\mathcal{F}^* \subset \mathcal{F}$. Let $\mathrm{proj} : \prod_{H \in \mathcal{F}} F(H) \to \prod_{H \in \mathcal{F}^*} F(H)$ be the canonical projection. This gives the homomorphisms $\mathrm{proj}_L : L_{\mathcal{F}}(F) \to L_{\mathcal{F}^*}(F)$ and $\mathrm{proj}_B : B_{\mathcal{F}}(F) \to B_{\mathcal{F}^*}(F)$ as well as the homomorphism $\rho_Q : Q_{\mathcal{F}}(F) \to Q_{\mathcal{F}^*}(F)$. Clearly, the diagram

$$\begin{tikzcd}
B_{\mathcal{F}}(F) \arrow[r, hook] \arrow[d, "\mathrm{proj}_B"'] & L_{\mathcal{F}}(F) \arrow[r, two heads] \arrow[d, "\mathrm{proj}_L"] & Q_{\mathcal{F}}(F) \arrow[d, "\rho_Q"] \\
B_{\mathcal{F}^*}(F) \arrow[r, hook] & L_{\mathcal{F}^*}(F) \arrow[r, two heads] & Q_{\mathcal{F}^*}(F)
\end{tikzcd}$$

commutes.

**Proposition 2.2.** *In the diagram above,* $\mathrm{proj}_B$, $\mathrm{proj}_L$, *and* $\rho_Q$ *all are isomorphisms.*

*Proof.* By definition of inverse limit, $\mathrm{proj}_L$ is an isomorphism. Since $\mathrm{proj}_B$ is surjective and $\mathrm{proj}_L$ is injective, $\mathrm{proj}_B$ is bijective. As $\mathrm{proj}_L$ and $\mathrm{proj}_B$ are isomorphisms, $\rho_Q$ is an isomorphism. □

## 3. Definition of $Q_{\mathcal{F}}(G,\mathcal{G})$ and interpretation of $L_{\mathcal{F}}(G,\mathcal{G})$

Let $\mathcal{G}$ be a subset of $\mathcal{S}(G)$ closed under taking conjugations by elements in $G$. For a subgroup $H$ of $G$, let $\mathcal{G}(\cap H)$ denote the set consisting of all $K \cap H$, where $K$ ranges over $\mathcal{G}$. We have the restriction homomorphism $\mathrm{res}^G_H : A(G,\mathcal{G}) \to A(H,\mathcal{G}(\cap H))$. Let $P_{\mathcal{F}}(G,\mathcal{G})$ denote the cartesian product $\prod_{H\in\mathcal{F}} A(H,\mathcal{G}(\cap H))$. We have the restriction homomorphism $\mathrm{res}^G : A(G,\mathcal{G}) \to P_{\mathcal{F}}(G,\mathcal{G})$. Let $L_{\mathcal{F}}(G,\mathcal{G})$ denote the inverse limit $\varprojlim_{\mathcal{F}} \mathcal{A}$ of the Burnside ring functor $\mathcal{A} : H \longmapsto A(H,\mathcal{G}(\cap H))$, where $H \in \mathcal{S}(G)$. Since the image of $\mathrm{res}^G$ above is contained in $L_{\mathcal{F}}(G,\mathcal{G})$, we obtain the restriction homomorphism $\mathrm{res}^G_{\mathcal{F}} : A(G,\mathcal{G}) \to L_{\mathcal{F}}(G,\mathcal{G})$. Let $B_{\mathcal{F}}(G,\mathcal{G})$ denote the image of $\mathrm{res}^G_{\mathcal{F}}$ and let $Q_{\mathcal{F}}(G,\mathcal{G})$ denote the cokernel of $\mathrm{res}^G_{\mathcal{F}}$, i.e. $Q_{\mathcal{F}}(G,\mathcal{G}) = L_{\mathcal{F}}(G,\mathcal{G}) \,/\, B_{\mathcal{F}}(G,\mathcal{G})$.

