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第2062巻 2018年 68-78

Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces (Review)

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Abstract

Let $(E, \overline{\partial}_E, \theta)$ be a stable Higgs bundle of degree 0 on a compact Riemann surface X. We have the associated family of harmonic bundles $(E, \overline{\partial}_E, t\theta, h_t)$ $(t \in \mathbb{C}^*)$. We briefly review the study of the asymptotic behaviour of this family when t goes to ∞.

1 Introduction

1.1 Harmonic bundles

Let X denote a Riemann surface. Let $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle on X, i.e., $(E, \overline{\partial}_E)$ denotes a holomorphic vector bundle, and θ denotes an $\mathrm{End}(E)$-valued holomorphic one form.

Let h be a Hermitian metric of E. We have the Chern connection ∇_h, which is the unitary connection of E whose $(0,1)$-part is equal to $\overline{\partial}_E$. The curvature of ∇_h is denoted by $R(\nabla_h)$. We also have the adjoint $\theta_h^\dagger$ of the Higgs field with respect to the metric, which is a C^∞-section of $\mathrm{End}(E) \otimes \Omega_X^{0,1}$.

The metric h is called a harmonic metric of the Higgs bundle $(E, \overline{\partial}_E, \theta)$ if the Hitchin equation is satisfied:

$$R(\nabla_h) + [\theta, \theta_h^\dagger] = 0. \tag{1}$$

A Higgs bundle with a harmonic metric is called a harmonic bundle. We remark that the Hitchin equation means the connection $\mathbb{D}_h^1 := \nabla_h + \theta + \theta_h^\dagger$ is flat.

1.2 Basic examples

Harmonic bundles are solutions of the Hitchin equation which is a complicated non-linear partial differential equation, hence it is difficult to see them, in general. Let us mention rather easy examples.

Harmonic bundles of rank one A Higgs bundle of *rank one* is just a holomorphic line bundle with a holomorphic one form and a flat Hermitian metric.

Indeed, if the rank of the vector bundle is one, the Higgs field θ is just a holomorphic one form, and we always have $[\theta, \theta_h^\dagger] = 0$. Hence, the Hitchin equation is equivalent to the vanishing $R(\nabla_h) = 0$.

We can study harmonic bundles of rank one by the method of the classical harmonic analysis. In this sense, they are rather easy to study.

Complex variations of Hodge structure A harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ is called a polarized complex variation of Hodge structure if moreover we have the orthogonal decomposition $E = \bigoplus E^i$ such that $\theta(E^i) \subset E^{i-1} \otimes \Omega^1$.

Polarized variations of Hodge structure naturally appear in algebraic geometry. When we are given a smooth projective morphism of complex manifolds $f : Y \longrightarrow X$ with a relatively ample line bundle, the cohomology groups of the fibers $H^j(f^{-1}(x), \mathbb{C})$ $(x \in X)$ give a flat bundle on X, and it is naturally enriched to a polarized variation of Hodge structure.

1.3 Flows on the moduli spaces

Homeomorphisms between moduli spaces By definition, a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ has the underlying Higgs bundle $(E, \overline{\partial}_E, \theta)$. As remarked, it also has the underlying flat bundle $(E, \mathbb{D}_h^1)$. The following is the most fundamental in the study of harmonic bundles on compact Riemann surfaces.

Theorem 1.1 (Corlette [4], Donaldson [5], Hitchin [9], Simpson [17]) *Suppose that X is compact and connected. The above correspondences induce homeomorphisms of the moduli spaces of harmonic bundles, polystable Higgs bundles of degree 0, and semisimple flat bundles.*

It is easy to see the correspondences

$$(E, \overline{\partial}_E, \theta, h) \longmapsto (E, \overline{\partial}_E, \theta), \quad (E, \overline{\partial}_E, \theta, h) \longmapsto (E, \mathbb{D}_h^1).$$

But, in general, the inverse correspondences are difficult because we have to solve complicated non-linear partial differential equations.

