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# 樟脳船の集団運動のモデル方程式に現れる線形化作用素に対するレゾルベント評価

明治大学総合数理学部 池田 幸太 (Kota Ikeda)
School of Interdisciplinary Mathematical Sciences,
Meiji University

北海道大学大学院理学研究院数学部門 栄 伸一郎 (EI, Shin-Ichiro)
Department of Mathematics,
Hokkaido University

## 1 Introduction

Self-driven motion of animal and inanimal organisms is observed in several fields, e.g., biology [10], chemistry [1], and nonlinear physics [5], [14], [15]. Organisms move spontaneously to aggregate and form self-organized structures. In many cases, the individual members do not interact directly, but rather change their surroundings in ways that have an influence on the behavior of other members, which implies that the organisms have long-range interactions [4], [8]. Therefore it is important not only to clarify the mechanism of the self-sustaining motion of each organism but also to study how organisms behave as a whole system.

Spatiotemporal collective motions in chemical experiments with camphor have been investigated in [11], [12], [13]. A camphor scraping at an air–water surface exhibits several motions, e.g., clockwise / counterclockwise rotation, and translation ([12]). Also, it was shown in [11] and [13] that unidirectional motion can be observed if we put a camphor boat in an annular water channel. In an experimental setup, a camphor boat is composed of a plastic disk and a camphor disk stuck on the edge of the plastic disk with an adhesive. Camphor boats constitute a system for changing the number of particles and with simple interaction. In this system we find two different states depending on the number. It was reported in [13] that when the number of boats is less than 30, camphor boats move with a constant velocity and spatially disperse with the same spacing between the boats, which is called a homogeneous state. On the other hand, when the number is larger than 30, the velocities of the boats change with temporal oscillation, and the shock wave appears in the line of the boats, which is an inhomogeneous state.

Various motions which a camphor boat exhibit have been studied mathematically. In this article we are based on [11] and [13], and introduce the following mathematical model for the self-sustaining motion of a camphor boat:

$$\begin{cases} x_\infty'' = -\mu x_\infty' + \gamma(u(x_\infty+\rho,t)) - \gamma(u(x_\infty-\rho,t)), \\ \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \alpha u + f(x - x_\infty, s), \end{cases} \tag{1.1}$$

where $\alpha, \mu, \rho$ are positive constants. The first equation is described by the Newtonian equation with the surface tension of water given by $\gamma(u)$ as a smooth function of $u$. In this model, a camphor scraping is regarded as a particle, and the position of a camphor boat is denoted by $x_\infty = x_\infty(t)$. The surface concentration, denoted by $u$, of a camphor molecular layer is supposed to yield to the reaction-diffusion equation with the function $f(x,s)$ defined by

$$f(x,s) = \begin{cases} 1, & 0 < x < \rho, \\ s, & -\rho < x < 0, \\ 0, & \text{otherwise}, \end{cases}$$

which represents that camphor molecules are supplied only from $(-\rho, \rho)$ where a camphor boat contacts the water surface. Let $s \in [0,1]$, which means that a camphor boat considered in this model is an inhomogeneous medium and the amount of the supply on $(-\rho, 0)$ is not larger than on $(0, \rho)$. This parameter $s$ does not appear in models proposed previously in [11] and [13], and provides a new type of a bifurcation structure in (1.1), which will be described soon later.

The spontaneous motion of a camphor boat can be characterized by a traveling wave solution of

$$\begin{cases} z'_\infty = y_\infty, & t > 0, \\ y'_\infty = -\mu(y_\infty - c) + \gamma(u(z_\infty + \rho, t)) - \gamma(u(z_\infty - \rho, t)), & t > 0, \\ \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial z^2} - c\dfrac{\partial u}{\partial z} - \alpha u + f(z - z_\infty, s), \quad -\infty < z < \infty, & t > 0, \end{cases} \tag{1.2}$$

where we set $z_\infty = z_\infty(t) = x_\infty(t) + ct$, $y_\infty = y_\infty(t) = z'_\infty(t)$, $z = x + ct$ in (1.1). We consider that ${}^t(z_\infty(t), y_\infty(t), u(z,t)) \in \mathbb{R}^2 \times H^1(\mathbb{R})$, and denote the right-hand side of (1.2) by $\mathcal{L}_\infty(z_\infty(t), y_\infty(t), u(z,t); c, \mu, s)$ for simplicity, where ${}^t$ denotes the transpose. A stationary solution defined by $(z_\infty, y_\infty, u(z)) = (0, 0, p(z))$ of (1.2) is called a traveling wave solution, and the parameter $c$ is called a wave speed.

As shown in [11], there is a critical value such that (1.2) for $s = 1$ has a stable traveling wave solution with a positive wave speed only in the case that $\mu$ is smaller than the critical value. Hence the pair of the critical value and $c = 0$ generates a supercritical pitchfork bifurcation. Actually, there is a pitchfork bifurcation point in (1.2) for a parameter set and $s < 1$. For example, there is a bifurcation point $(\mu_0, s_0, c_0)$ such that $0.298 < \mu_0 < 0.3$, $0.12 < s_0 < 0.13$, and $0.271 < c_0 < 0.273$ for $\gamma(u) = \gamma_1/(1 + au)$ and $(a, \gamma_1, \alpha, \rho) = (0.64, 1.7, 0.011, 0.84)$. In [11], there is no asymmetric structure in a model like $s < 1$ in (1.2) so that it is easy to verify the existence of a bifurcaiton point related to a pitchfork bifurcation. On the other hand, the assumption of $s < 1$ in our model have a possibility to provide an imperfection bifurcation instead of a pitchfork bifurcation. From this viewpoint, both $\mu$ and $s$ should change simultaneously in order to consider a pitchfork bifurcation in (1.2). Under a more general setting in the nonlinearity $\gamma(u)$, we do not know whether the system (1.2) exhibits such a bifurcation structure.

The existence of the bifurcation point affects the linearized operator

$$\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)\Phi \equiv \begin{pmatrix} Y_\Phi \\ -\mu_0 Y_\Phi + \gamma'(p(\rho))(p'(\rho)Z_\Phi + \phi(\rho)) - \gamma'(p(-\rho))(p'(-\rho)Z_\Phi + \phi(-\rho)) \\ \phi'' - c_0\phi' - \alpha\phi - Z_\Phi[s_0\delta_{-\rho} - (s_0-1)\delta_0 - \delta_\rho] \end{pmatrix} \tag{1.3}$$

for $\Phi = {}^t(Z_\Phi, Y_\Phi, \phi) \in \mathbb{R}^2 \times H^1(\mathbb{R})$, where we denote the Dirac delta function giving unit mass to the point $z_0$ by $\delta_{z_0}$. It is obvious that $\Phi = {}^t(-1,0,\phi)$ satisfies $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)\Phi = 0$, that is, $\Phi$ is an eigenfunction of $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)$ for the zero eigenvalue, where $\phi$ can be defined by $\phi = p'$.