**Lemma 3.1.** *Let $\mathcal{F}$ be a subset of $\mathcal{S}(G)$ closed under taking subgroups, and let $\mathcal{G}$ be a subset of $\mathcal{S}(G)$ closed under taking subgroups and conjugations by elements of $G$. Then $L_{\mathcal{F}}(G,\mathcal{G})$ coincides with $\overline{B_{\mathcal{F}}(G,\mathcal{G})}$, where*

$$\overline{B_{\mathcal{F}}(G,\mathcal{G})} = \{x \in P_{\mathcal{F}}(G,\mathcal{G}) \mid n\,x \in B_{\mathcal{F}}(G,\mathcal{G}) \text{ for some } n \in \mathbb{N}\}.$$

*Proof.* First recall that $L_{\mathcal{F}}(G,\mathcal{G})$ is a direct summand of the $\mathbb{Z}$-free module $P_{\mathcal{F}}(G,\mathcal{G})$. The lemma above follows from the facts that $B_{\mathcal{F}}(G,\mathcal{G}) \subset L_{\mathcal{F}}(G,\mathcal{G})$ and that $\mathrm{rank}_{\mathbb{Z}} B_{\mathcal{F}}(G,\mathcal{G}) = \mathrm{rank}_{\mathbb{Z}} L_{\mathcal{F}}(G,\mathcal{G})$. □

## 4. Definition of groups $R(-)$ and remarks

Let $W$ be a finitely generated free $\mathbb{Z}$-module. For a submodule $U$ of $W$, we define the submodule $\overline{U}$ of $W$ as $\{x \in W \mid n\,x \in U \text{ for some } n \in \mathbb{N}\}$. Therefore $\overline{U}$ is the smallest direct summand of $W$ containing $U$. We define a finite module $R_W(U)$ by

$$R_W(U) = \overline{U}/U. \tag{4.1}$$

Clearly, $R_W(U)$ coincides with $R_{\overline{U}}(U)$. We readily see

**Proposition 4.1.** *Let $U = U_1 \oplus U_2$ be a submodule of $W$.*

(1) *$\overline{U}$ contains $\overline{U_1} + \overline{U_2}$.*

(2) *If $\overline{U} = \overline{U_1} + \overline{U_2}$ holds then $R_W(U) = R_W(U_1) \oplus R_W(U_2)$.*

Let $f : V \to W$ be a homomorphism between finitely generated free $\mathbb{Z}$-modules $V$ and $W$. We define a finite module $R(f)$ by

$$R(f) = R_W(f(V)) \quad (= \overline{f(V)}/f(V)). \tag{4.2}$$

We immediately obtain

**Proposition 4.2.** *Let $V$, $W$, $V'$, and $W'$ be finitely generated free $\mathbb{Z}$-modules, and let $f : V \to W$ be a homomorphism.*

(1) *If $\alpha : V' \to V$ is an epimorphism then $R(f) = R(f \circ \alpha)$.*

(2) *If $\beta : W \to W'$ is a homomorphism such that $\beta|_{\overline{f(V)}} : \overline{f(V)} \to W'$ is split injective then $R(f) \cong R(\beta \circ f)$.*

Let $M = [u_{ij}]$ be an $m \times n$-matrix with entries in $\mathbb{Z}$, i.e. $M \in \mathrm{M}_{m,n}(\mathbb{Z})$. Then we have the homomorphism $f_M : \mathbb{Z}^m \to \mathbb{Z}^n$ defined by $f_M(\boldsymbol{x}) = \boldsymbol{x}\, M$, where $\boldsymbol{x} = [x_1, \dots, x_m] \in \mathbb{Z}^m$. We define the finite module $R(M)$ by

$$R(M) = R(f_M). \tag{4.3}$$

Therefore $R(M)$ coincides with $R_{\mathbb{Z}^n}(U)$, where $U = \langle \boldsymbol{u}_1, \dots, \boldsymbol{u}_m \rangle_{\mathbb{Z}}$ with $\boldsymbol{u}_i = [u_{i1}, \dots, u_{in}]$ $(i = 1, \dots, m)$.

**Example 4.3.** For natural numbers $p_1, \dots, p_n$ and the matrix

$$M = \begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & p_n & 0 & \cdots & 0 \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix},$$

$R(M)$ is isomorphic to $\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_n}$.