Flows on the moduli spaces Let us recall that we have a natural $\mathbb{C}^*$-action on the moduli spaces, which was originally studied by Hitchin and Simpson.

Suppose that we are given a stable Higgs bundle $(E, \overline{\partial}_E, \theta)$ of degree 0 on a compact Riemann surface X. For any non-zero complex number t, we have a new Higgs bundle $(E, \overline{\partial}_E, t\theta)$, which is also stable of degree 0. Then, we have the corresponding harmonic bundle $(E, \overline{\partial}_E, t\theta, h_t)$, and the flat bundle $(E, \mathbb{D}_{h_t}^1)$. This procedure gives a $\mathbb{C}^*$-action on the moduli spaces.

Basic examples Let us look at the rank one case. Because the condition for the metric and the Higgs field is separated, the metrics h_t can be independent of the parameter t, i.e., the associated family of harmonic bundles is given as $(E, \overline{\partial}_E, t\theta, h)$. Then, the corresponding family of flat bundles is given as

$(E, \nabla_h + 2\,\mathrm{Re}(t\theta))$. In this way, the dependence on t is easily described in the rank one case.

Let us look at the case of variation of complex Hodge structure. If the harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ is a complex variation of Hodge structure, we can easily observe the underlying Higgs bundles $(E, \overline{\partial}_E, t\theta)$ $(t \neq 0)$ are isomorphic to the original Higgs bundle $(E, \overline{\partial}_E, \theta)$. In other words, the underlying Higgs bundle of a complex variation of Hodge structure is a fixed point in the moduli space of Higgs bundles. Hence, we also have $(E, \overline{\partial}_E, t\theta, h_t) \simeq (E, \overline{\partial}_E, \theta, h)$, and $(E, \mathbb{D}^1_{h_t}) \simeq (E, \mathbb{D}^1_h)$.

Slightly more generally, if a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ is the direct sum of tensor products of rank one harmonic bundles $(L_i, \overline{\partial}_{L_i}, \theta_{L_i}, h_{L_i})$ and complex variation of Hodge structure $(E_i, \overline{\partial}_{E_i}, \theta_{E_i}, h_{E_i})$, then the dependence of the associated families on t is described easily up to isomorphisms:

$$(E, \overline{\partial}_E, t\theta, h_t) \simeq \bigoplus (L_i, \overline{\partial}_{L_i}, t\theta_{L_i}, h_{L_i}) \otimes (E_i, \overline{\partial}_{E_i}, \theta_{E_i}, h_{E_i}),$$

$$(E, \mathbb{D}^1_h) \simeq \bigoplus (L_i, \nabla_{h_{L_i}} + 2\,\mathrm{Re}(t\theta_{L_i})) \otimes (E_i, \mathbb{D}^1_{h_{E_i}}).$$

1.4 General Issue

In general, it is not easy to see the family $(E, \overline{\partial}_E, t\theta, h_t)$. Recently, the behaviour of this family for large parameter has been studied by several groups of mathematicians from several viewpoints. That is the topic of this brief review.

Roughly, it is motivated by the interest to the asymptotic of various structures of the moduli spaces around infinity. For instance, it seems useful to understand the asymptotic behaviour of the induced homeomorphism between the moduli spaces of Higgs bundles and flat bundles.

Mazzeo-Swoboda-Weiss-Witt In [12, 13], important contributions were given by Mazzeo, Swoboda, Weiss and Witt. Among others, they introduced the concepts of *asymptotic decoupling* and *limiting configuration.*

Let us recall the decoupling of the Hitchin equation. The Hitchin equation (1) is given as the vanishing of the sum of $R(\nabla_h)$ and $[\theta, \theta^\dagger_h]$. The decoupling of the Hitchin equation is the vanishing of both $R(\nabla_h)$ and $[\theta, \theta^\dagger_h]$:

$$R(\nabla_h) = 0, \quad [\theta, \theta^\dagger_h] = 0. \tag{2}$$

If the decoupled Hitchin equation (2) is satisfied, the harmonic bundle is *locally* isomorphic to a direct sum of harmonic bundles of rank one.

