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Gradient estimates for mean curvature flow with Neumann boundary conditions

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1. Mean curvature flow

We consider the following parabolic partial differential equation:

$$\text{(MCF)} \qquad \begin{cases} \dfrac{\partial_t u}{\sqrt{1+|du|^2}} = \operatorname{div}\left(\dfrac{du}{\sqrt{1+|du|^2}}\right) + \boldsymbol{F}(x,u,t)\cdot\boldsymbol{n}, & x\in\Omega,\ t>0,\\ du\cdot\nu\big|_{\partial\Omega} = 0, & t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain in an n-dimensional euclidean space with a smooth boundary, ν is the outer unit normal vector field on $\partial\Omega$, $u = u(x,t) : \Omega \times [0,\infty) \to \mathbb{R}$ is an unknown function, $d := (\partial_{x_1},\ldots,\partial_{x_n})$ and div are the usual gradient and divergence on Ω, $\boldsymbol{n} := \frac{1}{\sqrt{1+|du|^2}}(-du,1)$ is the unit normal vector field on the graph of u, $\boldsymbol{F} : \Omega\times\mathbb{R}\times[0,\infty) \to \mathbb{R}^{n+1}$ is a given transport term, and $u_0 = u_0(x) : \Omega \to \mathbb{R}$ is a given initial data. If u is a solution of (MCF), then the graph of u satisfies mean curvature flow with the transport $\boldsymbol{F}$ and with the Neumann boundary condition, namely

$$(1) \qquad \begin{cases} \boldsymbol{V} = \boldsymbol{H} + \boldsymbol{F}^{\perp}, & \text{on } \Gamma_t,\ t>0,\\ \Gamma_t \perp \partial(\Omega\times\mathbb{R}), & t>0,\\ \Gamma_t := \{(x,u(x,t)) : x\in\Omega\}, & t>0, \end{cases}$$

where $\boldsymbol{V} := \frac{\partial_t u}{\sqrt{1+|du|^2}}\boldsymbol{n}$ is a normal velocity vector of Γ_t, $\boldsymbol{H} := \operatorname{div}(\frac{du}{\sqrt{1+|du|^2}})\boldsymbol{n}$ is the mean curvature vector of Γ_t, and $\boldsymbol{F}^{\perp} = (\boldsymbol{F}\cdot\boldsymbol{n})\boldsymbol{n}$ is the transport term(see Figure. 1).

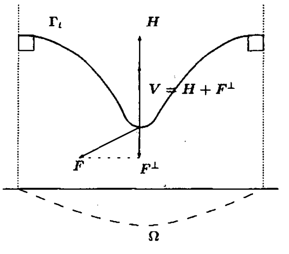

FIGURE 1. Mean curvature flow with the transport $\boldsymbol{F}$ and with the Neumann boundary condition.

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We consider an up-to-boundary regularity problem for (MCF). In particular, we study a relationship between up-to-boundary estimates for $v := \sqrt{1+|du|^2}$ and the regularity for the transport term $\boldsymbol{F}$. From the point of partial differential equations, (MCF) is a nonlinear degenerate parabolic equation of non-divergence type, hence regularity for solutions of (MCF) is not clear. When the gradient of solutions is bounded, then the Schauder estimates for (MCF) is applicable thus existence of classical solutions of (MCF) can be deduced. It is also interesting to obtain the gradient estimates under reasonable conditions of the transport $\boldsymbol{F}$. I will later discuss blowup/scaling arguments for (MCF) and deduce the reasonable condition of the transport $\boldsymbol{F}$.

From the point of geometry, especially geometric measure theory, the volume measure $d\mu_t = \sqrt{1+|du|^2}\,dx$ is important to study regularity for Γ_t. The boundedness of $|du|$ implies rectifiability of μ_t (besides integrality in our setting). Rectifiability is a basic regularity concept for geometric measure theory, hence we need to show the gradient estimates for (MCF).