We remark that in general, the module $R\left(\begin{bmatrix} M_{11} & M_{12} \\ O & M_{22} \end{bmatrix}\right)$ is not isomorphic to $R(M_{11}) \oplus R(M_{22})$, nor to $R\left(\begin{bmatrix} M_{11} & M_{12} \end{bmatrix}\right) \oplus R(M_{22})$.

Let $\mathcal{A} = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ and $\mathcal{B} = \{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n\}$ be $\mathbb{Z}$-bases of $V$ and $W$, respectively. For a homomorphism $f : V \to W$, the matrix $M_f = [u_{ij}] \in \mathrm{M}_{m,n}(\mathbb{Z})$ is defined by

$$f(\boldsymbol{a}_i) = \sum_{j=1}^{n} u_{ij}\, \boldsymbol{b}_j \quad (i = 1, \ldots, m).$$

The matrix $M_f$ is called the *matrix presentation* of $f$ with respect to $\mathcal{A}$ and $\mathcal{B}$. We readily obtain

**Proposition 4.4.** *Let $f : V \to W$ be a homomorphism and let $M_f = [u_{ij}] \in \mathrm{M}_{m,n}(\mathbb{Z})$ be the matrix presentation of $f$. Then the module $R(f)$ is isomorphic to $R(M_f)$, and hence to $R_{\mathbb{Z}^n}(U)$, where $U = \langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \rangle_{\mathbb{Z}}$ and $\boldsymbol{u}_i = [u_{i1}, \ldots, u_{in}]$ $(i = 1, \ldots, m)$.*

For matrices $M$, $N \in \mathrm{M}_{m,n}(\mathbb{Z})$, we say that $M$ is *similar* to $N$, and write $M \sim N$, if there exist $X \in \mathrm{GL}_m(\mathbb{Z})$ and $Y \in \mathrm{GL}_n(\mathbb{Z})$ such that $N = X \cdot M \cdot Y$. By Proposition 4.2, we get

**Proposition 4.5.** *Let $M$ and $N$ be matrices in $\mathrm{M}_{m,n}(\mathbb{Z})$. If $M$ is similar to $N$ then $R(M)$ is isomorphic to $R(N)$.*

**Corollary 4.6.** *Let $X \in \mathrm{M}_{p,q}(\mathbb{Z})$, $Y \in \mathrm{M}_{s,t}(\mathbb{Z})$, and $Z \in \mathrm{M}_{q,t}(\mathbb{Z})$. Then*

$$R\left(\begin{bmatrix} X & X \cdot Z \\ O & Y \end{bmatrix}\right) \cong R\left(\begin{bmatrix} X & O \\ O & Y \end{bmatrix}\right) \cong R(X) \oplus R(Y).$$

*Proof.* First note

$$\begin{bmatrix} I & Z \\ O & I \end{bmatrix} \in \mathrm{GL}_{q+t}(\mathbb{Z}).$$

The corollary above follows from the equality

$$\begin{bmatrix} X & X \cdot Z \\ O & Y \end{bmatrix} = \begin{bmatrix} X & O \\ O & Y \end{bmatrix} \begin{bmatrix} I & Z \\ O & I \end{bmatrix}$$

and Proposition 4.5. $\square$

We give a computational example of Proposition 4.5.

**Example 4.7.** Let $p$ be a natural number and $M$ the $(p+3)\times(2p+2)$-matrix

$$\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & 0 \\ p & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots & \cdots & 1 \\ 0 & p & 0 & & 0 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & p & 0 & \vdots & & \ddots & 0 & 1 \\ 0 & \cdots & \cdots & 0 & p & 1 & \cdots & \cdots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & p & \cdots & \cdots & p & p \end{bmatrix}.$$

We readily check that $M$ is similar to the matrix

$$N = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & & & \vdots \\ \vdots & & \ddots & \ddots & 0 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & p & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Therefore we get $R(M) \cong R(N) \cong \mathbb{Z}_p$.