Actually, (1.3) has a degeneracy condition, and there is a solution $\Psi = {}^t(0,1,\psi)$ of $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)\Psi = -\Phi$. Here we assume that the first component of $\Psi$ is 0 without loss of generality. From this condition, we note that $\psi \in H^2(\mathbb{R})$ can be detemined uniquely if it exists. The existence of $\Psi$ generically means that the multiplicity of the zero eigenvalue of $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)$ is equal to or more than 2.

From the viewpoint above, we suppose that there is $(\mu_0, s_0, c_0)$ such that the following conditions hold true;

(A1) There is $p = p(z) \in H^2(\mathbb{R})$ such that $(0,0,p(z))$ satisfies $\mathcal{L}_\infty(0,0,p;c_0,\mu_0,s_0) = 0$.

(A2) There is a unique solution $\Psi \in \mathbb{R}^2 \times H^2(\mathbb{R})$ of $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)\Psi = -\Phi$.

(A3) Any spectrum $\lambda$ of $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)$ is contained in $\mathbb{C}\backslash\Sigma$ except for $\lambda = 0$, where

$$\Sigma = \{\lambda \in \mathbb{C} \mid -\frac{\alpha}{2} \le \mathrm{Re}\lambda\} \cup \{\lambda \in \mathbb{C} \mid |\arg\lambda| \le \theta\}$$

for some $\theta > \pi/2$. Re$\lambda$ and $\arg\lambda$ denote the real part and the argument of a complex value $\lambda$, respectively, where $\arg\lambda$ is assumed to satisfy $-\pi < \arg\lambda \le \pi$.

(A4) The generalized eigenspace associated to 0 is spanned by $\Phi$ and $\Psi$.

Now we consider the collective motion of $(N+1)$-camphors on a one-dimensional circuit $(0,L)$. Our system is described by

$$\begin{cases} z_i' = y_i, & t > 0, \\ y_i' = -(\mu_0 + \kappa_1)(y_i - c_0) + \gamma(u(z_i+\rho,t)) - \gamma(u(z_i-\rho,t)), & t > 0, \\ \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial z^2} - c_0\dfrac{\partial u}{\partial z} - \alpha u + \displaystyle\sum_{i=0}^{N} \overline{f}(z - z_i, s_0 + \kappa_2), & z \in \mathbb{T} \quad t > 0, \end{cases} \tag{1.4}$$

for $N \ge 1$ and $i = 0, \ldots, N$, where $\mathbb{T} = \mathbb{R}/L\mathbb{Z}$, $z_i = z_i(t)$, $y_i = y_i(t)$, and $u = u(z,t) \in H^1(\mathbb{T})$. Here we denote by $H^1(\mathbb{T})$ a functional space defined by the closure of the space of infinitely differentiable functions $\varphi : \mathbb{T} \to \mathbb{R}$ in $H^1(0,L)$. In addition, $(H^1(\mathbb{T}))^*$ is the dual space of $H^1(\mathbb{T})$. For simplicity, we denote an operator associated with the right-hand sides of the first, second, and third equations of (1.4) by $\mathcal{L}(U,\kappa) = {}^t(\mathcal{L}_0^z(U), \mathcal{L}_0^y(U,\kappa_1),$

$\ldots, \mathcal{L}_N^z(U),\ \mathcal{L}_N^y(U,\kappa_1), \mathcal{L}^u(U,\kappa_2))$, where $U = {}^t(z_0, y_0, \ldots, z_N, y_N, u) \in X^1 \equiv \mathbb{R}^{2N+2} \times H^1(\mathbb{T})$, $\kappa = (\kappa_1, \kappa_2) = (\mu - \mu_0, s - s_0)$, and $\mathcal{L}_i^z, \mathcal{L}_i^y$ $(i = 0, \ldots, N)$, $\mathcal{L}^u$ are given by

$$\begin{aligned}
&\mathcal{L}_i^z(U) = y_i,\\
&\mathcal{L}_i^y(U,\kappa_1) = -(\mu_0+\kappa_1)(y_i - c_0) + \gamma(u(z_i+\rho)) - \gamma(u(z_i-\rho)),\\
&\mathcal{L}^u(U,\kappa_2) = \frac{\partial^2 u}{\partial z^2} - c_0\frac{\partial u}{\partial z} - \alpha u + \sum_{i=0}^{N} \overline{f}(z - z_i, s_0 + \kappa_2).
\end{aligned}$$

For a function $\varphi$ defined on $(-\infty,\infty)$, $\overline{\varphi}$ denotes an $L$-periodic function given by

$$\overline{\varphi}(z) = \chi(z-nL)\varphi(z-nL), \quad \left(n - \frac{1}{2}\right)L < z < \left(n + \frac{1}{2}\right)L, \quad n \in \mathbb{Z}, \tag{1.5}$$

where $\chi \in C^\infty(\mathbb{R})$ is a cut-off function defined by

$$\chi(z) = \begin{cases} 1, & |z| < \dfrac{L}{4},\\ 0, & |z| > \dfrac{3L}{8}. \end{cases}$$

We see $\mathcal{L}(\cdot,\kappa) : X^1 \to X \equiv \mathbb{R}^{2N+2} \times (H^1(\mathbb{T}))^*$.

In the same way as in (1.2), we can show that there exists a traveling wave solution of (1.4) such that $z_i = iL/(N+1)$, which corresponds to the homogeneous state observed in the experiment of [13]. According to the result obtained in [13], the traveling wave solution is expected to be unstable for a large number $N$. In order to prove the instability of the traveling wave solution, it is necessary to find an unstable eigenvalue. However the linearized eigenvalue problem is too difficult to analyze theoretically even in the case of $N = 1$. From this point of view, we need to reduce (1.4) and derive a new system.

Let $l$ be a position of the 0th particle and denote the distance between the $i$-th and the $(i+1)$-th particles by $h_i$ for $i = 0, \ldots, N-1$. Since we consider a one-dimensional circuit, $h_N$ is defined by the distance between the $N$-th and the 0th particles. Since $L$ is the length of the circuit, $h_N$ should be equal to $L - \sum_{i=0}^{N-1} h_i$. Thus we define a relative position of the $i$-th particle by $\overline{z}_i = \overline{z}_i(\mathbf{h}) \equiv \sum_{j=0}^{i-1} h_j$, where $\mathbf{h} = {}^t(h_0, \ldots, h_{N-1})$. We put $\overline{z}_0 = 0$. Note that the position of the $i$-th particle is given by $z_i = \overline{z}_i + l$. Here we assume that one particle is sufficiently separated from any others. In other words, $h_i$ is assumed to satisfy $h_i > h^*$ for any $i = 0, \ldots, N$, where $h^*$ is assumed to be sufficiently large. Define $H(h^*) = \{\mathbf{h} = (h_0, \ldots, h_{N-1}) \in \mathbb{R}^N \mid h^* < h_j \ (j = 0, \ldots, N)\}$, where $h_N = L - \sum_{j=0}^{N-1} h_j$. Set $\varepsilon = e^{-\beta_1 h^*}$, where a constant $\beta_1 > 0$ will be given in the next section. Note that if $h^*$ is large, $\varepsilon$ is small.