Interior gradient estimates for (MCF) under $\boldsymbol{F} \equiv 0$ ware studied by Ecker-Huisken [6] when the initial surface is C^1, and by Colding-Minicozzi II [3] when u_0 is bounded. Takasao [14] studied the interior gradient estimates for (MCF) when u_0 is C^1 and the transport $\boldsymbol{F}$ is bounded in time and space variables. Huisken [7] studied (MCF) with the Neumann boundary condition and without the transport $\boldsymbol{F}$. He showed the existence of a classical solution of (MCF) under $\boldsymbol{F} \equiv 0$. To show the existence of the solution, it is important to derive up-to-boundary a priori gradient estimates of (MCF). Huisken showed the gradient estimates when the initial data u_0 is $C^{2,\alpha}$ up to boundary and $\partial\Omega$ is of class $C^{2,\alpha}$. Stahl [12] also considered the gradient estimates of (MCF) without the transport and obtained some blow-up criterion of the classical solution of (MCF) under $\boldsymbol{F} \equiv 0$. Our arguments are similar to Ecker's or Takasao's work [4, 14]. Ecker deduced the interior gradient estimates for (MCF) under $\boldsymbol{F} \equiv 0$ via Huisken's monotonicity formula [8]. Takasao obtained an interior monotonicity formula with the bounded transport $\boldsymbol{F}$. In our setting, we need to derive a boundary monotonicity formula for (MCF). From this point, Buckland [1] obtained the boundary monotonicity formula for (MCF) without the transport.

2. **Main results**

For fixed $T_0 > 0$ and $p, q \geq 1$, we define

$$\|\boldsymbol{F}\|_{L^q_t L^p_x(\Gamma_t)} := \left(\int_0^{T_0} \left(\int_{\Gamma_t} |\boldsymbol{F}(x,z,t)|^p \, d\mathscr{H}^n(x,z) \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} \tag{2}$$

where $\mathscr{H}^n$ is the n-dimensional Hausdorff measure on $\overline{\Omega}$. We give the following assumption to the transport.

Assumption 1. There exist $p, q \geq 1$ such that $\frac{n}{p} + \frac{2}{q} < 1$ and $\|\boldsymbol{F}\|_{L^q_t L^p_x(\Gamma_t)} < \infty$.

Remark 2. By the Meyer-Ziemer inequality (cf. [16, p. 266, Theorem 5.12.4]),

$$\int_{\Gamma_t} |\boldsymbol{F}(x,z,t)|^p \, d\mathscr{H}^n(x,z) \leq C_1 \|\boldsymbol{F}(\cdot,t)\|^p_{W^{1,p}_{(x,z)}(\Omega\times\mathbb{R})}$$

for some positive constant $C_1 > 0$. Therefore if $\frac{n}{p} + \frac{2}{q} < 1$ and $\boldsymbol{F} \in L^q(0, T_0 : W^{1,p}_{(x,z)}(\Omega \times \mathbb{R}))$, then Assumption 1 is fulfilled.

We give the boundary gradient estimates for (MCF) with the transports under Assumption 1.

Theorem 3 (A priori estimates for the gradient). *Let $\Omega \subset \mathbb{R}^n$ be a convex domain. For $u_0 \in W^{1,\infty}(\Omega)$ and for the transport $\boldsymbol{F}$ under Assumption 1, let u be a classical solution of* (MCF). *Then there exists $T > 0$ depending only on n, p, q and $\|\boldsymbol{F}\|_{L^q_t L^p_x(\Gamma_t)}$ such that*

$$\sup_{0<t<T,\ x\in\overline{\Omega}} \sqrt{1+|du(x,t)|^2} \le 2(1+\|du_0\|_\infty^2). \tag{3}$$

With the aid of the Leray-Schauder fixed point theorem, we can show the time local existence of a classical solution of (MCF). For $0 < \alpha \le 1$, define

$$\|\boldsymbol{F}\|_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times\mathbb{R}\times(0,T_0))} := \sup_{((x,z),t),((y,w),s)\in(\Omega\times\mathbb{R})\times(0,T_0)} \frac{|\boldsymbol{F}(x,z,t)-\boldsymbol{F}(y,w,s)|}{|x-y|^\alpha+|z-w|^\alpha+|t-s|^{\alpha/2}}.$$