## 5. Coincidence of $R(\mathrm{res}^G_{\mathcal{F}})$ and $R(\mathrm{res}^G_{\mathcal{F}_{\max}{}^*})$

For $\mathcal{F} \subset \mathcal{S}(G)$, let $\mathcal{F}_{\max}$ be the set of all maximal elements of $\mathcal{F}$ and let $\mathcal{F}^*$ be a complete set of representatives of conjugacy classes of subgroups belonging to $\mathcal{F}$ such that $\mathcal{F}^* \subset \mathcal{F}$. If $\mathcal{F}_{\max}$ is closed under taking conjugations by elements of $G$, the equality $\mathcal{F}_{\max} = \bigcup_{H\in\mathcal{F}_{\max}{}^*}(H)$ holds, where $(H) = \{gHg^{-1} \mid g \in G\}$. Let $\mathcal{G}$ be a subset of $\mathcal{S}(G)$ closed under taking conjugations by elements in $G$. We have the commutative diagram

$$\begin{tikzcd} A(G,\mathcal{G}) \arrow[r, "\mathrm{res}^G_{\mathcal{F}}"] \arrow[dr, "\mathrm{res}^G_{\mathcal{F}_{\max}{}^*}"'] & L_{\mathcal{F}}(G,\mathcal{G}) \arrow[r, "j_1"] \arrow[d, "\eta"] & P_{\mathcal{F}}(G,\mathcal{G}) \arrow[d, "\mathrm{proj}"] \\ & L_{\mathcal{F}_{\max}{}^*}(G,\mathcal{G}) \arrow[r, "j_2"'] & P_{\mathcal{F}_{\max}{}^*}(G,\mathcal{G}) \end{tikzcd}$$

consisting of canonical homomorphisms.

**Proposition 5.1.** *For the diagram above, the following holds.*

(1) *The homomorphisms $j_1$ and $j_2$ are split injective.*

(2) *The composition* $\mathrm{proj} \circ j_1 : L_{\mathcal{F}}(G,\mathcal{G}) \to P_{\mathcal{F}_{\max}{}^*}(G,\mathcal{G})$ *is split injective.*

(3) $R(\mathrm{res}^G_{\mathcal{F}}) \cong R(\mathrm{res}^G_{\mathcal{F}_{\max}{}^*}) \cong R(j_2 \circ \mathrm{res}^G_{\mathcal{F}_{\max}{}^*})$.

*Proof.* Since $A(H)$ is a free $\mathbb{Z}$-module for each $H \in \mathcal{F}_{\max}$, $j_1$ and $j_2$ are split injective, which implies $R(\mathrm{res}^G_{\mathcal{F}_{\max}{}^*}) \cong R(j_2 \circ \mathrm{res}^G_{\mathcal{F}_{\max}{}^*})$. Since $j_2$ and $\eta$ both are split injective, $\mathrm{proj} \circ j_1$ is split injective and hence $R(\mathrm{res}^G_{\mathcal{F}}) \cong R(\mathrm{proj} \circ j_1 \circ \mathrm{res}^G_{\mathcal{F}})$. Thus we conclude $R(\mathrm{res}^G_{\mathcal{F}}) \cong R(\mathrm{res}^G_{\mathcal{F}_{\max}{}^*})$. □

The next corollary immediately follows from the proposition above.

**Corollary 5.2.** *For the commutative diagram*

$$\begin{tikzcd}
 & L_{\mathcal{F}}(G,\mathcal{G}) \arrow[d, "\eta_1"] & \\
A(G,\mathcal{G}) \arrow[ur, "\mathrm{res}^G_{\mathcal{F}}"] \arrow[r, "\mathrm{res}^G_{\mathcal{F}_{\max}}"] \arrow[dr, "\mathrm{res}^G_{\mathcal{F}_{\max}{}^*}"'] & L_{\mathcal{F}_{\max}}(G,\mathcal{G}) \arrow[d, "\eta_2"] & \\
 & L_{\mathcal{F}_{\max}{}^*}(G,\mathcal{G}) \arrow[r, "j"'] & P_{\mathcal{F}_{\max}{}^*}(G,\mathcal{G})
\end{tikzcd}$$