Due to the interactions described by (1.4), the positions of the particles will be varied. Then the position $l$ and the distance $h_i$ are regarded as functions in time, denoted by $l(t)$ and $h_i(t)$. In the statement of our result, we use the several notation. Put $\mathbf{r} = {}^t(r_0, \ldots, r_N)$, $P(z, l, \mathbf{h}) = \sum_{i=0}^N [(\overline{z}_i + l)e_{2i+1} + \overline{p}(z - \overline{z}_i - l)e_{2N+3}]$, $\xi(z, l, \mathbf{h}, \mathbf{r}) = \sum_{i=0}^N r_i[e_{2i+2} + \overline{\psi}(z - \overline{z}_i - l)e_{2N+3}]$, and $S(z, l, \mathbf{h}, \mathbf{r}) = P(z, l, \mathbf{h}) + \xi(z, l, \mathbf{h}, \mathbf{r})$, where $e_i$ is the unit vector in $\mathbb{R}^{2N+3}$ given by $e_i = {}^t(0, \ldots, 0, \overset{\overset{i}{\vee}}{1}, 0, \ldots, 0)$. Note that $\overline{\varphi}$ is identically equal to 0 in a neighborhood of $z = (n + 1/2)L$ for any $n \in \mathbb{Z}$. Then, if $\varphi \in C^1(\mathbb{R})$, $\overline{\varphi}$ is supposed to belong to $C^1(\mathbb{R})$. Therefore we see $S(z, l, \mathbf{h}, \mathbf{r}) \in \mathbb{R}^{2N+2} \times H^2(\mathbb{T})$.

Denote the solution of (1.4) by $U(t)$. Under (A1)–(A4) and more conditions for parameters, we can show the following statement;

**Claim.** *Assume that $h^*$ is sufficiently large and $r^*$ is sufficieltly small. If the initial data $U_0$ of (1.4) can be approximated by $S(z, l_0, \mathbf{h}_0, \mathbf{r}_0)$ for some $l_0 \in \mathbb{R}, \mathbf{h}_0 \in H(h^*), \mathbf{r}_0 \in \mathbb{R}^{N+1}$ with $|\mathbf{r}_0| < r^*$, then there exist $l(t), \mathbf{h}(t), \mathbf{r}(t)$ such that $U(z,t)$ can be approximated by $S(z, l(t), \mathbf{h}(t), \mathbf{r}(t))$ as long as $\mathbf{h} \in H(h^*), |\mathbf{r}| < r^*$ and the parameters in (1.4) are near the bifurcation point.*

This statement says that the behavior of $U(t)$ can be determined by $l(t), \mathbf{h}(t), \mathbf{r}(t)$. Actually, these functions yields to a finite-dimensional dynamical system. Our claim is similar to one as in [3], where the authors considered the interaction between two pulses with very small velocity near a bifurcation point in a reaction-diffusion system. In that article each pulse can be approximated by a stationary solution, which implies that all eigenfunctions $\Phi, \Psi, \Phi^*, \Psi^*$ are expected to be symmetric, that is, odd or even functions. Here $\Phi^*$ and $\Psi^*$ will be introduced in the next section. As a result, several calculations in the reduction process become easier than in our case. In our previous works [6], [7], we formally derive a reduced model.

In order to prove the claim above, we have to study a linearized operator of $\mathcal{L}(U,0)$ at $U = P(z, 0, \mathbf{h})$, denoted by $L(\mathbf{h}) = \mathcal{L}'(P(z,0,\mathbf{h}),0)$. Our aim in this article is to study all eigenvalues and estimate the resolvent operator for $L(\mathbf{h})$ in $\Sigma$. Our main results are as follows.

**Theorem 1.** *Under (A1)-(A4) and for sufficiently large $h^*$, one has*

$$\|(\lambda I - L(\mathbf{h}))^{-1} U\|_{X^1} \leq C \left(1 + \frac{1}{|\lambda|}\right)^2 \|U\|_X \tag{1.6}$$

*uniformly in $\mathbf{h} \in H(h^*)$, $\lambda \in \Sigma$ with $C\varepsilon^{1/8} \leq |\lambda|$ and $U \in X$. In addtition, there exists $M > 0$ independent of $L, h^*$ such that $L(\mathbf{h})$ satisfies*

$$\|(\lambda I - L(\mathbf{h}))^{-1}\| \leq \frac{C}{|\lambda|}, \quad \|(\lambda I - L(\mathbf{h}))^{-1} U\|_{X^1} \leq C\|U\|_X \tag{1.7}$$

*uniformly in $\mathbf{h} \in H(h^*)$, $\lambda \in \Sigma$ with $|\lambda| \geq M$ and $U \in X$.*

**Theorem 2.** *Under the same condtions as in Theorem 1, there exists a positive constant $C$ independent of $h^*$ such that the operator $L(\mathbf{h})$ has $2(N+1)$ semi-simple eigenvalues $\lambda_j(\mathbf{h})$ for $j = 1, \ldots, 2(N+1)$ with $|\lambda_j(\mathbf{h})| \leq C\varepsilon^{1/8}$ in $\mathbf{h} \in H(h^*)$. Multiple eigenvalues are repeated as many times as their multiplicity indicates. Other spectra of $L(\mathbf{h})$ are in $\mathbb{C} \backslash \Sigma$.*

The rest of this article is essentially devoted to the proofs of the theorems above. First of all, we estimate the reslvent operators of $\mathcal{L}'_\infty(0, 0, p; c_0, \mu_0, s_0)$ and $L(\mathbf{h}) = \mathcal{L}'(P(z,0,\mathbf{h}),0)$. In the proofs of Lemmas 1–3, the assumptions (A2)–(A4) are unnecessary. We will define a projection maps $Q(\mathbf{h})$ by the Dunford's integral such as

$$Q(\mathbf{h}) \equiv \frac{1}{2\pi i} \int_\Gamma (\lambda I - L(\mathbf{h}))^{-1} d\lambda$$

and $E(\mathbf{h}) \equiv Q(\mathbf{h})X$, where $I : X \to X$ is the identity map and $\Gamma = \{\lambda \in \mathbb{C} \mid |\lambda| = \sigma\} \subset \Sigma$ for sufficiently small $\sigma > 0$ independent of $h^*, L$. We will show that $E(\mathbf{h})$ is a $2(N+1)$-dimensional space which consists of the eigenfunctions of $L(\mathbf{h})$.

