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GENERALIZED ALTERNATIVE THEOREMS BASED ON SET-RELATIONS

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ABSTRACT. This paper contains generalized alternative theorems via a set scalarization method based on set-relations. We introduce semidefinite system and its dual case as a supplementary result.

1. INTRODUCTION

Gordan's theorem was appeared in 1873 [2]. This theorem focuses on the geometry of finitely many vectors and the origin. This paper is originally motivated by a generalized Gordan's theorem for vector-valued functions [4] in 1986. Vector-valued functions are able to be replaced with matrices under a convex assumption. This was interestingly extended to the case of set-valued maps [7] in 1999 and [9] in 2000. They found that set-valued maps can be used in the form of alternative theorem with a similar convexity.

Their theorems based on some separation theorems require linear scalarizaions such as bilinear forms. In 2005, Nishizawa, Onodsuka, and Tanaka gave generalization forms [8] by using sublinear scalarizations for vectors inspired by ones introduced in [3]. Sublinear separation enables to eliminate assumptions of theorems related to linearity and convexity.

Kuwano, Tanaka, and Yamada established sublinear scalarizations for sets [6] in 2009 by use of set-relations [5]. These functions are composed of six infimum types and six supremum types. The number of the functions for set is much larger than that for vectors which consist of just four types. This comes from the complexity of ways where we consider a set preceding another.

In this paper, we show 12 alternative theorems by using the scalarizing functions for sets, considering the set-relations between an image and a reference set. If the reference set is the origin, our theorems will be the result of the previous research. This reference set gives flexibility and let problems free themselves from the origin.

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2. Preliminaries

Unless otherwise specified, we let X be a nonempty set, Y a real ordered topological vector space with $\leq_C$ induced by a convex solid (i.e., there exists nonempty interior) cone C as follows: $x \leq_C y$ if $y - x \in C$ for $x, y \in Y$, and F a set-valued map from X to 2^Y.

Definition 2.1 (Kuroiwa, Tanaka, Ha (1997), [5]). Let $A, B \in 2^Y \backslash \{\emptyset\}$.

(i) $A \leq_C^{(1)} B \overset{\text{def}}{\Longleftrightarrow} A \subset \bigcap_{b \in B}(b - C) \Leftrightarrow B \subset \bigcap_{a \in A}(a + C)$;
(ii) $A \leq_C^{(2)} B \overset{\text{def}}{\Longleftrightarrow} A \cap \bigcap_{b \in B}(b - C) \neq \emptyset$;
(iii) $A \leq_C^{(3)} B \overset{\text{def}}{\Longleftrightarrow} B \subset (A + C)$;
(iv) $A \leq_C^{(4)} B \overset{\text{def}}{\Longleftrightarrow} \bigcap_{a \in A}(a + C) \cap B \neq \emptyset$;
(v) $A \leq_C^{(5)} B \overset{\text{def}}{\Longleftrightarrow} A \subset (B - C)$;
(vi) $A \leq_C^{(6)} B \overset{\text{def}}{\Longleftrightarrow} A \cap (B - C) \neq \emptyset \Leftrightarrow (A + C) \cap B \neq \emptyset$.

Definition 2.2 (Kuwano, Tanaka, Yamada, (2009), [6]). Let $A, B \in 2^Y \backslash \{\emptyset\}$ and $k \in \text{int}C$. For each $j = 1, ..., 6$, scalarizing functions $I_{k,B}^{(j)}$ and $S_{k,B}^{(j)}$ from $2^Y \backslash \{\emptyset\}$ to $\mathbb{R} \cup \{\pm\infty\}$ are defined by $I_{k,B}^{(j)}(A) := \inf\{t \in \mathbb{R} \mid A \leq_C^{(j)} (tk + B)\}$, $S_{k,B}^{(j)}(A) := \sup\{t \in \mathbb{R} \mid (tk + B) \leq_C^{(j)} A\}$.