Roughly, *asymptotic decoupling* means, when t is very large, $R(h_t)$ and $[\theta, \theta^\dagger_{h_t}]$ become small outside the discriminant $D(E, \theta)$ of θ, under some genericity assumption. (See Theorem 2.1 for more details.) Hence, $(E, \overline{\partial}_E, t\theta, h_t)$ should be close to a direct sum of rank one harmonic bundles on $X \setminus D(E, \theta)$.

Moreover, when t goes to ∞, the family of the holomorphic bundles equipped with Hermitian metric $(E, \overline{\partial}_E, h_t)_{|X \setminus D(E,\theta)}$ should be convergent to a direct sum of line bundles with flat metrics. The limit metric and the corresponding parabolic bundle are called *the limiting configuration.* It is desirable to prove the convergence and to obtain an explicit description of the limit metric.

Interestingly, Mazzeo, Swoboda, Weiss and Witt established these claims under the assumptions that the rank of the bundle is 2 and that the spectral curve is smooth. Let us mention that their study [12, 13] was inspired by the work of physicists Gaiotto, Moore and Neitzke [7, 8].

Katzarkov-Noll-Pandit-Simpson A different viewpoint was provided by Katzarkov, Noll, Pandit and Simpson [10]. They are interested in the asymptotic behaviour of the parallel transports of $\mathbb{D}^1_{h_t}$ when t is large, which is called the Hitchin-WKB problem. They proposed a conjectural estimate to describe how the parallel transports are far from unitary. (See Corollary 2.2.)

They related it to the following interesting picture. A harmonic bundle of rank r on the Riemann surface X is equivalent to a harmonic map from the universal covering $\widetilde{X}$ of X to the symmetric space $\mathrm{GL}(r,\mathbb{C})/U(r)$, which is equivariant with respect to the action of the fundamental group of X. The family of harmonic bundles give a family of harmonic maps $\varphi_t : \widetilde{X} \longrightarrow \mathrm{GL}(r,\mathbb{C})/U(r)$. In general, we cannot expect that this sequence is convergent to any map $\widetilde{X} \longrightarrow \mathrm{GL}(r,\mathbb{C})/U(r)$. But, Katzarkov, Noll, Pandit and Simpson discovered that if the target space is replaced to the affine building B of A_{r-1}-type, and if their conjectural estimate is verified, then any sequence φ_{t_i} contains a subsequence which is convergent to a harmonic map $\widetilde{X} \longrightarrow B$.

Moreover, they have been developing the theory of universal building associated to Higgs bundles [10, 11].

Collier-Li, Dai-Li Collier and Li [2] closely studied these issues for some cyclic type harmonic bundles. They established the asymptotic decoupling for large parameters and the convergence to the limiting configuration for such harmonic bundles. It is remarkable that their results are quite precise. They also studied the Hitchin-WKB problem, and they established the conjectural estimate in such cases.

Relatedly, in a more recent study [3], Dai and Li obtained interesting estimates for such harmonic bundles, for example, interesting bounds for $\mathrm{tr}(\theta\theta_h^\dagger)$, nowhere vanishing result of $[\theta, \theta_h^\dagger]$, etc.

Our result After the interesting works [2, 10, 12, 13], I studied the issues and partially generalized the previous results by using the different methods in [16]. Our result can be summarized as follows.

First, the asymptotic decoupling holds in any rank case under the genericity assumption that the eigenvalues of the Higgs field are generically multiplicity-free. We do not need the assumption on the smoothness of the spectral curve, nor the compactness of Riemann surface. As an application, we proved the conjectural estimate in the Hitchin-WKB problem.

Second, we proved the convergence to the limiting configuration if the rank of the harmonic bundle is 2. We do not need to impose the smoothness or the irreducibility to the spectral curve. We also obtained a rather explicit description of the limiting configuration.