We introduce several notation. Let $B$ be a Banach space and denote the norm equipped with $B$ by $\|\cdot\|_B$. Let $B^*$ be the dual space of $B$, and $\langle\cdot,\cdot\rangle_B$ be a paring between $B$ and $B^*$. The norm of $B^*$ can be given by $\|\varphi^*\|_{B^*} = \sup_{\varphi\in B, \|\varphi\|_B=1} |\langle\varphi^*, \varphi\rangle_B|$ for $\varphi^* \in B^*$. In particular, we denote a pairing between $X = \mathbb{R}^{2N+2} \times (H^1(\mathbb{T}))^*$ and $X^1 = \mathbb{R}^{2N+2} \times H^1(\mathbb{T})$ by $\langle\cdot,\cdot\rangle$ for simplicity. Note that $X = (X^1)^*$, where we think of $X^1$ as a usual Hilbert space. In addtion, we represent a usual operator norm by $\|\cdot\|$ simply.

The usual Lebesgue space $L^2(\mathbb{R})$ can be embedded into $(H^1(\mathbb{R}))^*$ by the inclusion map $I$. More precisely, $Iu \in (H^1(\mathbb{R}))^*$ can be defined by

$$\langle Iu, \phi\rangle_{H^1(\mathbb{R})} \equiv \langle u, \phi\rangle_{L^2(\mathbb{R})}$$

for $\phi \in H^1(\mathbb{R})$. Since $L^2(\mathbb{R}) \subset (H^1(\mathbb{R}))^*$ in this sense, we will represent $Iu$ by $u$ in this article for simplicity. Similarly, $L^2(\mathbb{T}) \subset (H^1(\mathbb{R}))^*$ holds true.

## 2 Spectra and resolvent estimate of $L(\mathbf{h})$

In this section, we first study spectra and estimate a resolvent operator of $L(\mathbf{h})$. We first stufy $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)$.

**Lemma 1.** *Under (A1), there is $\theta > \pi/2$ such that for $\lambda \in \Sigma$, $\lambda I - \mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)$ is a Fredholm operator, where $\theta$ appears in the definition of $\Sigma$ in the assumption (A2).*

*Proof.* The operator $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)$ is decomposed into $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0) = A + B$, where $A$ and $B$ are defined by

$$AU = \begin{pmatrix} 0 \\ 0 \\ u'' - c_0u' - \alpha u \end{pmatrix},$$

$$BU = \begin{pmatrix} Y \\ -\mu_0 Y + \gamma'(p(\rho))(p'(\rho)Z + u(\rho)) - \gamma'(p(-\rho))(p'(-\rho)Z + u(-\rho)) \\ -Z[s_0\delta_{-\rho} - (s_0-1)\delta_0 - \delta_\rho] \end{pmatrix}$$

for $U = {}^t(Z, Y, u)$ with the domains $D(A) = \mathbb{R}^2 \times H^1(\mathbb{R})$ and $D(B) = \mathbb{R}^2 \times ((H^1(\mathbb{R}))^* \cap C_b(\mathbb{R}))$, where $C_b(\mathbb{R})$ denotes a set of bounded and continuous functions on $\mathbb{R}$. The Sobolev's imbedding theorem implies $D(A) \subset D(B)$.

Put $\lambda \in \Sigma$. We first prove that $\lambda I - A$ is Fredholm. It is easy to see that $D(A)$ is dense in $\mathbb{R}^2 \times (H^1(\mathbb{R}))^*$ and $\lambda I - A$ is a closed operator. It is well-known that for any $f \in (H^1(\mathbb{R}))^*$, there is a unique solution $u \in H^1(\mathbb{R})$ of $\lambda u - (u'' - c_0u' - \alpha u) = f$ and there is $C > 0$ independent of $\lambda, u, f$ such as

$$\|u\|_{(H^1(\mathbb{R}))^*} \le \frac{C}{|\lambda+\alpha|}\|f\|_{(H^1(\mathbb{R}))^*}, \quad \|u\|_{H^1(\mathbb{R})} \le C\|f\|_{(H^1(\mathbb{R}))^*}, \tag{2.1}$$

from which we see that $R(\lambda I - A)$ is closed. In the case of $\lambda = 0$, the dimension of the null space of $\lambda I - A$ is 2 and the codimension of the range of $\lambda I - A$ is 2 while $\lambda$ belongs to the resolvent set of $A$. Therefore $\lambda I - A$ is Fredholm and its Fredholm index is equal to 0.

We next prove that $B$ is $(\lambda I - A)$-compact. Suppose that for $\Phi_n = {}^t(Z_n, Y_n, u_n) \in D(A)$, there is a constant $C > 0$ independent of $n$ such that

$$\|\Phi_n\|_{\mathbb{R}^2\times(H^1(\mathbb{R}))^*} + \|(\lambda I - A)\Phi_n\|_{\mathbb{R}^2\times(H^1(\mathbb{R}))^*} \le C.$$

Set $f_n = \lambda u_n - (u_n'' - c_0 u_n' - \alpha u_n) \in (H^1(\mathbb{R}))^*$. From the inequality above and (2.1), we see that $|Z_n|$, $|Y_n|$, and $\|u_n\|_{H^1(\mathbb{R})}$ are uniformly bounded in $n$. Then there are a subsequence $(Z_{n_k}, Y_{n_k}, u_{n_k})$ and $(Z, Y, u) \in \mathbb{R}^2 \times ((H^1(\mathbb{R}))^* \cap C_b(\mathbb{R})) = D(B)$ such as $(Z_{n_k}, Y_{n_k}, u_{n_k}) \to (Z, Y, u)$ in $\mathbb{R}^2 \times ((H^1(\mathbb{R}))^* \cap C_b(\mathbb{R}))$ as $k \to \infty$ due to the Sobolev's imbedding theorem. Therefore $B\Phi_{n_k}$ converges to $B\Phi$. Hence we complete the proof of Lemma 1. □

**Remark 1.** *For $f \in (H^1(\mathbb{T}))^*$, let $u \in H^1(\mathbb{T})$ be a solution of $\lambda u - (u'' - c_0 u' - \alpha u) = f$. It is well-known that the following resolvent estimates hold true;*

$$\|u\|_{(H^1(\mathbb{T}))^*} \le \frac{C}{|\lambda + \alpha|}\|f\|_{(H^1(\mathbb{T}))^*}, \quad \|u\|_{H^1(\mathbb{T})} \le C\|f\|_{(H^1(\mathbb{T}))^*} \tag{2.2}$$

*for $\lambda \in \Sigma$ and a constant $C$ independent of $L, \lambda, u, f$.*

In the following, $\theta$ denotes a constant given in Lemma 1 throughout this article.