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Let $u_0 \in W^{1,\infty}(\Omega)$ satisfy compatibility conditions $du_0 \cdot \nu|_{\partial\Omega} = 0$ and let the transport $\boldsymbol{F}$ satisfy $\|\boldsymbol{F}\|_{C^{\alpha,\frac{\alpha}{2}}(\Omega\times\mathbb{R}\times(0,T_0))} < \infty$ for some $0 < \alpha \le 1$. Then, there exists a unique classical solution u of* (MCF) *on $\Omega \times \mathbb{R} \times [0,T)$ for some $0 < T < T_0$.*

In this note, we focus on Theorem 3 since proof of Theorem 4 is more or less standard arguments once we show Theorem 3. Before proving Theorem 3, we discuss Assumption 1 from the point of blowup/scaling arguments. Consider the scale transform for $\lambda > 0$

$$x = \lambda y, \quad u(x,t) = \lambda w(y,s), \quad t = \lambda^2 s.$$

Then $d_y w = (\partial_{y_1} w, \ldots, \partial_{y_n} w) = du$ and (MCF) is transformed into

$$\frac{\partial_s w}{\sqrt{1+|d_y w|^2}} = \operatorname{div}_y\left(\frac{d_y w}{\sqrt{1+|d_y w|^2}}\right) + \lambda \boldsymbol{F}(\lambda y, \lambda w, \lambda^2 s)\cdot \boldsymbol{n}.$$

To study regularity of du, it is important to investigate the asymptotic behavior of w as $\lambda \downarrow 0$. When $\frac{n}{p} + \frac{2}{q} < 1$, then

$$\|\lambda \boldsymbol{F}(\lambda y, \lambda w, \lambda^2 s)\|_{L^q_s L^p_y} = \lambda^{1-\frac{n}{p}-\frac{2}{q}} \|\boldsymbol{F}\|_{L^q_t L^p_x} \to 0 \quad \text{as } \lambda \downarrow 0,$$

thus the transport is small from the point of blowup/scaling arguments and Assumption 1 is the reasonable condition to obtain the gradient estimates. We remark that Assumption 1 is same assumption to an inhomogeneous term $f = f(x,t)$ to obtain gradient estimates for inhomogeneous heat equations $\partial_t u - \Delta u = f$ (see Ladyženskaja-Solonnikov-Ural'ceva [9, Theorem 11.1 in p.211]).

Figure 2 illustrates the region of (p,q) in Assumption 1. Takasao's result is almost equivalent to the condition $p > n+1$ and $q = \infty$. In fact, for $\boldsymbol{F} \in L^\infty(0,T_0 : W^{1,p}_{(x,z)}(\Omega\times\mathbb{R}))$ under $p > n+1$, we obtain $\boldsymbol{F} \in L^\infty_{(x,z,t)}(\Omega\times\mathbb{R}\times(0,T_0))$ by the Sobolev inequality. On the other hand, the transport $\boldsymbol{F}$ might not be bounded in time and space variables in our Assumption 1. Therefore, our results can be regarded as some extension of Takasao's early result even for the interior gradient estimates.

3. Monotonicity of the metric

Our main task is to establish the up-to-the-boundary monotonicity formula of the Huisken type. Let $R := \|\text{principal curvature of } \partial\Omega\|^{-1}_{L^\infty(\partial\Omega)}$. For $r < R$, define $N_r := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) < r\}$. Then for $x \in \partial\Omega$, there uniquely exists $\zeta(x) \in \partial\Omega$ such that $\operatorname{dist}(x,\partial\Omega) = |x - \zeta(x)|$. We define the reflection point of x with respect to $\partial\Omega$ as $\tilde{x} = 2\zeta(x) - x$. Remark that $x + \tilde{x} = 2\zeta(x)$ hence $\zeta(x)$ is midpoint between x and $\tilde{x}$(see Figure 3).

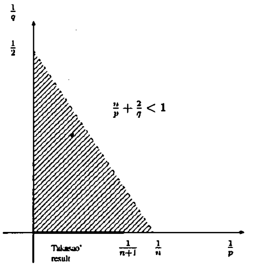

FIGURE 2. Region of (p, q) in Assumption 1. Our assumption is the region of oblique lines and Takasao's assumption is almost equivalent to the bold line.

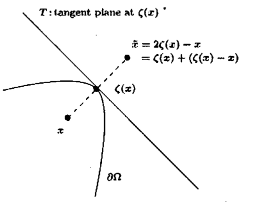

FIGURE 3. The reflection point of $x \in \Omega \cap N_r$ with respect to $\partial\Omega$ is denoted by $\tilde{x}$.