Study of the behaviour when t goes to 0 Before finishing the introduction, let us mention the study of the family $(E, \overline{\partial}_E, t\theta, h_t)$ for small t. It is a classical and celebrated result of Hitchin and Simpson that the family is convergent to a complex variation of Hodge structure when t goes to 0. This is an interesting fact, and some applications were given. More recently, the study for small parameter has been renewed by Dumitrescu, Fredrickson, Kydonakis, Mazzeo, Mulase and Neitzke [6], inspired by a conjecture of Gaiotto. They described the convergence of the associated family of flat connections explicitly. We expect that it will lead us to a new exciting development.

2 Asymptotic decoupling

In the rest of this review, we shall explain our result in the case $\operatorname{rank} E = 2$. See [16] for more detailed and precise results.

2.1 Preliminary

Spectral curve Let $(E, \overline{\partial}_E, \theta, h)$ be a harmonic bundle of rank 2 on the compact Riemann surface X. Then, we have the spectral curve $\Sigma(E, \theta)$ of the Higgs bundle. It is a complex curve in the cotangent bundle T^*X of X, and it is always finite over X. Roughly, the fiber $\Sigma(E, \theta)_P$ over P is the set of the eigenvalues of the Higgs field at P. Because $\operatorname{rank} E = 2$, we have $|\Sigma(E, \theta)_P| \leq 2$ for any $P \in X$, and we have the following two cases.

(i) We have $|\Sigma(E, \theta)_P| = 1$ for any $P \in X$.

(ii) We have the 0-dimensional subset $D(E, \theta)$ called the discriminant, such that $|\Sigma(E, \theta)_P| = 2$ for any $P \in X \setminus D(E, \theta)$. In other words, the Higgs field generically has two distinct eigenvalues.

The case (i) Let us remark that the case (i) is easier. We have the description

$$(E, \overline{\partial}_E, \theta, h) = (L, \overline{\partial}_L, \theta_L, h_L) \otimes (E', \overline{\partial}_{E'}, \theta', h'),$$

where $\operatorname{rank} L = 1$, and θ' is nilpotent. Then, the associated family is described as follows:

$$(E, \overline{\partial}_E, t\theta, h_t) = (L, \overline{\partial}_L, t\theta_L, h_L) \otimes (E', \overline{\partial}_{E'}, t\theta', h'_t).$$

Because the Higgs field θ' is nilpotent, we have a uniform bound of the energy of the harmonic bundles $(E', \overline{\partial}_{E'}, t\theta', h'_t)$. Hence, it is classical that this family of harmonic bundles is convergent when t goes to ∞, and the limit $(E'_\infty, \overline{\partial}_{E'_\infty}, \theta'_\infty, h'_\infty)$ is a complex variation of Hodge structure. Hence, the family $(E, \overline{\partial}_E, t\theta, h_t)$ is asymptotically close to the following family up to isomorphisms:

$$(L, \overline{\partial}_L, t\theta_L, h_L) \otimes (E'_\infty, \overline{\partial}_{E'_\infty}, \theta'_\infty, h'_\infty).$$

The associated family of flat bundles is close to the following family up to isomorphisms:

$$(L, \nabla_{h_L} + 2\,\mathrm{Re}(t\theta_L)) \otimes (E'_\infty, \mathbb{D}^1_{h'_\infty}).$$

2.2 First main result

Asymptotic decoupling We are more interested in the case (ii). We can take a ramified covering $p : X' \longrightarrow X$ such that the spectral curve of the pull back of the Higgs bundle is the union of the image of the one forms ϕ_i $(i = 1, 2)$. Here, ϕ_i are holomorphic one forms on X', and we have $\phi_1 \neq \phi_2$. Once we know the asymptotic behaviour of the pull back of the family $p^*(E, \overline{\partial}_E, t\theta, h_t)$, we can understand the asymptotic behaviour of the original family, so we may assume the existence of the decomposition of the spectral curve $\Sigma(E, \theta) = \mathrm{Im}(\phi_1) \cup \mathrm{Im}(\phi_2)$ from the beginning. It implies that we have the decomposition into a direct sum of Higgs bundles of rank one

$$(E, \overline{\partial}_E, \theta)_{|X \setminus D(E,\theta)} = \bigoplus_{i=1,2} (E_i, \overline{\partial}_{E_i}, \phi_i \,\mathrm{id}_{E_i}), \tag{3}$$

and that $\phi_1 - \phi_2$ is nowhere vanishing on $X \setminus D(E, \theta)$. Then, our first main result is the following.