**Lemma 2.** *Under (A1), there is $M > 0$ such that any $\lambda \in \Sigma$ with $|\lambda| \ge M$ belongs to the resolvent set of $\mathcal{L}'_\infty(0, 0, p; c_0, \mu_0, s_0)$. In addtition, the following resolvent estimate holds true for any $\lambda \in \Sigma$ with $|\lambda| \ge M$;*

$$\|(\lambda I - \mathcal{L}'_\infty(0, 0, p; c_0, \mu_0, s_0))^{-1}\| \le \frac{C}{|\lambda|}. \tag{2.3}$$

*Proof.* Suppose that there is a solution of $(\lambda I - \mathcal{L}'_\infty(0, 0, p; c_0, \mu_0, s_0))U = W$ for $W \in \mathbb{R}^2 \times (H^1(\mathbb{R}))^*$. Put $U = {}^t(Z, Y, u)$. From (2.1), we have

$$|Z| \le \frac{1}{|\lambda|}(|Y| + \|W\|_{\mathbb{R}^2\times(H^1(\mathbb{R}))^*}), \qquad |Y| \le \frac{C}{|\lambda + \mu_0|}(|Z| + \|u\|_{H^1(\mathbb{R})} + \|W\|_{\mathbb{R}^2\times(H^1(\mathbb{R}))^*}),$$

$$\|u\|_{H^1(\mathbb{R})} \le C(\|W\|_{\mathbb{R}^2\times(H^1(\mathbb{R}))^*} + |Z|), \quad \|u\|_{(H^1(\mathbb{R}))^*} \le \frac{C}{|\lambda + \alpha|}(\|W\|_{\mathbb{R}^2\times(H^1(\mathbb{R}))^*} + |Z|).$$

From these inequalities, there is a constant $C$ independent of $\lambda, W, M$ such as

$$\|U\|_{\mathbb{R}^2\times H^1(\mathbb{R})} \le C\|W\|_{\mathbb{R}^2\times(H^1(\mathbb{R}))^*}, \quad \|U\|_{\mathbb{R}^2\times(H^1(\mathbb{R}))^*} \le \frac{C}{|\lambda|}\|W\|_{\mathbb{R}^2\times(H^1(\mathbb{R}))^*}$$

because of $|\lambda| \ge M$ for sufficiently large $M$. The latter inequality above implies $N(\lambda I - \mathcal{L}'_\infty(0, 0, p; c_0, \mu_0, s_0)) = \{0\}$ for any $\lambda \in \Sigma$ with $|\lambda| \ge M$. As described in the proof of Lemma 1, the Fredholm index of $\lambda I - \mathcal{L}'_\infty(0, 0, p; c_0, \mu_0, s_0)$ is 0 so that we have $R(\lambda I - \mathcal{L}'_\infty(0, 0, p; c_0, \mu_0, s_0)) = \mathbb{R}^2 \times (H^1(\mathbb{R}))^*$. Hence any $\lambda \in \Sigma$ with $|\lambda| \ge M$ belongs to the resolvent set of $\mathcal{L}'_\infty(0, 0, p; c_0, \mu_0, s_0)$ because of the iniqualities above. □

Lemma 1 implies that $\lambda \in \Sigma$ either belongs to the resolvent set or is a spectrum with a finite multiplicity, which is called an "eigenvalue", of $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)$. Lemma 2 implies that all spectra in $\Sigma$ of $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)$ should satisfy $|\lambda| \le M$. The assumptions (A1)–(A4) imply that $\lambda \in \Sigma$ with $|\lambda| \le M$ belongs to the resolvent set of $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)$ except for $\lambda = 0$, and $\lambda = 0$ is an eigenvalue with a finite multiplicity. From these facts, a decomposition theorem described in [9] implies the following resolvent estimates;

$$\|(\lambda I - \mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0))^{-1}\| \le \frac{C}{|\lambda|^2} \tag{2.4}$$

in $\lambda \in \Sigma$ with $|\lambda| \le M$ and $\lambda \ne 0$.

We will define $E(\mathbf{h})$ by using the generalized eigenfunctions of the linearized operator $\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)$ and its adjoint operator $(\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0))^*$, which can be explicitly given by

$$(\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0))^*\Phi^* \equiv \begin{pmatrix} \gamma'(p(\rho))p'(\rho)Y^* - \gamma'(p(-\rho))p'(-\rho)Y^* - [s_0\phi^*(-\rho) - (s_0-1)\phi^*(0) - \phi^*(\rho)] \\ Z^* - \mu_0 Y^* \\ \dfrac{\partial^2\phi^*}{\partial z^2} + c_0\dfrac{\partial\phi^*}{\partial z} - \alpha\phi^* + (\gamma'(p(\rho))\delta_\rho - \gamma'(p(\rho))\delta_{-\rho})Y^* \end{pmatrix}$$

for $\Phi^* = (Z^*, Y^*, \phi^*) \in \mathbb{R}^2 \times H^1(\mathbb{R})$. Using this expression, we have

$$\langle \mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0)\Phi, \Phi^*\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})} = \langle (\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0))^*\Phi^*, \Phi\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})}$$

for $\Phi, \Phi^* \in \mathbb{R}^2 \times H^1(\mathbb{R})$. Since $\mathbb{R}^2 \times H^1(\mathbb{R})$ is a Hilbert space, it is reflective. Hence we think of $(\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0))^*$ as an operator in $\mathbb{R}^2 \times (H^1(\mathbb{R}))^*$ with a domain $\mathbb{R}^2 \times H^1(\mathbb{R})$.

Let $\Phi^*, \Psi^*$ be the generalized eigenfunctions of $(\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0))^*$ which satisfy $(\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0))^*\Phi^* = 0$ and $(\mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0))^*\Psi^* = -\Phi^*$, respectively. Under (A1)–(A4), $\Phi^*, \Psi^*$ do exist thanks to the Fredholm's alternative. Then, $\Phi^*$ and $\Psi^*$ are uniquely determined by orthogonal conditions such as $\langle \Psi, \Phi^*\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})} = 1$ and $\langle \Psi, \Psi^*\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})} = 0$. Then we note that $\langle \Phi, \Psi^*\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})} = 1$ and $\langle \Phi, \Phi^*\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})} = 0$ automatically hold. Put $\Phi^* = {}^t(Z^*_\Phi, Y^*_\Phi, \phi^*)$ and $\Psi^* = {}^t(Z^*_\Psi, Y^*_\Psi, \psi^*)$. Note that $\phi, \psi, \phi^*, \psi^*$ decay exponentially as $|z| \to \infty$. More precisely, there is $\beta_1 > 0$ such that $|\varphi(z)| \le Ce^{-\beta_1|z|}$ and $|\varphi'(z)| \le Ce^{-\beta_1|z|}$ in $z \in \mathbb{R}$ for $\varphi = p, \phi, \psi, \phi^*, \psi^*$.

To construct $E(\mathbf{h})$, we introduce $L(\mathbf{h}) = \mathcal{L}'(P(z,0,\mathbf{h}),0)$, where $\mathcal{L}'(P(z,l,\mathbf{h}),\kappa)$ denotes the linearized operator of $\mathcal{L}(U,\kappa)$ at $U = P(z,l,\mathbf{h})$. First we prove that $\lambda I - L(\mathbf{h})$ is Fredholm in $X$ for $\lambda \in \Sigma$. The proof of Lemma 3 is almost the same as that of Lemam 1. So we omit the detail.