Fix a radially symmetric smooth cut-off function $\eta = \eta(r) = \eta(|(x, z)|)$ such that

$$0 \le \eta \le 1, \quad \frac{\partial \eta}{\partial r} \le 0, \quad \operatorname{spt} \eta \subset B_{R/2}, \quad \eta = 1 \text{ on } B_{R/4}.$$

For $0 < t < s < T_0$ and (x, z), $(y, w) \in N_R \times \mathbb{R}$, define the n-dimensional backward and reflected backward heat kernels as

$$\begin{aligned} \rho_{(y,w,s)}(x, z, t) &:= \frac{1}{(4\pi(s-t))^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2 + |z-w|^2}{4(s-t)}\right), \\ \tilde{\rho}_{(y,w,s)}(x, z, t) &:= \rho_{(y,w,s)}(\tilde{x}, z, t) = \frac{1}{(4\pi(s-t))^{\frac{n}{2}}} \exp\left(-\frac{|\tilde{x}-y|^2 + |z-w|^2}{4(s-t)}\right). \end{aligned} \tag{4}$$

For fixed $0 < t < s$ and (x, z), $(y, w) \in N_R \times \mathbb{R}$, define a truncated version of ρ and $\tilde{\rho}$ as

$$\begin{aligned} \rho_1 = \rho_1(x, z, t) &:= \eta(|(x, z) - (y, w)|)\rho_{(y,w,s)}(x, z, t), \\ \rho_2 = \rho_2(x, z, t) &:= \eta(|(\tilde{x}, z) - (y, w)|)\tilde{\rho}_{(y,w,s)}(x, z, t). \end{aligned} \tag{5}$$

To derive Huisken's monotonicity formula,

$$\frac{(e \cdot D\rho)^2}{\rho} + \operatorname{tr}((I - e \otimes e)D^2\rho) + \partial_t \rho = 0 \tag{6}$$

is the crucial identity, where $e \in \mathbb{R}^{n+1}$ with $|e| = 1$. In [11], a similar inequality for the reflected backward heat kernel was obtained.

Lemma 5 ([11]). *There is a constant $C_2 > 0$ such that for $e \in \mathbb{R}^{n+1}$ with $|e| = 1$ and $\tilde{\rho} = \tilde{\rho}_{(y,w,s)}(x,z,t)$,*

$$(7)\quad \frac{(e \cdot D\tilde{\rho})^2}{\tilde{\rho}} + \mathrm{tr}((I - \boldsymbol{e} \otimes \boldsymbol{e})D^2\tilde{\rho}) + \partial_t \tilde{\rho} \le C_2 \left(\frac{|\tilde{x} - y| + |z - w|}{s - t} + \frac{(|\tilde{x} - y| + |z - w|)^3}{(s-t)^2} \right) \tilde{\rho}$$

for $0 < t < s$ and (x,z), $(y,w) \in N_{R/2} \times \mathbb{R}$.

We also use the following identity about the second fundamental form of Γ_t.

Lemma 6. *Let u be a classical solution of* (MCF) *and let $v := \sqrt{1 + |du|^2}$. Then*

$$(8)\qquad \partial_t v - \Delta_{\Gamma_t} v - \left(\frac{du}{v} \cdot dv \right) \frac{\partial_t u}{v} = -|A_t|^2 v - \frac{2|D_{\Gamma_t} v|^2}{v} + du \cdot d(\boldsymbol{F} \cdot \boldsymbol{n}),$$

where D_{Γ_t}, Δ_{Γ_t}, and A_t are the surface gradient, the surface Laplacian, and the second fundamental form associated to Γ_t, respectively.

Proof of Lemma 6 is given in Takasao [14]. A key identity is

$$-\Delta_{\Gamma_t} v + |A_t|^2 v + \frac{2|D_{\Gamma_t} v|^2}{v} - v^2 (D_{\Gamma_t} h \cdot e_{n+1}) = 0$$

given by Ecker-Huisken [5].