Theorem 2.1 ([16]) *We take any neighbourhood N of $D(E, \theta)$ and any Kähler metric g_X of X. Then, we have positive constants C_1 and ϵ_1, such that the following estimates hold on $X \setminus N$:*

- *Let v_1 and v_2 be sections of E_1 and E_2 on $X \setminus N$, then we have*

  $$\big|h_t(v_1, v_2)\big| \leq C_1 \exp\big(-\epsilon_1 |t|\big) \cdot |v_1|_{h_t} \cdot |v_2|_{h_t}.$$

  This means that E_1 and E_2 are almost orthogonal.

- *We have $\big|R(\nabla_{h_t})\big|_{h_t, g_X} = \big|[\theta, \theta^\dagger_{h_t}]_{h_t, g_X}\big| \leq C_1 \exp\big(-\epsilon_1|t|\big)$ on $X \setminus N$. We also have the estimate for higher derivatives.*

- *Let $h_{E_i,t}$ $(i = 1, 2)$ denote the restriction of h_t to E_i. Then, the curvature of the Chern connections of $(E_i, h_{E_i,t})$ are dominated by $C_1 \exp(-\epsilon_1 t)$.*

Similar claims hold in the higher rank case if the Higgs bundle is generically regular semisimple, i.e., the eigenvalues of the Higgs field is generically multiplicity-free.

As mentioned in the introduction, this type of estimates were proved by Mazzeo, Swoboda, Weiss and Witt [12] in the case where the spectral curve is smooth and the rank of the harmonic bundle is 2, and by Collier and Li [2] in the case of some cyclic type harmonic bundles.

This theorem implies that the decomposition (3) is close to a decomposition of harmonic bundles of rank one on $X \setminus N$ when the parameter t is large. Hence, the structure of harmonic bundles on $X \setminus D(E, \theta)$ becomes easier when t is large.

For the proof, we use a technique called "Simpson's main estimate". It consists of two types of estimates for harmonic bundles on any disc. One is the estimate of the norm of the Higgs field on any strictly smaller disc in terms of the spectral curve. The other is concerned with the almost orthogonality of the decomposition of the bundle according to the decomposition of the spectral curve. Such estimates were pioneered by Simpson [18] in his study of tame harmonic bundles with the inspiration from [1], and further pursued by myself [14, 15] in the study of tame and wild harmonic bundles. A slightly new estimate was given in [16].

Application to Hitchin WKB-problem We can apply Theorem 2.1 to the study of the parallel transport of the associated flat connections $\mathbb{D}^1_{h_t}$, and we can establish the conjectural estimate in the Hitchin-WKB-problem.

To explain the estimate, we recall a terminology in [10]. Let V be any r-dimensional complex vector space, and let h_1 and h_2 be two Hermitian metrics of V. We have a basis of V which is orthogonal with respect to both h_1 and h_2. We have the real numbers $\kappa_j := \log |e_j|_{h_2} - \log |e_j|_{h_1}$ $(j = 1, \ldots, r)$. We impose that the sequence of the numbers κ_j is decreasing. Then, the sequence $\vec{d}(h_1, h_2) := (\kappa_1, \ldots, \kappa_r)$ is called the vector distance of h_1 and h_2.