**Lemma 3.** *Under (A1), $\lambda I - L(\mathbf{h})$ is a Fredholm operator in $\lambda \in \Sigma$.*

Now we are in a position to prove Theorem 1. Here we introduce

$$\begin{aligned} [\hat{\Phi}_k(\mathbf{h})](z) &= -e_{2k+1} + \overline{\phi}(z - \overline{z}_k)e_{2N+3}, \\ [\hat{\Psi}_k(\mathbf{h})](z) &= e_{2k+2} + \overline{\psi}(z - \overline{z}_k)e_{2N+3}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} [\hat{\Phi}^*_k(\mathbf{h})](z) &= Z^*_\Phi e_{2k+1} + Y^*_\Phi e_{2k+2} + \overline{\phi^*}(z - \overline{z}_k)e_{2N+3}, \\ [\hat{\Psi}^*_k(\mathbf{h})](z) &= Z^*_\Psi e_{2k+1} + Y^*_\Psi e_{2k+2} + \overline{\psi^*}(z - \overline{z}_k)e_{2N+3}. \end{aligned} \tag{2.6}$$

Throughout the following proof, $C$ denotes general constants independent of $\lambda, L, h^*$ and $\mathbf{h}$.

*Proof.* We first suppose that there exists a solution $U$ of $(\lambda I - L(\mathbf{h}))U = \hat{\Phi}_k(\mathbf{h})$ for $\lambda \in \Sigma$. Lemma 4. We set $U(z) = {}^t(z_0, y_0, \ldots, z_N, y_N, u(z))$ and

$$U_i(z) = z_i e_{2i+1} + y_i e_{2i+2} + \overline{\zeta}(\iota z) u(z + \overline{z}_i) e_{2N+3},$$
$$\tilde{U}(z) = \left(1 - \sum_{i=0}^{N} \overline{\zeta}(\iota(z - \overline{z}_i))\right) u(z) e_{2N+3},$$

where $\zeta$ is a cut-off function satisfying

$$\zeta(z) = \begin{cases} 1, & |z| \leq 1, \\ 0, & |z| \geq 2, \end{cases}$$

and $\iota = 4/h^*$. Then we have $U(z) = \sum_{i=0}^{N} U_i(z - \overline{z}_i) + \tilde{U}(z)$.

We first estimate $\tilde{U}$. Put $\tilde{u}(z) = (1 - \sum_{i=0}^{N} \overline{\zeta}(\iota(z - \overline{z}_i)))u(z)$. Then $\tilde{u}$ satisfies

$$\lambda \tilde{u} - (\tilde{u}'' - c_0 \tilde{u}' - \alpha \tilde{u}) = F(z),$$

where $F \equiv F(z)$ is defined by

$$F(z) \equiv \sum_{i=0}^{N} [\iota^2 \overline{\zeta}''(\iota(z - \overline{z}_i))u(z) + 2\iota \overline{\zeta}'(\iota(z - \overline{z}_i))u'(z) - c_0 \iota \overline{\zeta}'(\iota(z - \overline{z}_i))u(z)]$$
$$+ \left(1 - \sum_{i=0}^{N} \overline{\zeta}(\iota(z - \overline{z}_i))\right) \overline{\phi}(z - \overline{z}_k).$$

By (2.2), we have

$$\|\tilde{u}\|_{H^1(\mathbb{T})} \leq C\|F\|_{(H^1(\mathbb{T}))^*} \leq C(\varepsilon^{1/4} + \iota\|u\|_{H^1(\mathbb{T}\setminus B)}).$$

Here we set $B = \cup_{i=0}^{N} B_{1/\iota}(\overline{z}_i)$, where $B_r(z)$ denotes a ball with the radius $r$ and the center $z$.

Next we estimate $U_i$. Put $u_i(z) = \overline{\zeta}(\iota z)u(z + \overline{z}_i)$. For $i = k$, we have

$$\begin{aligned} &\lambda z_k - y_k = -1, \\ &\lambda y_k - (-\mu_0 y_k + \gamma'(p(\rho))(p'(\rho)z_k + u_k(\rho)) - \gamma'(p(-\rho))(p'(-\rho)z_k + u_k(-\rho))) = F_1, \quad (2.7) \\ &\lambda u_k - (u_k'' - c_0 u_k' - \alpha u_k - z_k[s_0\delta_{-\rho} - (s_0 - 1)\delta_0 - \delta_\rho]) = \overline{\zeta}(\iota z)\overline{\phi}(z) + F_2(z), \end{aligned}$$

where $F_1, F_2(z)$ are defined by

$$F_1 \equiv \gamma' \left(\sum_{i=0}^{N} \overline{p}(\overline{z}_i - \overline{z}_k + \rho)\right) \left(\sum_{i=0}^{N} \overline{p}'(\overline{z}_i - \overline{z}_k + \rho) z_i + u_k(\rho)\right)$$
$$- \gamma' \left(\sum_{i=0}^{N} \overline{p}(\overline{z}_i - \overline{z}_k - \rho)\right) \left(\sum_{i=0}^{N} \overline{p}'(\overline{z}_i - \overline{z}_k - \rho) z_i + u_k(-\rho)\right)$$
$$- (\gamma'(p(\rho))(p'(\rho)z_k + u_k(\rho)) - \gamma'(p(-\rho))(p'(-\rho)z_k + u_k(-\rho))),$$

$$F_2(z) \equiv -\iota^2\overline{\zeta}''(\iota z)u(z+\overline{z}_k) - 2\iota\overline{\zeta}'(\iota z)u'(z+\overline{z}_k) + c_0\iota\overline{\zeta}'(\iota z)u(z+\overline{z}_k).$$

Here we naturally extend the domains of $u_k(z)$ and $F_2(z)$ to a whole line without loss of generality. Then (2.7) is represented by $(\lambda I - \mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0))\tilde{U}_k = Y_1 + Y_2$, where $\tilde{U}_k = {}^t(z_k, y_k, u_k)$, $Y_1 = {}^t(-1, 0, \zeta(\iota z)\phi(z))$, $Y_2 = {}^t(0, F_1, F_2(z))$. Here we have

$$\begin{aligned}
&\left|\langle Y_1, \Phi^*\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})}\right| \le C\sqrt{\varepsilon}, \quad \left|\langle Y_1, \Psi^*\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})} - 1\right| \le C\sqrt{\varepsilon},\\
&\left|\langle Y_2, \Phi^*\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})}\right| \le C(\varepsilon^{1/4}\iota\|u\|_{H^1(\mathbb{T}\backslash B)} + \varepsilon\|U\|_{X^1}),\\
&\left|\langle Y_2, \Psi^*\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})}\right| \le C(\varepsilon^{1/4}\iota\|u\|_{H^1(\mathbb{T}\backslash B)} + \varepsilon\|U\|_{X^1}).
\end{aligned}$$