*consisting of canonical homomorphisms, it holds that*

$$R(\mathrm{res}^G_{\mathcal{F}}) \cong R(\mathrm{res}^G_{\mathcal{F}_{\max}}) \cong R(\mathrm{res}^G_{\mathcal{F}_{\max}{}^*}) \cong R(j \circ \mathrm{res}^G_{\mathcal{F}_{\max}{}^*}).$$

**Example 5.3** (cf. [3, Proposition 2.2]). Let $p$ be a prime, $C_p$ a cyclic group of order $p$, $G = C_p \times C_p$, $\mathcal{F} = \mathcal{F}_G$, and $M$ the matrix given in Example 4.7. Then it holds that

$$Q_{\mathcal{F}}(G, \mathcal{S}(G)) \cong R(\mathrm{res}^G_{\mathcal{F}}) \cong R(j \circ \mathrm{res}^G_{\mathcal{F}_{\max}{}^*}) \cong R(M) \cong \mathbb{Z}_p. \tag{5.1}$$

## 6. Decomposition of $R(\mathrm{res}^G_{\mathcal{F}})$

In this section, let $N$ be a normal subgroup of $G$ and let $\mathcal{F}$ and $\mathcal{G}$ be subsets of $\mathcal{S}(G)$ closed under taking conjugations by elements of $G$. We use the notation

$$\mathcal{F}(\geq N) = \{H \in \mathcal{F} \mid H \supset N\},$$
$$\mathcal{F}(\geq N)/N = \{H/N \mid H \in \mathcal{F}(\geq N)\},$$
$$\mathcal{F}(\geq N)' = \mathcal{F} \smallsetminus \mathcal{F}(\geq N).$$

Let

$$\begin{aligned}
&\mathrm{res}^G_{\mathcal{F}} : A(G,\mathcal{G}) \to L_{\mathcal{F}}(G,\mathcal{G}),\\
&\mathrm{res}_1 : A(G,\mathcal{G}(\geq N)) \to L_{\mathcal{F}(\geq N)}(G,\mathcal{G}(\geq N)),\\
&\mathrm{res}_2 : A(G,\mathcal{G}(\geq N)') \to L_{\mathcal{F}}(G,\mathcal{G}(\geq N)'), \text{ and}\\
&\overline{\mathrm{res}}_1 : A(G/N,\mathcal{G}(\geq N)/N) \to L_{\mathcal{F}(\geq N)/N}(G/N,\mathcal{G}(\geq N)/N)
\end{aligned}$$

denote the restriction homomorphisms, respectively.

**Theorem 6.1.** *If the condition $N \subset \bigcap_{L\in\mathcal{F}_{\max}} L$ is satisfied then it holds that*

$$R(\mathrm{res}^G_{\mathcal{F}}) \cong R(\mathrm{res}_1) \oplus R(\mathrm{res}_2) \cong R(\overline{\mathrm{res}}_1) \oplus R(\mathrm{res}_2).$$

*Proof.* We can readily see $R(\mathrm{res}_1) \cong R(\overline{\mathrm{res}}_1)$. Observe the commutative diagram

$$\begin{tikzcd}
A(G,\mathcal{G}(\geq N)) \arrow[r, "\mathrm{res}_1"] \arrow[d, "j_1"'] & L_{\mathcal{F}(\geq N)}(G,\mathcal{G}(\geq N)) \arrow[r, "\rho_1"] \arrow[d, "\eta_1"] & P_{\mathcal{F}_{\max}{}^*}(G,\mathcal{G}(\geq N)) \arrow[d, "\iota_1"] \\
A(G,\mathcal{G}) \arrow[r, "\mathrm{res}^G_{\mathcal{F}}"] & L_{\mathcal{F}}(G,\mathcal{G}) \arrow[r, "\rho"] & P_{\mathcal{F}_{\max}{}^*}(G,\mathcal{G}) \\
A(G,\mathcal{G}(\geq N)') \arrow[r, "\mathrm{res}_2"'] \arrow[u, "j_2"] & L_{\mathcal{F}}(G,\mathcal{G}(\geq N)') \arrow[r, "\rho_2"'] \arrow[u, "\eta_2"] & P_{\mathcal{F}_{\max}{}^*}(G,\mathcal{G}(\geq N)') \arrow[u, "\iota_2"]
\end{tikzcd} \tag{6.1}$$