Let γ be any path in X. We have two families of the metrics of $E_{|\gamma(0)}$. Let $\Pi_{t,\gamma} : E_{|\gamma(0)} \longrightarrow E_{|\gamma(1)}$ be the isomorphism obtained as the parallel transport of $\mathbb{D}^1_{h_t}$ along γ. Let $\Pi^*_{t,\gamma} h_{t|\gamma(1)}$ be the family of metrics of $E_{|\gamma(0)}$ obtained as the pull back of $h_{t|\gamma(1)}$ by $\Pi_{t,\gamma}$. We also have the family of metrics $h_{t|\gamma(0)}$ of $E_{|\gamma(0)}$. Katzarkov, Noll, Pandit and Simpson proposed a conjectural estimate for the vector distances of these two families of metrics.

We impose a condition to the path γ as in [10]. First, we assume that γ is a path in $X \setminus D(E, \theta)$. Recall the decomposition (3) on $X \setminus D(E, \theta)$. We say a path in $X \setminus D(E, \theta)$ is non-critical, if $\gamma^* \operatorname{Re}(\phi_1 - \phi_2)$ is nowhere vanishing on the interval. Then, we have the following corollary of Theorem 2.1.

Corollary 2.2 *Suppose that γ is non-critical. Set $\alpha_i := -\int_\gamma \operatorname{Re}(\phi_i)$. We assume that $\alpha_1 > \alpha_2$. Then, we have $C_2 > 0$ and $\epsilon_2 > 0$ such that*

$$\left| \frac{1}{t} \vec{d}\big(h_{t|\gamma(0)}, \Pi^*_{t,\gamma} h_{t|\gamma(1)}\big) - (2\alpha_1, 2\alpha_2) \right| \leq C_2 \exp(-\epsilon_2 t). \tag{4}$$

We also have a similar estimate in any rank case under the assumption that the Higgs bundle is generically regular semisimple.

As mentioned, this kind of estimate was conjectured by Katzarkov, Noll, Pandit and Simpson [10]. Some cases were verified by Collier and Li [2].

The proof of this corollary is not so difficult, once we obtain the asymptotic decoupling (Theorem 2.1). The estimate (4) is obvious for harmonic bundles of rank one. Our asymptotic decoupling ensures that along any non-critical path the harmonic bundles $(E, \overline{\partial}_E, t\theta, h_t)$ for large t are close to a direct sum of a direct sum of harmonic bundles of rank one. Hence, by using a standard technique of singular perturbation theory, we can obtain the desired estimate (4).

3 Limiting configuration

3.1 Rough statement of the second main result

We continue to consider a harmonic bundle $(E, \overline{\partial}_E, \theta, h)$ of rank 2 on a compact Riemann surface X with the decomposition (3). We have the associated family of harmonic bundles $(E, \overline{\partial}_E, t\theta, h_t)$ $(t \in \mathbb{C}^*)$. According to Theorem 2.1, the bundles E_1 and E_2 are asymptotically orthogonal for large t, and the metrics $h_{E_i,t} := h_{t|E_i}$ on $X \setminus D(E,\theta)$ are asymptotically flat. The next issue is to study the existence of the limit of the metrics $h_i^{\lim} = \lim_{t\to\infty} h_{t|E_i}$ after gauge transforms on $X \setminus D(E,\theta)$. We can prove it (Theorem 3.1 below). Hence, the family of harmonic bundles $(E, \overline{\partial}_E, t\theta, h_t)$ is asymptotically close to the following singular but much simpler family of harmonic bundles on $X \setminus D(E,\theta)$:

$$\bigoplus_{i=1,2} (E_i, \overline{\partial}_{E_i}, t\phi_i, h_i^{\lim}).$$

The family of flat bundles $(E, \mathbb{D}^1_{h_t})$ is asymptotically close to

$$\bigoplus_{i=1,2} (E_i, \nabla_{h_i^{\lim}} + 2\operatorname{Re}(t\phi_i))$$

on $X \setminus D(E,\theta)$. Moreover, we have an explicit description of $(E_i, \overline{\partial}_{E_i}, h_i^{\lim})$, which we will explain in the rest.

Mazzeo, Swoboda, Weiss and Witt [12] proved such convergence under the assumptions that the rank of the bundle is 2 and that the spectral curve is smooth. Collier and Li [2] proved it in the case of some cyclic type harmonic bundles with higher rank. They gave explicit descriptions of the limit metric in their cases.