We use the following identities which give a relationship between the normal derivative of $|du|^2$ and the second fundamental form of the boundary. It is a simple observation and has been used in a number of papers (see for [2, 10, 13]). Proof of the following Lemma 7 is given in [11, Lemma 4.2] or Tonegawa [15, Lemma 3.1] for instance.

Lemma 7. *Let Ω be a convex domain and let $u \in C^2(\overline{\Omega})$ satisfy $du \cdot \nu\big|_{\partial\Omega} = 0$. Then*

$$(9)\qquad (d|du|^2 \cdot \boldsymbol{\nu})|_{\partial(\Omega \times \mathbb{R})} \le 0.$$

Now we give the boundary monotonicity formula of Huisken type with the transport $\boldsymbol{F}$.

Proposition 8 (Boundary monotonicity formula). *Let u be a classical solution of* (MCF) *and let $v := \sqrt{1 + |du|^2}$. Then for $(y,w) \in N_{R/4} \times \mathbb{R}$ and $0 < t < s$,*

$$(10)\qquad \begin{aligned} \frac{d}{dt} \int_{\Gamma_t} v(\rho_1 + \rho_2)\, d\mathscr{H}^n \le & - \int_{\Gamma_t} (\rho_1 + \rho_2) \left(|A_t|^2 v + \frac{2|D_{\Gamma_t} v|^2}{v} - du \cdot d(\boldsymbol{F} \cdot \boldsymbol{n}) \right) d\mathscr{H}^n \\ & + \frac{1}{4} \int_{\Gamma_t} v(\rho_1 + \rho_2)(\boldsymbol{F} \cdot \boldsymbol{n})^2\, d\mathscr{H}^n \\ & + C_3 \mathscr{H}^n(\Gamma_t) + C_4 (s-t)^{-\frac{3}{4}} \int_{\Gamma_t} v(\rho_1 + \rho_2)\, d\mathscr{H}^n \\ & + C_5 \int_{\Gamma_t \cap \operatorname{spt} \rho_2} v\, d\mathscr{H}^n, \end{aligned}$$

where C_3, C_4, $C_5 > 0$ are positive constants depending only on n and R.

Proof. For $i = 1, 2$

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma_t} v\rho_i \, d\mathscr{H}^n &= \frac{d}{dt}\int_\Omega v\rho_i v\, dx \\
&= \int_\Omega \partial_t v\rho_i v\, dx + \int_\Omega v\partial_{x_{n+1}}\rho_i \partial_t u v\, dx \\
&\quad + \int_\Omega v\partial_t\rho_i v\, dx + \int_\Omega v\rho_i\partial_t v\, dx \\
&= \int_{\Gamma_t} (\partial_t v\rho_i + v\partial_t\rho_i + v\partial_{x_{n+1}}\rho_i\partial_t u)\, d\mathscr{H}^n + \int_\Omega \rho_i du\cdot d(\partial_t u)\, dx,
\end{aligned} \tag{11}$$

where we denote $\rho_i = \rho_i(x, u(x,t), t)$ for simplicity. Using the integration by parts,

$$\begin{aligned}
\int_\Omega \rho_i du\cdot d(\partial_t u)\, dx &= \int_\Omega \rho_i\frac{du}{v}\cdot d(\partial_t u) v\, dx \\
&= -\int_\Omega dv\cdot\frac{du}{v}\rho_i\partial_t u\, dx - \int_\Omega \left(d\rho_i + \partial_{x_{n+1}}\rho_i du\right)\cdot\frac{du}{v}\partial_t u\, v dx \\
&\quad - \int_\Omega \rho_i \operatorname{div}\left(\frac{du}{v}\right)\partial_t u\, v dx \\
&= -\int_{\Gamma_t}\left(dv\cdot\frac{du}{v}\right)\rho_i\frac{\partial_t u}{v}\, d\mathscr{H}^n - \int_{\Gamma_t}\left((d\rho_i + \partial_{x_{n+1}}\rho_i du)\cdot\frac{du}{v}\right)\partial_t u\, d\mathscr{H}^n \\
&\quad + \int_{\Gamma_t}\rho_i h\partial_t u\, d\mathscr{H}^n,
\end{aligned}$$