consisting of canonical homomorphisms. By Proposition 5.1, we have $R(\mathrm{res}^G_{\mathcal{F}}) \cong R(\rho\circ\mathrm{res}^G_{\mathcal{F}})$, $R(\mathrm{res}_1) \cong R(\rho_1\circ\mathrm{res}_1)$, and $R(\mathrm{res}_2) \cong R(\rho_2\circ\mathrm{res}_2)$. There are canonical direct sum decompositions

$$\begin{aligned}
&A(G,\mathcal{G}) = A(G,\mathcal{G}(\geq N)) \oplus A(G,\mathcal{G}(\geq N)'), \text{ and}\\
&P_{\mathcal{F}_{\max}{}^*}(G,\mathcal{G}) = P_{\mathcal{F}_{\max}{}^*}(G,\mathcal{G}(\geq N)) \oplus P_{\mathcal{F}_{\max}{}^*}(G,\mathcal{G}(\geq N)').
\end{aligned}$$

With respect to these direct sums, $\rho\circ\mathrm{res}^G_{\mathcal{F}}$ coincides with $(\rho_1\circ\mathrm{res}_1)\oplus(\rho_2\circ\mathrm{res}_2)$. Thus we get $R(\rho\circ\mathrm{res}^G_{\mathcal{F}}) \cong R(\rho_1\circ\mathrm{res}_1)\oplus R(\rho_2\circ\mathrm{res}_2)$. $\square$

**Example 6.2.** Let $p$ be a prime and $m$ a natural number $\geq 2$. Let $a$ and $b$ be generators of $C_{p^m}$ and $C_p$, respectively, and let $G = C_{p^m}\times C_p$ be the group generated by $a$ and $b$. Let $N$ be the subgroup of $G$ generated by $a^{p^{m-1}}$. We regard $N = C_p \times E$ as the subgroup of $C_{p^m}\times C_p$. Let $\mathcal{F} = \mathcal{F}_G$, $K = G/N$, and $\mathcal{H} = \mathcal{F}_K$. We remark that $K \cong C_{p^{m-1}} \times C_p$. Let $M$ be the $(p+1) \times (2p+1)$-matrix

$$\begin{bmatrix} p & 0 & \cdots & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & p & \ddots & & \vdots & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & p & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & p & p & p & \cdots & p \end{bmatrix}.$$

We readily show that $M$ is similar to the matrix

$$\begin{bmatrix} p & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & p & \ddots & & \vdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & p & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus we get $R(M) \cong \mathbb{Z}_p{}^p$. It follows that

$$\begin{aligned} Q_{\mathcal{F}}(G, \mathcal{S}(G)) &\cong R(\mathrm{res}^G_{\mathcal{F}}) \cong R(\mathrm{res}^G_{\mathcal{F}_{\max}{}^*}) \cong R(\mathrm{res}^K_{\mathcal{H}_{\max}{}^*}) \oplus R(M) \\ &\cong Q_{\mathcal{H}}(K, \mathcal{S}(K)) \oplus \mathbb{Z}_p{}^p. \end{aligned} \tag{6.2}$$

## 7. Decomposition of $R(\mathrm{res}^G_{\mathcal{F}})$ for $G = C_{p^m} \times C_{p^n}$

Throughout this section, let $G = C_{p^m} \times C_{p^n}$ with $m \geq n \geq 2$ for a prime $p$. Let $a$ and $b$ be generators of the cyclic groups $C_{p^m}$ and $C_{p^n}$, respectively. Let $N$ denote the subgroup generated by $a^{p^{m-1}}$ and $b^{p^{n-1}}$. Thus $N$ is isomorphic to $C_p \times C_p$. Let $H_0$ denote the subgroup generated by $a^{p^{m-1}}$. We can regard $H_0$ as the subgroup $C_p \times E$ of $C_{p^m} \times C_{p^n}$, where $E$ is the trivial group. The group $G$ also contains subgroups $H_i$ ($i = 1, \ldots, p$) of order $p$ generated by $a^{ip^{m-1}} b^{p^{n-1}}$. We can regard $H_p$ as the subgroup $E \times C_p$ of $C_{p^m} \times C_{p^n}$. Let