3. Generalized Alternative Theorems

3.1. Infimum Types.

Theorem 3.1 (Infimum type (1)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. If F is compact-valued on X and V is compact, then exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $F(x) \leq_{\text{int}C}^{(1)} V$;
(ii) there exists $k \in \text{int}C$ such that $(I_{k,V}^{(1)} \circ F)(x) \geq 0$ for all $x \in X$.

Proof. [(i)$\Rightarrow$not (ii)] Fix $k \in \text{int}C$. Assume that (i) is consistent. Then, there exists $x \in X$ such that $y \in \bigcap_{v \in V}(v - \text{int}C)$ for all $y \in F(x)$. Since $\bigcap_{v \in V}(v - \text{int}C)$ is open, there exists $\lambda_y > 0$ such that $y + \lambda_y k \in \bigcap_{v \in V}(v - \text{int}C)$. Therefore, $y \in S_y := \bigcap_{v \in -\lambda_y k + V}(v - \text{int}C)$. This implies $\{S_y\}_{y \in F(x)}$ is a cover of $F(x)$. Since $F(x)$ is compact, there exists $\{y_1, \ldots, y_n\} \subset F(x)$ such that $F(x) \subset \bigcup_{i \in I} S_{y_i}$ where $I := \{1, \ldots, n\}$.

Taking $\bar{\lambda} := \min\{\lambda_{y_1}, \ldots, \lambda_{y_n}\} > 0$, then $F(x) \subset \bigcup_{i \in I} \bigcap_{v \in -\lambda_{y_i} k + V}(v - \text{int}C) = \bigcap_{v \in -\bar{\lambda} k + V}(v - \text{int}C) \subset \bigcap_{v \in -\bar{\lambda} k + V}(v - C)$. Thus, $\inf\{t \in \mathbb{R} \mid F(x) \subset \bigcap_{v \in tk + V}(v - C)\} \leq -\bar{\lambda} < 0$.

[Not (i)$\Rightarrow$(ii)] Assume that (i) is not consistent. Then, for all $x \in X$, there exists $\bar{y} \in F(x)$ such that $\bar{y} \notin \bigcap_{v \in V}(v - \text{int}C)$. Since $\bigcap_{v \in \lambda k + V}(v - C) \subset \bigcap_{v \in V}(v - \text{int}C)$ for all

$k \in \text{int}C$ and $\lambda < 0$, then $\bar{y} \notin \bigcap_{v\in\lambda k+V}(v-C)$. Thus, $\inf\{t \in \mathbb{R} \mid F(x) \subset \bigcap_{v\in tk+V}(v-C)\}$ is non-negative. □

Remark 3.1. We let $S := \bigcap_{v\in V}(v-C)$. Then $S + (-C) \subset S$ so that for all $x \in X$, $y \in F(x)$, and $k \in \text{int}C$, there exists $t_y > 0$ such that $y \in \bigcap_{v\in t_y k+V}(v-C)$. Thus, there exists $\bar{t} \geq 0$ such that $(I^{(1)}_{k,V} \circ F)(x) = \inf\{t \in \mathbb{R} \mid F(x) \subset \bigcap_{v\in tk+V}(v-C)\} \leq \bar{t} < +\infty$.

Theorem 3.2 (Infimum type (2)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. If V is compact, then exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $F(x) \leq^{(2)}_{\text{int}C} V$;

(ii) there exists $k \in \text{int}C$ such that $(I^{(2)}_{k,V} \circ F)(x) \geq 0$ for all $x \in X$.