where $h = -\operatorname{div}(\frac{du}{v})$. Note that

$$\partial_{x_{n+1}}\rho_i v - \partial_{x_{n+1}}\rho_i\frac{|du|^2}{v} - d\rho_i\cdot\frac{du}{v} = \partial_{x_{n+1}}\rho_i\frac{1}{v} - d\rho_i\cdot\frac{du}{v} = (D\rho_i\cdot\boldsymbol{n}),$$

where $D = (d, \partial_{x_{n+1}}) = (\partial_{x_1}, \ldots, \partial_{x_{n+1}})$. Hence we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma_t} v\rho_i\, d\mathscr{H}^n = \int_{\Gamma_t}\partial_t v\rho_i\, d\mathscr{H}^n + \int_{\Gamma_t} v\partial_t\rho_i\, d\mathscr{H}^n + \int_{\Gamma_t} v(D\rho_i\cdot\boldsymbol{n})\frac{\partial_t u}{v}\, d\mathscr{H}^n \\
- \int_{\Gamma_t}\left(dv\cdot\frac{du}{v}\right)\rho_i\frac{\partial_t u}{v}\, d\mathscr{H}^n + \int_{\Gamma_t} v\rho_i h\frac{\partial_t u}{v}\, d\mathscr{H}^n.
\end{aligned}$$

Using (MCF), we obtain $\partial_t u = (-(\boldsymbol{H}\cdot\boldsymbol{n}) + (\boldsymbol{F}\cdot\boldsymbol{n}))v$ and

$$\begin{aligned}
&(v(D\rho_i\cdot\boldsymbol{n}) + v\rho_i(\boldsymbol{H}\cdot\boldsymbol{n}))\,\frac{\partial_t u}{v} \\
&= (v(D\rho_i\cdot\boldsymbol{n}) + v\rho_i(\boldsymbol{H}\cdot\boldsymbol{n}))\,(-(\boldsymbol{H}\cdot\boldsymbol{n}) + \boldsymbol{F}\cdot\boldsymbol{n}) \\
&= -v\rho_i\left|\boldsymbol{H} - \frac{D^\perp\rho_i}{\rho_i}\right|^2 + v\frac{|D^\perp\rho_i|^2}{\rho_i} - v(D^\perp\rho_i\cdot\boldsymbol{H}) \\
&\quad + v\rho_i\left(\left(\frac{D^\perp\rho_i}{\rho_i} - \boldsymbol{H}\right)\cdot\boldsymbol{n}\right)(\boldsymbol{F}\cdot\boldsymbol{n}) \\
&\le v\frac{|D^\perp\rho_i|^2}{\rho_i} - v(D^\perp\rho_i\cdot\boldsymbol{H}) + \frac{1}{4}v\rho_i(\boldsymbol{F}\cdot\boldsymbol{n})^2,
\end{aligned}$$

where $D^{\perp}\rho_i = (D\rho_i \cdot \boldsymbol{n})\boldsymbol{n}$. Therefore,

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma_t} v\rho_i \, d\mathscr{H}^n \le& \int_{\Gamma_t} \partial_t v \rho_i \, d\mathscr{H}^n - \int_{\Gamma_t}\left(dv \cdot \frac{du}{v}\right)\rho_i \frac{\partial_t u}{v}\, d\mathscr{H}^n \\ &+ \int_{\Gamma_t} v\left(\partial_t \rho_i + \frac{|D^{\perp}\rho_i|^2}{\rho_i} - (D^{\perp}\rho_i \cdot \boldsymbol{H})\right) d\mathscr{H}^n \\ &+ \frac{1}{4}\int_{\Gamma_t} v\rho_i (\boldsymbol{F}\cdot \boldsymbol{n})^2 \, d\mathscr{H}^n.\end{aligned}$$