$$\begin{aligned} \mathcal{G} &= \mathcal{S}(G), \\ \mathcal{G}_1 &= \{H \in \mathcal{S}(G) \mid H \supset N\}, \\ \mathcal{G}_{2,i} &= \{H \in \mathcal{S}(G) \smallsetminus \mathcal{G}_1 \mid H \supset H_i\}, \end{aligned}$$

where $i = 0, \ldots, p$. Then the equality

$$\mathcal{G} = \mathcal{G}_1 \amalg \mathcal{G}_{2,0} \amalg \left( \coprod_{i=1}^{p} \mathcal{G}_{2,i} \right) \amalg \{E\} \tag{7.1}$$

holds, which gives the decomposition formula

$$(7.2)\qquad A(G,\mathcal{G}) = A(G,\mathcal{G}_1) \oplus A(G,\mathcal{G}_{2,0}) \oplus \bigoplus_{i=1}^{p} A(G,\mathcal{G}_{2,i}) \oplus A(G,\{E\})$$

of the Burnside module. In addition, we have the canonical identifications

$$(7.3)\qquad \begin{aligned} \mathcal{G}_1 &= \mathcal{S}(G/N), \\ \mathcal{G}_1 \cup \mathcal{G}_{2,0} &= \mathcal{S}(G/H_0), \\ G/N &= C_{p^{m-1}} \times C_{p^{n-1}}, \text{ and} \\ G/H_0 &= C_{p^{m-1}} \times C_{p^n}. \end{aligned}$$

Set $X_{m,n} = A(G,\mathcal{G}_1)$, $Y_{m,n,i} = A(G,\mathcal{G}_{2,i})$ $(i = 0, \ldots, p)$, and $Z_{m,n} = A(G,\{E\})$. By (7.2) we get

$$(7.4)\qquad A(C_{p^m} \times C_{p^n}) = X_{m,n} \oplus \bigoplus_{i=0}^{p} Y_{m,n,i} \oplus Z_{m,n}.$$

It follows from (7.3) that

$$(7.5)\qquad \begin{aligned} A(C_{p^{m-1}} \times C_{p^{n-1}}) &\cong X_{m,n}, \\ A(C_{p^{m-1}} \times C_{p^n}) &\cong X_{m,n} \oplus Y_{m,n,0} \end{aligned}$$

In addition, we have

$$(7.6)\qquad Y_{m,n,i} \cong Y_{n,n,i} \cong Y_{n,n,0} \quad (i = 1, \ldots, p).$$

Let $\mathcal{F} = \mathcal{S}(G) \smallsetminus \{G\}$, $\mathcal{F}_1 = \{H \in \mathcal{F} \mid H \supset N\}$, $\mathcal{F}_{2,i} = \{H \in \mathcal{F} \mid H \supset H_i\}$ $(i = 0, \ldots, p)$, and let $f_{m,n}$, $g_{m,n}$, $h_{m,n,i}$, $k_{m,n}$ be the restriction homomorphisms

$$\begin{aligned} &\mathrm{res}^G_{\mathcal{F}} : A(G) \to L_{\mathcal{F}}(G,\mathcal{S}(G)), \\ &\mathrm{res}^G_{\mathcal{F}_1} : X_{m,n} \to L_{\mathcal{F}_1}(G,\mathcal{G}_1), \\ &\mathrm{res}^G_{\mathcal{F}_{2,i}} : Y_{m,n,i} \to L_{\mathcal{F}_{2,i}}(G,\mathcal{G}_{2,i}), \\ &\mathrm{res}^G_{\mathcal{F}} : Z_{m,n} \to L_{\mathcal{F}}(G,\{E\}), \end{aligned}$$

respectively.