We remark that they also proved the exponential decay of the difference between the harmonic metrics h_t and the limit metric. In contrast, our method gives only the convergence and the description of the limit.

3.2 Explicit description of the limit

Let us explain the explicit description of the limit.

We may assume $\phi_1 = \omega$ and $\phi_2 = -\omega$ for a holomorphic one form $\omega \neq 0$. We have the line bundles L_1 and L_2 on X, not only on $X \setminus D(E,\theta)$, with an inclusion ι of E into $L_1 \oplus L_2$ such that (i) $\iota \circ \theta = \Big(\omega \,\mathrm{id}_{L_1} \oplus (-\omega)\,\mathrm{id}_{L_2}\Big) \circ \iota$, (ii) the induced morphisms $E \longrightarrow L_i$ are epimorphisms. The restriction of L_i to $X \setminus D(E,\theta)$ are the same as E_i in (3).

The limit metric should be flat metrics of L_i which are singular at $D(E,\theta)$. Such flat metrics are determined by the parabolic weights at the point of $D(E,\theta)$.

Singular flat metrics on line bundles Let us recall a general theory for singular flat metrics on holomorphic line bundles. Suppose that we are given a holomorphic line bundle L on X, and real numbers $\boldsymbol{b} = (b_P \,|\, P \in D(E,\theta))$ such that

$$\deg(L) - \sum_{P \in D(E,\theta)} b_P = 0.$$

Then, we have a flat metric $h_L^{\boldsymbol{b}}$ of $L_{|X \setminus D(E,\theta)}$ such that at any point $P \in D(E,\theta)$, if we take a holomorphic coordinate z_P centered at P, then $|z_P|^{2b_P} h_L^{\boldsymbol{b}}$ gives a C^∞-metric of L around P. Such a flat metric is unique up to the multiplication of positive constants. Conversely, any flat singular metric is described in this way.

Description of the limit metrics To describe the limit metrics, it is enough to indicate the parabolic weights.

We set $d_i := \deg(L_i)$. We may assume $d_1 \leq d_2$.

To any point P of $D(E,\theta)$, two integers are attached. Let m_P denote the order of zero of ω at P, i.e., $\omega \sim z_P^{m_P} dz_P$. Another integer ℓ_P is given as follows. We have the induced morphism $\det(E) \longrightarrow L_1 \otimes L_2$. It is generically an isomorphism. The support of the cokernel is contained in $D(E,\theta)$. Let ℓ_P denote the length of the cokernel at P.

We have the function $\chi_P : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\leq 0}$, given as follows:

$$\chi_P(a) = \begin{cases} (m_P + 1)a - \ell_P/2 & (0 \leq a \leq \ell_P/2(m_P+1)) \\ 0 & (a > \ell_P/2(m_P+1)). \end{cases}$$

We have a unique non-negative number $a_{E,\theta}$ determined by the conditions

$$0 \leq a_{E,\theta} \leq \max_{P \in D(E,\theta)} \{\ell_P/2(m_P+1)\}, \quad d_1 + \sum_{P \in D(E,\theta)} \chi_P(a_{E,\theta}) = 0.$$

We have flat metrics $h_{L_i}^{\lim}$ of L_i on $X \setminus D(E,\theta)$ satisfying the following conditions.

- For each point of $D(E,\theta)$, take a holomorphic coordinate (U_P, z_P) with $z_P(P) = 0$. Then, $h_{L_1}^{\lim} |z_P|^{-2\chi_P(a_{E,\theta})}$ and $h_{L_2}^{\lim} |z_P|^{2\chi_P(a_{E,\theta}) + 2\ell_P}$ are C^∞.

The following is our second main result.

Theorem 3.1 $\lim_{t\to\infty} h_{t|E_i} = h_{L_i}^{\lim}$ *on* $X \setminus D(E,\theta)$.