Proof. [(i)⇒not (ii)] Fix $k \in \text{int}C$. Assume that (i) is consistent. Then, there exist $x \in X$ and $\bar{y} \in F(x)$ such that $\bar{y} \in \bigcap_{v\in V}(v - \text{int}C)$. Since $\bigcap_{v\in V}(v - \text{int}C)$ is open, there exists $\bar{\lambda} > 0$ such that $\bar{y}+\bar{\lambda}k \in \bigcap_{v\in V}(v-\text{int}C) \subset \bigcap_{v\in V}(v-C)$. Therefore, $\bar{y} \in \bigcap_{v\in -\bar{\lambda}k+V}(v-C)$. Then, $F(x) \cap \bigcap_{v\in -\bar{\lambda}k+V}(v-C) \neq \emptyset$. Thus, $\inf\{t \in \mathbb{R} \mid F(x) \cap \bigcap_{v\in tk+V}(v-C) \neq \emptyset\} \leq -\bar{\lambda} < 0$.

[Not (i)⇒(ii)] Assume that (i) is not consistent. Then, $y \notin \bigcap_{v\in V}(v - \text{int}C)$ for all $x \in X$ and $y \in F(x)$. That means, $y \notin \bigcap_{v\in\lambda k+V}(v-C)$ for all $k \in \text{int}C$ and $\lambda < 0$. Thus, $\inf\{t \in \mathbb{R} \mid F(x) \cap \bigcap_{v\in tk+V}(v-C) \neq \emptyset\}$ is non-negative. □

Remark 3.2. We let $S := \bigcap_{v\in V}(v-C)$ like Remark 3.1. For all $x \in X$, $y \in F(x)$, and $k \in \text{int}C$, there exists $\bar{t} > 0$ such that $y \in \bigcap_{v\in \bar{t}k+V}(v-C)$. Thus, $(I^{(2)}_{k,V} \circ F)(x) = \inf\{t \in \mathbb{R} \mid F(x) \cap \bigcap_{v\in tk+V}(v-C) \neq \emptyset\} \leq \bar{t} < +\infty$.

Theorem 3.3 (Infimum type (3)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. If V is compact, then exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $F(x) \leq^{(3)}_{\text{int}C} V$;

(ii) there exists $k \in \text{int}C$ such that $(I^{(3)}_{k,V} \circ F)(x) \geq 0$ for all $x \in X$.

Proof. [(i)⇒not (ii)] Fix $k \in \text{int}C$. Assume that (i) is consistent. Then, there exists $x \in X$ such that $v \in F(x) + \text{int}C$ for all $v \in V$. Since $F(x) + \text{int}C$ is open, there exists $\lambda_v > 0$ such that $v - \lambda_v k \in F(x) + \text{int}C$. We let $S_v := \lambda_v k + F(x) + \text{int}C$ then, $\{S_v\}_{v\in V}$ is a cover of V. Since V is compact, there exists $\{v_1, \ldots, v_n\} \subset V$ such that $V \subset \bigcup_{i\in I} S_{v_i}$ where $I := \{1, \ldots, n\}$.

Taking $\bar{\lambda} := \min\{\lambda_{v_1}, \ldots, \lambda_{v_n}\} > 0$, then $V \subset \bigcup_{i \in I}(\lambda_{v_i}k + F(x) + \text{int}C) = \bar{\lambda}k + F(x) + \text{int}C \subset \bar{\lambda}k + F(x) + C$. Thus, $\inf\{t \in \mathbb{R} \mid (tk + V) \subset (F(x) + C)\} \leq -\bar{\lambda} < 0$.

[Not (i)$\Rightarrow$(ii)] Assume that (i) is not consistent. Then, for all $x \in X$, there exists $\bar{v} \in V$ such that $\bar{v} \notin F(x) + \text{int}C$. Since $-\lambda k + F(x) + C \subset F(x) + \text{int}C$ for all $k \in \text{int}C$ and $\lambda < 0$, then $\bar{v} \notin -\lambda k + F(x) + C$. Thus, $\text{int}\{t \in \mathbb{R} \mid (tk + V) \subset (F(x) + C)\}$ is non-negative. □

Remark 3.3. We let $S := F(x) + C$. Then $S + C \subset S$ so that for all $v \in V$ and $k \in \text{int}C$, there exists $\bar{t} > 0$ such that $\bar{t}k + v \in F(x) + C$. Thus, $(I^{(3)}_{k,V} \circ F)(x) = \inf\{t \in \mathbb{R} \mid (tk + V) \subset (F(x) + C)\} \leq \bar{t} < +\infty$.