Thanks to the assumption, we can expand $\tilde{U}_k$ as $\tilde{U}_k = a\Phi + b\Psi + \tilde{U}_k^\perp$ with

$$\begin{aligned}
&a = \left\langle \tilde{U}_k, \Psi^*\right\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})}, \quad b = \left\langle \tilde{U}_k, \Phi^*\right\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})},\\
&\left\langle \tilde{U}_k^\perp, \Phi^*\right\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})} = \left\langle \tilde{U}_k^\perp, \Psi^*\right\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})} = 0.
\end{aligned}$$

Substituting it into the above equation, we have

$$\begin{aligned}
|\lambda a - 1| &\le C\left(1 + \frac{1}{|\lambda|}\right)(\sqrt{\varepsilon} + \varepsilon^{1/4}\iota\|u\|_{H^1(\mathbb{T}\backslash B)} + \varepsilon\|U\|_{X^1}),\\
|\lambda b| &\le C(\sqrt{\varepsilon} + \varepsilon^{1/4}\iota\|u\|_{H^1(\mathbb{T}\backslash B)} + \varepsilon\|U\|_{X^1})
\end{aligned}$$

if $\lambda \ne 0$, and

$$\begin{aligned}
&(\lambda I - \mathcal{L}'_\infty(0,0,p;c_0,\mu_0,s_0))\tilde{U}_k^\perp\\
&= Y_1 + Y_2 - \langle Y_1 + Y_2, \Psi^*\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})}\Phi - \langle Y_1 + Y_2, \Phi^*\rangle_{\mathbb{R}^2\times H^1(\mathbb{R})}\Psi.
\end{aligned}$$

From the assumption, Lemma 1 and a similar result of Theorem 6.17 in [9], we have

$$\|\tilde{U}_k^\perp\|_{\mathbb{R}^2\times(H^1(\mathbb{R}))^*} \le C(\varepsilon^{1/4} + \iota\|u\|_{H^1(\mathbb{T}\backslash B)} + \varepsilon\|U\|_{X^1}).$$

From the inequality above and (2.1), we also have an estimate such as

$$\|\tilde{U}_k^\perp\|_{\mathbb{R}^2\times H^1(\mathbb{R})} \le C(\varepsilon^{1/4} + \iota\|u\|_{H^1(\mathbb{T}\backslash B)} + \varepsilon\|U\|_{X^1}).$$

Hence we have

$$\begin{aligned}
&\|\tilde{U}_k - \frac{1}{\lambda}\Phi\|_{\mathbb{R}^2\times H^1(\mathbb{R})}\\
&\le C\left(1 + \frac{\varepsilon^{1/4}}{|\lambda|}\left(1 + \frac{1}{|\lambda|}\right)\right)(\varepsilon^{1/4} + \iota\|u\|_{H^1(\mathbb{T}\backslash B)}) + C\left(1 + \frac{1}{|\lambda|}\right)^2\varepsilon\|U\|_{X^1}.
\end{aligned}$$

By the similar argument, we have

$$\begin{aligned}
&\|\tilde{U}_i\|_{\mathbb{R}^2\times H^1(\mathbb{R})}\\
&\le C\left(1 + \frac{\varepsilon^{1/4}}{|\lambda|}\left(1 + \frac{1}{|\lambda|}\right)\right)(\varepsilon^{1/4} + \iota\|u\|_{H^1(\mathbb{T}\backslash B)}) + C\left(1 + \frac{1}{|\lambda|}\right)^2\varepsilon\|U\|_{X^1}.
\end{aligned}$$

for any $i \neq k$. Hence we have

$$\|U - \frac{1}{\lambda}\hat{\Phi}_k(\mathbf{h})\|_{X^1}$$
$$\leq C\left(1+\frac{\varepsilon^{1/4}}{|\lambda|}\left(1+\frac{1}{|\lambda|}\right)\right)(\varepsilon^{1/4}+\iota\|u\|_{H^1(\mathbb{T}\setminus B)}) + C\left(1+\frac{1}{|\lambda|}\right)^2 \varepsilon\|U\|_{X^1}.$$

Here we have

$$\|U - \frac{1}{\lambda}\hat{\Phi}_k(\mathbf{h})\|_{X^1} \geq \|u\|_{H^1(\mathbb{T}\setminus B)} - \frac{C}{|\lambda|}\varepsilon^{1/4}.$$

Since $\iota$ is sufficiently small, we have

$$\|u\|_{H^1(\mathbb{T}\setminus B)} \leq C\varepsilon^{1/8} + C\left(1+\frac{1}{|\lambda|}\right)^2 \varepsilon\|U\|_{X^1},$$
$$\|U - \frac{1}{\lambda}\hat{\Phi}_k(\mathbf{h})\|_{X^1} \leq C\varepsilon^{1/8} + C\left(1+\frac{1}{|\lambda|}\right)^2 \varepsilon\|U\|_{X^1}$$

for any $\lambda$ with $C\varepsilon^{1/8} \leq |\lambda|$. Therefore we have

$$\left\|U - \frac{1}{\lambda}\hat{\Phi}_k(\mathbf{h})\right\|_{X^1} \leq C\varepsilon^{1/8}. \tag{2.8}$$

By the similar argument, the solution $U$ of $(\lambda I - L(\mathbf{h}))U = \hat{\Psi}_k(\mathbf{h})$ satisfies

$$\left\|U - \left(\frac{1}{\lambda}\hat{\Psi}_k(\mathbf{h}) - \frac{1}{\lambda^2}\hat{\Phi}_k(\mathbf{h})\right)\right\|_{X^1} \leq C\varepsilon^{1/8}. \tag{2.9}$$

for $\lambda \in \Sigma$ with $C\varepsilon^{1/16} \leq |\lambda|$.

Next we prove that any $\lambda \in \Sigma$ with $C\varepsilon^{1/8} \leq |\lambda|$ belongs to the resolvent set of $L(\mathbf{h})$. We assume that there is a solution $U$ of $(\lambda I - L(\mathbf{h}))U = 0$. Setting $\|U\|_{X^1} = 1$ and carrying out the same calculation above, we have $\|U\|_{X^1} \leq C\varepsilon^{1/4}$, which is a contradiction. Since $\lambda$ is not an eigenvalue of $L(\mathbf{h})$, $\lambda$ belongs to the resolvent set of $L(\mathbf{h})$ due to Lemma 3. In other words, if $\lambda \in \Sigma$ is an eigenvalue of $L(\mathbf{h})$, then $|\lambda| \leq C\varepsilon^{1/8}$.