**Theorem 7.1.** *Under the situation above, the direct sum decomposition formula*

$$R(f_{m,n}) \cong R(g_{m,n}) \oplus R(h_{m,n,0}) \oplus \bigoplus_{i=1}^{p} R(h_{m,n,i}) \oplus R(k_{m,n})$$

*holds.*

*Proof.* Similarly to the diagram (6.1), we have diagrams

$$\begin{tikzcd}
X_{m,n} \arrow[r, "g_{m,n}"] \arrow[d] & L_{\mathcal{F}_1}(G,\mathcal{G}_1) \arrow[r] \arrow[d] & P_{\mathcal{F}_{\max}}(G,\mathcal{G}_1) \arrow[d] \\
A(G) \arrow[r, "f_{m,n}"] & L_{\mathcal{F}}(G,\mathcal{G}) \arrow[r] & P_{\mathcal{F}_{\max}}(G,\mathcal{G}) \\
A(G,\mathcal{G}_1{}') \arrow[r, "g_{m,n}{}'"'] \arrow[u] & L_{\mathcal{F}}(G,\mathcal{G}_1{}') \arrow[r] \arrow[u] & P_{\mathcal{F}_{\max}}(G,\mathcal{G}_1{}'), \arrow[u]
\end{tikzcd}$$

$$\begin{tikzcd}
Y_{m,n,0} \arrow[r, "h_{m,n,0}"] \arrow[d] & L_{\mathcal{F}_{2,0}}(G,\mathcal{G}_{2,0}) \arrow[r] \arrow[d] & P_{\mathcal{F}_{\max}}(G,\mathcal{G}_{2,0}) \arrow[d] \\
A(G,\mathcal{G}_1{}') \arrow[r, "g_{m,n}{}'"] & L_{\mathcal{F}}(G,\mathcal{G}_1{}') \arrow[r] & P_{\mathcal{F}_{\max}}(G,\mathcal{G}_1{}') \\
A(G,\mathcal{G}_{2,0}{}') \arrow[r, "h_{m,n,0}{}'"'] \arrow[u] & L_{\mathcal{F}}(G,\mathcal{G}_{2,0}{}') \arrow[r] \arrow[u] & P_{\mathcal{F}_{\max}}(G,\mathcal{G}_{2,0}{}'), \arrow[u]
\end{tikzcd}$$

and so on, where $\mathcal{G}_1{}' = \mathcal{G} \smallsetminus \mathcal{G}_1$ and $\mathcal{G}_{2,0}{}' = \mathcal{G}_1{}' \smallsetminus \mathcal{G}_{2,0}$. The theorem above is obtained by iteration of use of Theorem 6.1. □

Concerning with Theorem 7.1, we remark

**Proposition 7.2.**

(1) $R(f_{m-1,n-1}) \cong R(g_{m,n})$.

(2) $R(f_{m-1,n}) \cong R(g_{m,n}) \oplus R(h_{m,n,0})$.

(3) $R(f_{m,n}) \cong R(g_{m,n}) \oplus R(h_{m,n,0}) \oplus R(h_{n,n,0})^{\oplus p} \oplus \mathbb{Z}_p$.

*Proof.* The claims (1) and (2) follow from (7.5). Since $R(k_{m,n}) \cong \mathbb{Z}_p$, the claim (3) follows from Theorem 7.1 and (7.6). □

The next fact immediately follows from the proposition above.

**Corollary 7.3.**

(1) $R(f_{n-1,n}) \cong R(f_{n-1,n-1}) \oplus R(h_{n,n,0})$.

(2) $R(f_{m,n}) \cong R(f_{m-1,n}) \oplus R(h_{n,n,0})^{\oplus p} \oplus \mathbb{Z}_p$.

Now recall Lemma 3.1 and Sugimura's theorem described in the introduction. By induction arguments on $m$ and $n$ with the corollary above, we can readily prove

**Proposition 7.4.** *Any element $x \in Q_{\mathcal{F}_G}(G, \mathcal{S}(G))$ has exponent $p$, i.e. $px = 0$, and hence $Q_{\mathcal{F}_G}(G, \mathcal{S}(G))$ is isomorphic to a direct sum of copies of $\mathbb{Z}_p$.*

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