Theorem 3.4 (Infimum type (4)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. If F is compact-valued on X, then exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $F(x) \leq^{(4)}_{\text{int}C} V$;

(ii) there exists $k \in \text{int}C$ such that $(I^{(4)}_{k,V} \circ F)(x) \geq 0$ for all $x \in X$.

Proof. [(i)$\Rightarrow$not (ii)] Fix $k \in \text{int}C$. Assume that (i) is consistent. Then, there exist $x \in X$ and $\bar{v} \in V$ such that $\bar{v} \in \bigcap_{y \in F(x)}(y + \text{int}C)$. Since $\bigcap_{y \in F(x)}(y + \text{int}C)$ is open, there exists $\bar{\lambda} > 0$ such that $-\bar{\lambda}k + \bar{v} \in \bigcap_{y \in F(x)}(y + \text{int}C) \subset \bigcap_{y \in F(x)}(y + C)$.
Thus, $\inf\{t \in \mathbb{R} \mid (tk + V) \cap \bigcap_{y \in F(x)}(y + C) \neq \emptyset\} \leq -\bar{\lambda} < 0$.

[Not (i)$\Rightarrow$(ii)] Assume that (i) is not consistent. Then, $v \notin \bigcap_{y \in F(x)}(y + \text{int}C)$ for all $x \in X$ and $v \in V$. Since $\bigcap_{y \in F(x)}(-\lambda k + y + C) \subset \bigcap_{y \in F(x)}(y + \text{int}C)$ for all $k \in \text{int}C$ and $\lambda < 0$, then $v \notin \bigcap_{y \in F(x)}(-\lambda k + y + C)$.
Thus, $\inf\{t \in \mathbb{R} \mid (tk + V) \cap \bigcap_{y \in F(x)}(y + C) \neq \emptyset\}$ is non-negative. □

Remark 3.4. We let $S := \bigcap_{y \in F(x)}(y + C)$. Then $S + C \subset S$ so that for all $v \in V$ and $k \in \text{int}C$, there exists $t_v > 0$ such that $t_v k + v \in \bigcap_{y \in F(x)}(y + C)$. Then, there exists $\bar{t} \geq 0$ such that $(\bar{t}k + V) \cap \bigcap_{y \in F(x)}(y + C) \neq \emptyset$. Thus, $(I^{(4)}_{k,V} \circ F)(x) = \inf\{t \in \mathbb{R} \mid (tk + V) \cap \bigcap_{y \in F(x)}(y + C) \neq \emptyset\} \leq \bar{t} < +\infty$.

Theorem 3.5 (Infimum type (5)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. If F is compact-valued on X, then exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $F(x) \leq^{(5)}_{\text{int}C} V$;

(ii) there exists $k \in \text{int}C$ such that $(I^{(5)}_{k,V} \circ F)(x) \geq 0$ for all $x \in X$.

Proof. [(i)$\Rightarrow$not (ii)] Fix $k \in \mathrm{int}C$. Assume that (i) is consistent. Then, there exists $x \in X$ such that $y \in V - \mathrm{int}C$ for all $y \in F(x)$. Since $V - \mathrm{int}C$ is open, there exists $\lambda_y > 0$ such that $y + \lambda_y k \in V - \mathrm{int}C$. We let $S_y := -\lambda_y k + V - \mathrm{int}C$, then $\{S_y\}_{y \in F(x)}$ is a cover of $F(x)$. Since $F(x)$ is compact, there exists $\{y_1, \ldots, y_n\} \subset F(x)$ such that $F(x) \subset \bigcup_{i \in I} S_{y_i}$ where $I := \{1, \ldots, n\}$.