Finally we have a resolvent estimate. For $W \in X$, $U$ is supposed to be a solution of $(\lambda I - L(\mathbf{h}))U = W$. Setting $\alpha_k = \left\langle W, \hat{\Phi}_k^*(\mathbf{h})\right\rangle$ and $\beta_k = \left\langle W, \hat{\Psi}_k^*(\mathbf{h})\right\rangle$, we obtain

$$\left\|U - \sum_{k=0}^{N}\left(\left(-\frac{\alpha_k}{\lambda^2}+\frac{\beta_k}{\lambda}\right)\hat{\Phi}_k(\mathbf{h}) + \frac{\alpha_k}{\lambda}\hat{\Psi}_k(\mathbf{h})\right)\right\|_{X^1} \leq C\|W\|_X \tag{2.10}$$

for $\lambda \in \Sigma$ with $C\varepsilon^{1/8} \leq |\lambda|$ by the same argument as above. From this inequality, we can obtain a resolvent estimate (1.6).

The remainder of the statement of Theorem 1 can be proved in a similar way to the proof of Lemma 2. So we omit the detail. □

In the proof above, we have already shown that eigenvalues in $\Sigma$ are smaller than $C\varepsilon^{1/8}$ if exist. Therefore we only have to prove that such eigenvalues do exist and the total number is exactly equivalent to $2(N+1)$. The folloiwing lemma implies these facts. Note that the dimension of $E(\mathbf{h})$ corresponds to the number of eigenvalues in a domain surrounded by the closed curve $\Gamma$.

**Lemma 4.** *Set* $\Phi_k(\mathbf{h}) = Q(\mathbf{h})\hat{\Phi}_k(\mathbf{h})$, *and* $\Psi_k(\mathbf{h}) = Q(\mathbf{h})\hat{\Psi}_k(\mathbf{h})$. *Then, it holds true that* $\{\Phi_k(\mathbf{h}), \Psi_k(\mathbf{h})\}_{k=0}^N$ *is a basis of* $E(\mathbf{h})$, *and* $\|\Phi_k(\mathbf{h}) - \hat{\Phi}_i(\mathbf{h})\|_{X^1} \le C\varepsilon^{1/4}$ *and* $\|\Psi_k(\mathbf{h}) - \hat{\Psi}_i(\mathbf{h})\|_{X^1} \le C\varepsilon^{1/4}$.

*Proof.* By the similar argument to the proof above, there is a constant $C$ independent of $\lambda, \sigma, L, h^*, \mathbf{h}$ such as $\|(\lambda I - L(\mathbf{h}))^{-1}\hat{\Phi}_k(\mathbf{h}) - \hat{\Phi}_k/\lambda\|_{X^1} \le C\varepsilon^{1/4}$ for $|\lambda| = \sigma$. Then we have

$$\|Q(\mathbf{h})\hat{\Phi}_k(\mathbf{h}) - \hat{\Phi}_k(\mathbf{h})\|_{X^1} = \left\| \frac{1}{2\pi i}\int_\Gamma \left( (\lambda I - L(\mathbf{h}))^{-1}\hat{\Phi}_k(\mathbf{h}) - \frac{1}{\lambda}\hat{\Phi}_k(\mathbf{h}) \right) d\lambda \right\|_{X^1} \le C\varepsilon^{1/4}.$$

Similarly, we have

$$\begin{aligned} &\|Q(\mathbf{h})\hat{\Psi}_k(\mathbf{h}) - \hat{\Psi}_k(\mathbf{h})\|_{X^1} \\ &= \left\| \frac{1}{2\pi i}\int_\Gamma \left( (\lambda I - L(\mathbf{h}))^{-1}\hat{\Psi}_k(\mathbf{h}) - \left( \frac{1}{\lambda}\hat{\Psi}_k(\mathbf{h}) - \frac{1}{\lambda^2}\hat{\Phi}_k(\mathbf{h}) \right) \right) d\lambda \right\|_{X^1} \le C\varepsilon^{1/4}. \end{aligned}$$

This result implies that the dimension of $E(\mathbf{h})$ is larger than or equal to $2(N+1)$.

Finally, we prove that it is equal to $2(N+1)$. Set $W \in E(\mathbf{h})$ and assume

$$\left\langle W, \hat{\Phi}_i^*(\mathbf{h}) \right\rangle = \left\langle W, \hat{\Psi}_i^*(\mathbf{h}) \right\rangle = 0$$

for all $i = 0, \dots, N$. Let $U$ be a solution of $(\lambda I - L(\mathbf{h}))U = W$ for $\lambda \in \Sigma$ with $|\lambda| = \sigma$. By (2.10) and the inequalities above, we have $\|U\|_{X^1} \le C\|W\|_X$ for a constant $C$ independent of $\lambda, \sigma, L, h^*, \mathbf{h}$. Hence we have

$$\|W\|_X = \|Q(\mathbf{h})W\|_X \le C\sigma\|W\|_X,$$

which implies $W = 0$ because $\sigma$ is small. Therefore the dimension of $E(\mathbf{h})$ is equal to $2(N+1)$. □

In the end of this article, we state the properties of generalized eigenfunctions of $L(\mathbf{h})^*$, an operator from $X^1 \to X$ defined by an equality

$$\langle L(\mathbf{h})\Phi, \Phi^* \rangle = \langle L(\mathbf{h})^*\Phi^*, \Phi \rangle$$

for any $\Phi, \Phi^* \in X^1$, which is associated with the adjoint operator of $L(\mathbf{h})$. Thanks to the Fredholm's alternative, $L(\mathbf{h})^*$ also has $2(N+1)$ eigenvalues close to 0. We denote the associated eigenspace of $L(\mathbf{h})^*$ by $E(\mathbf{h})^* \equiv Q(\mathbf{h})^*X$, where

$$Q(\mathbf{h})^* \equiv \frac{1}{2\pi i}\int_\Gamma (\lambda I - L(\mathbf{h})^*)^{-1} d\lambda.$$

By the similar argument of Lemma 4, we can show that $E(\mathbf{h})^*$ is a $2(N+1)$-dimensional space and give a basis of $E(\mathbf{h})^*$, denoted by $\{\tilde{\Phi}_i^*(\mathbf{h}), \tilde{\Psi}_i^*(\mathbf{h})\}_{i=0}^N$ as in the following lemma.

**Lemma 5.** *Set* $\tilde{\Phi}_k^*(\mathbf{h}) = Q(\mathbf{h})^*\hat{\Phi}_k^*(\mathbf{h})$ *and* $\tilde{\Psi}_k^*(\mathbf{h}) = Q(\mathbf{h})^*\hat{\Phi}_k^*(\mathbf{h})$. *Then, it holds true that* $\{\tilde{\Phi}_k^*(\mathbf{h}), \tilde{\Psi}_k^*(\mathbf{h})\}_{k=0}^N$ *is a basis of* $E(\mathbf{h})^*$, *and* $\|\tilde{\Phi}_k^*(\mathbf{h}) - \hat{\Phi}_k^*(\mathbf{h})\|_{X^1} \le C\varepsilon^{1/4}$ *and* $\|\tilde{\Psi}_k^*(\mathbf{h}) - \hat{\Psi}_k^*(\mathbf{h})\|_{X^1} \le C\varepsilon^{1/4}$.

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