More precisely, we need gauge transformations of $E_{|X\setminus D(E,\theta)}$. See [16] for a more precise statement.

Let us remark if $\deg(L_1) = \deg(L_2)$, then we have $a_{E,\theta} = 0$ and the parabolic weights are $-\chi_P(a_{E,\theta}) = \chi(a_{E,\theta}) + \ell_P = \ell_P/2$. But, if $\deg(L_1) \neq \deg(L_2)$, we may have $-\chi(a_{E,\theta}) \neq \ell_P/2$. Hence, the parabolic weights of the limit metrics can be more complicated.

Acknowledgments This review is based on the manuscripts of my talks given at the workshops "the 21th Symposium on Complex Geometry" in Kanazawa, "Modern Interactions between Algebra, Geometry and Physics" in Sendai, "Flat connections, Higgs bundles and Painlevé equations" in Taipei, "Topology and Analysis of Discrete Groups and Hyperbolic Spaces" in Kyoto, and "Workshop on New Perspectives on Moduli Spaces in Gauge Theory" in Singapore. I would like to express my gratitude to the organizers.

The work [16] was partially supported by the Grant-in-Aid for Scientific Research (S) (No. 24224001) and the Grant-in-Aid for Scientific Research (C) (No. 15K04843), Japan Society for the Promotion of Science.

References

[1] L. V. Ahlfors, *An extension of Schwarz's lemma*, Trans. Amer. Math. Soc. **43** (1938), 359–364.

[2] B. Collier, Q. Li. *Asymptotics of certain families of Higgs bundles in the Hitchin component*, arXiv:1405.1106

[3] S. Dai, Q. Li, *Minimal surfaces for Hitchin representations*, arXiv:1605.09596

[4] K. Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom. **28** (1988), 361–382.

[5] S. K. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. **55** (1987), 127–131.

[6] O. Dumitrescu, L. Fredrickson, G. Kydonakis, R. Mazzeo, M. Mulase, A. Neitzke, *Opers versus nonabelian Hodge*, arXiv:1607.02172.

[7] D. Gaiotto, G. Moore, A. Neitzke, *Four-dimensional wall-crossing via three-dimensional field theory*, Comm. Math. Phys. **299** (2010), 163–224

[8] D. Gaiotto, G. Moore, A. Neitzke, *Wall-crossing, Hitchin systems, and the WKB approximation*, Adv. Math. **234** (2013), 239–403.

[9] N. Hitchin, *The self-duality equations on a Riemann surface*, Proc. London Math. Soc. (3) **55** (1987), 59–126.

[10] L. Katzarkov, A. Noll, P. Pandit, C. Simpson, *Harmonic Maps to Buildings and Singular Perturbation Theory*, Comm. Math. Phys. **336** (2015), 853–903.

[11] L. Katzarkov, A. Noll, P. Pandit, C. Simpson, *Constructing Buildings and Harmonic Maps*, arXiv:1503.00989.

[12] R. Mazzeo, J. Swoboda, H. Weiss, F. Witt. *Ends of the moduli space of Higgs bundles*, Duke Math. J. **165** (2016), 2227–2271.

[13] R. Mazzeo, J. Swoboda, H. Weiss, F. Witt, *Limiting configurations for solutions of Hitchin's equation*, Semin. Theor. Spectr. Geom. **31** (2012–2014), 91–116.

[14] T. Mochizuki, *Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules I, II*, Mem. AMS. **185** (2007)

[15] T. Mochizuki, *Wild harmonic bundles and wild pure twistor D-modules*, Astérisque **340**, Société Mathématique de France, Paris, 2011.

[16] T. Mochizuki, *Asymptotic behaviour of certain families of harmonic bundles on Riemann surfaces*, to appear in Journal of Topology, doi: 10.1112/jtopol/jtw018

[17] C. T. Simpson, *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. **1** (1988), 867–918.

[18] C. T. Simpson, *Harmonic bundles on noncompact curves*, J. Amer. Math. Soc. **3** (1990), 713–770.

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