Taking $\bar{\lambda} := \min\{\lambda_{y_1}, \ldots, \lambda_{y_n}\} > 0$, then $F(x) \subset \bigcup_{i \in I}(-\lambda_{y_i} + V - \mathrm{int}C) = -\bar{\lambda}k + V - \mathrm{int}C \subset -\bar{\lambda}k + V - C$. Thus, $\inf\{t \in \mathbb{R} \mid F(x) \subset tk + V - C\} \leq -\bar{\lambda} < 0$.

[Not (i)$\Rightarrow$(ii)] Assume that (i) is not consistent. Then, for all $x \in X$, there exists $\bar{y} \in F(x)$ such that $\bar{y} \notin V - \mathrm{int}C$. Since $\lambda k + V - C \subset V - \mathrm{int}C$ for all $k \in \mathrm{int}C$ and $\lambda < 0$, then $\bar{y} \notin \lambda k + V - C$. Thus, $\inf\{t \in \mathbb{R} \mid F(x) \subset tk + V - C\}$ is non-negative. □

Remark 3.5. We let $S := V - C$. Then $S + (-C) \subset S$ so that for all $x \in X$, $y \in F(x)$, and $k \in \mathrm{int}C$, there exists $t_y > 0$ such that $y \in t_y k + V - C$. Then, there exists $\bar{t} \geq 0$ such that $F(x) \subset \bar{t}k + V - C$. Thus, $(I^{(5)}_{k,V} \circ F)(x) = \inf\{t \in \mathbb{R} \mid F(x) \subset (tk + V - C)\} \leq \bar{t} < +\infty$.

Theorem 3.6 (Infimum type (6)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. Then, exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $F(x) \leq^{(6)}_{\mathrm{int}C} V$;

(ii) there exists $k \in \mathrm{int}C$ such that $(I^{(6)}_{k,V} \circ F)(x) \geq 0$ for all $x \in X$.

Proof. [(i)$\Rightarrow$not (ii)] Fix $k \in \mathrm{int}C$. Assume that (i) is consistent. Then, there exist $x \in X$ and $\bar{y} \in F(x)$ such that $\bar{y} \in V - \mathrm{int}C$. Thus, there exists $\bar{\lambda} > 0$ such that $\bar{y} + \bar{\lambda}k \in V - \mathrm{int}C \subset V - C$. Then, $F(x) \cap (-\bar{\lambda}k + V - C) \neq \emptyset$. Therefore, $\inf\{t \in \mathbb{R} \mid F(x) \cap (tk + V - C) \neq \emptyset\} \leq -\bar{\lambda} < 0$.

[Not (i)$\Rightarrow$(ii)] Assume that (ii) is not consistent. Then, $y \in V - \mathrm{int}C$ for all $x \in X$ and $y \in F(x)$. Since $\lambda k + V - C \subset V - \mathrm{int}C$ for all $k \in \mathrm{int}C$ and $\lambda < 0$, then $y \notin \lambda k + V - C$. Thus, $\inf\{t \in \mathbb{R} \mid F(x) \cap (tk + V - C) \neq \emptyset\}$ is non-negative. □

Remark 3.6. We let $S := V - C$ like Remark 3.5. For all $x \in X$, $y \in F(x)$, and $k \in \mathrm{int}C$, there exists $t_y > 0$ such that $y \in t_y k + V - C$. Then, there exists $\bar{t} \geq 0$ such that $F(x) \cap (\bar{t}k + V - C) \neq \emptyset$. Thus, $(I^{(6)}_{k,V} \circ F)(x) = \inf\{t \in \mathbb{R} \mid F(x) \cap (tk + V - C) \neq \emptyset\} \leq \bar{t} < +\infty$.

3.2. **Supremum Types.** We obtain supremum types conversely by replacing an ordering cone C in Infimum types with $-C$. In addition, just like six remarks described above, $(S^{(j)}_{k,V} \circ F)(x) > -\infty$ for all $x \in X$ and $j = 1, \ldots, 6$.

Theorem 3.7 (Supremum type (1)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. If F is compact-valued on X and V is compact, then exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $V \leq^{(1)}_{\mathrm{int}C} F(x)$;

(ii) there exists $k \in \mathrm{int}C$ such that $(S^{(1)}_{k,V} \circ F)(x) \leq 0$ for all $x \in X$.

Theorem 3.8 (Supremum type (2)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. If V is compact, then exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $V \leq^{(2)}_{\mathrm{int}C} F(x)$;

(ii) there exists $k \in \mathrm{int}C$ such that $(S^{(2)}_{k,V} \circ F)(x) \leq 0$ for all $x \in X$.

Theorem 3.9 (Supremum type (3)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. If V is compact, then exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $V \leq^{(3)}_{\mathrm{int}C} F(x)$;

(ii) there exists $k \in \mathrm{int}C$ such that $(S^{(3)}_{k,V} \circ F)(x) \leq 0$ for all $x \in X$.

Theorem 3.10 (Supremum type (4)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. If F is compact-valued on X, then exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $V \leq^{(4)}_{\mathrm{int}C} F(x)$;

(ii) there exists $k \in \mathrm{int}C$ such that $(S^{(4)}_{k,V} \circ F)(x) \leq 0$ for all $x \in X$.

Theorem 3.11 (Supremum type (5)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. If F is compact-valued on X, then exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $V \leq^{(5)}_{\mathrm{int}C} F(x)$;

(ii) there exists $k \in \mathrm{int}C$ such that $(S^{(5)}_{k,V} \circ F)(x) \leq 0$ for all $x \in X$.

Theorem 3.12 (Supremum type (6)). Let X be a nonempty set, Y a topological vector space, C a convex solid cone in Y, $F : X \to 2^Y$ a set-valued map, and V a nonempty subset in Y. Then, exactly one of the following two systems is consistent;

(i) there exists $x \in X$ such that $V \leq^{(6)}_{\mathrm{int}C} F(x)$;

(ii) there exists $k \in \text{int}C$ such that $(S^{(6)}_{k,V} \circ F)(x) \leq 0$ for all $x \in X$.

4. Positive Semi-Definite System

Let S^n be the set of $n \times n$ symmetric matrices, S^n_+ the set of $n \times n$ positive semidefinite symmetric matrices, S^n_{++} the set of $n \times n$ positive definite symmetric matrices, $F : X \to 2^{S^n}$ a set-valued map, and V a nonempty subset in S^n. We denote the trace of $A \in S^n$ by $\text{tr}(A) := \sum_{i=1}^n a_{ii}$. Some notions are from pp.9–10 of [1].

Theorem 4.1. Exactly one of the following two systems is consistent;

(i) There exists $x \in X$ such that $F(x) \cap (V - S^n_{++}) \neq \emptyset$;

(ii) There exists $K \in S^n_{++}$ such that
$\inf\{t \in \mathbb{R} \mid F(x) \cap (tK + V - S^n_+) \neq \emptyset\} \geq 0$ for all $x \in X$.

This theorem is simply verified by $S^n_{++} = \text{int}S^n_+$.

Corollary 4.1. Let $f : \mathbb{R}^m \to S^n$ be a function. Then, exactly one of the following two systems is consistent;

(i) There exists $x \in \mathbb{R}^m$ such that $f(x) \in -S^n_{++}$;

(ii) There exists $B \in S^n_{++}$ such that
$\inf\{t \in \mathbb{R} \mid f(x) \in tB - S^n_+\} \geq 0$ for all $x \in \mathbb{R}^m$.

Let $A \in S^n$ and $B \in S^n_{++}$ be given. Consider the following semidefinite optimization problem and its dual problem:

$$\text{(SDP)} \quad \inf\{\text{tr}(-AX) \mid \text{tr}((-B)X) = -1,\ X \in S^n_+\}.$$

$$\text{(SDD)} \quad \sup\{-t \mid -tB + S = -A,\ S \in S^n_+\}.$$

Notice that $\inf\{t \in \mathbb{R} \mid A \in tB - S^n_+\} = -\sup\{-t \in \mathbb{R} \mid -tB + S = -A,\ S \in S^n_+\}$ for all $A \in S^n$ and $B \in S^n_{++}$, and we can calculate $\inf\{t \in \mathbb{R} \mid A \in tB - S^n_+\}$ with Matlab as a semidefinite optimization problem.

Lemma 4.1. Problems (SDP) and (SDD) are strictly feasible, that is, there exists $\tilde{X} \in S^n_{++}$ such that $\text{tr}((-B)\tilde{X}) = -1$, and there exist $\tilde{t} \in \mathbb{R}$ and $\tilde{S} \in S^n_{++}$ such that $-\tilde{t}B + \tilde{S} = -A$.

Proof. Let $X := \sum_{i=1}^n u_i q_i q_i^{\mathrm{T}} \in S^n_{++}$ where u_i are positive and $\{q_i\}_{i=1,\ldots,n}$ is a base of $\mathbb{R}^n$. Then, $\text{tr}(BX) = \text{tr}(B(\sum_{i=1}^n u_i q_i q_i^{\mathrm{T}})) = \sum_{i=1}^n u_i \text{tr}(B q_i q_i^{\mathrm{T}}) = \sum_{i=1}^n u_i q_i^{\mathrm{T}} B q_i > 0$. Thus, $\tilde{X} := (1/\text{tr}(BX))X \in S^n_{++}$ satisfies $\text{tr}(B\tilde{X}) = 1$.

Let $B \in S^n_{++}$. If $A = \theta_{S^n}$, we have $-tB + \tilde{S} = \theta_{S^n}$ by letting $\tilde{S} := tB$ for all $t > 0$. Now we assume that $A \neq \theta_{S^n}$. Since S^n_{++} is open, there exists an open neighborhood

$U \in N(\theta_{S^n})$ such that $B + U \subset S^n_{++}$. There exists $\tilde{t} > 0$ such that $(-1/\tilde{t})A \in U$. Therefore, we can find $\tilde{S} := \tilde{t}B - A = \tilde{t}(B - (1/\tilde{t})A) \in S^n_{++}$. □

Lemma 4.2. The optimal values of (SDP) and (SDD) coincide and their optimal solutions are nonempty.

Proof. Since (SDP) and (SDD) are strictly feasible, it follows from Corollary 2.1 in p.28 of [10] that the optimal values of (SDP) and (SDD) coincide and their optimal solutions are nonempty. □

Remark 4.1. Let $A \in S^n$ and $B \in S^n_{++}$ be given. Then, $h_{S^n_+}(A; B) := \min\{t \in \mathbb{R} \mid A \in tB - S^n_+\} = \max\{\mathrm{tr}(AX) \mid \mathrm{tr}(BX) = 1,\ X \in S^n_+\}$.

From Corollary 4.1, Lemma 4.2, and Remark 4.1, we obtain the following corollary.

Corollary 4.2. Let $f : \mathbb{R}^m \to S^n$ be a map. Then, exactly one of the following two systems is consistent;

(i) There exists $x \in \mathbb{R}^m$ such that $f(x) \in -S^n_{++}$;

(ii) There exists $B \in S^n_{++}$ such that
$\max\{\mathrm{tr}(f(x)X) \mid \mathrm{tr}(BX) = 1, X \in S^n_+\} \geq 0$ for all $x \in \mathbb{R}^m$.

This implies a test giving a criterion for satisfaction or dissatisfaction of Slater's condition.

5. Conclusion

We have extended some concepts of alternative theorems to set-valued case. Our theorems give a natural generalization to ones having been researched since before. In addition, these result can be used when we are to make sure a set precedes another. Note that there are several ways of reduction of scalarizing functions to ease calculation of their values.

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