According to the divergence theorem on Γ_t,

$$\begin{aligned}-\int_{\Gamma_t} v(D^{\perp}\rho_i \cdot \boldsymbol{H})\, d\mathscr{H}^n =& \int_{\Gamma_t} \operatorname{div}_{\Gamma_t}(vD\rho_i)\, d\mathscr{H}^n - \int_{\partial\Gamma_t} v(D\rho_i \cdot \boldsymbol{\nu})\, d\mathscr{H}^{n-1} \\ =& \int_{\Gamma_t} D_{\Gamma_t} v \cdot D\rho_i \, d\mathscr{H}^n + \int_{\Gamma_t} v \operatorname{tr}((I - \boldsymbol{n}\otimes\boldsymbol{n})D^2\rho_i)\, d\mathscr{H}^n \\ &- \int_{\partial\Gamma_t} v(D\rho_i\cdot\boldsymbol{\nu})\, d\mathscr{H}^{n-1} \\ =& -\int_{\Gamma_t} \rho_i \Delta_{\Gamma_t} v \, d\mathscr{H}^n + \int_{\Gamma_t} v \operatorname{tr}((I - \boldsymbol{n}\otimes\boldsymbol{n})D^2\rho_i)\, d\mathscr{H}^n \\ &+ \int_{\partial\Gamma_t} (\rho_i(D_{\Gamma_t} v\cdot\boldsymbol{\nu}) - v(D\rho_i\cdot\boldsymbol{\nu}))\, d\mathscr{H}^{n-1}\end{aligned}$$

where $\boldsymbol{\nu} = (\nu, 0)$, $\operatorname{div}_{\Gamma_t}$ is the surface divergence, Δ_{Γ_t} is the surface Laplacian, and $D_{\Gamma_t} = (I - \boldsymbol{\nu}\otimes\boldsymbol{\nu})D$ is the surface gradient associated to Γ_t. Remarking that

$$D_{\Gamma_t} v = (dv, 0) + \frac{(du, dv)}{\sqrt{1+|du|^2}}\boldsymbol{n},$$

we obtain $D_{\Gamma_t} v \cdot \boldsymbol{\nu} = dv \cdot \nu$ from the Neumann boundary condition. Using (6) and (7), we obtain

$$\frac{|D^{\perp}\rho_1|^2}{\rho_1} + \operatorname{tr}((I - \boldsymbol{n}\otimes\boldsymbol{n})D^2\rho_1) + \partial_t\rho_1 \le C_6 \tag{12}$$

and

$$\begin{aligned}\frac{|D^{\perp}\rho_2|^2}{\rho_2} &+ \operatorname{tr}((I - \boldsymbol{n}\otimes\boldsymbol{n})D^2\rho_2) + \partial_t\rho_2 \\ &\le C_7\left(\frac{|\tilde{x}-y| + |z-w|}{s-t} + \frac{(|\tilde{x}-y|+|z-w|)^3}{(s-t)^2}\right)\rho_2 + C_8\end{aligned} \tag{13}$$

for some constants $C_6, C_7, C_8 > 0$ depending only on n and R.

To compute the integrations of (13), we decompose the integrations as

$$\begin{aligned}\int_{\Gamma_t} v\frac{|\tilde{x}-y|+|z-w|}{s-t}\rho_2\, d\mathscr{H}^n \le& \int_{\Gamma_t\cap\{|\tilde{x}-y|+|z-w|\le(s-t)^{\frac14}\}} v\frac{|\tilde{x}-y|+|z-w|}{s-t}\rho_2\, d\mathscr{H}^n \\ &+ \int_{\Gamma_t\cap\{|\tilde{x}-y|+|z-w|\ge(s-t)^{\frac14}\}} v\frac{|\tilde{x}-y|+|z-w|}{s-t}\rho_2\, d\mathscr{H}^n \\ =:& \, I_1 + I_2,\end{aligned}$$

and

$$\int_{\Gamma_t} v\frac{(|\tilde{x}-y|+|z-w|)^3}{(s-t)^2}\rho_2\, d\mathscr{H}^n \le \int_{\Gamma_t\cap\{|\tilde{x}-y|+|z-w|\le(s-t)^{\frac{5}{12}}\}} v\frac{(|\tilde{x}-y|+|z-w|)^3}{(s-t)^2}\rho_2\, d\mathscr{H}^n$$
$$+\int_{\Gamma_t\cap\{|\tilde{x}-y|+|z-w|\ge(s-t)^{\frac{5}{12}}\}} v\frac{(|\tilde{x}-y|+|z-w|)^3}{(s-t)^2}\rho_2\, d\mathscr{H}^n$$
$$=: I_3+I_4.$$

I_1 and I_3 are estimated as

$$I_1 \le (s-t)^{-\frac{3}{4}}\int_{\Gamma_t} v\rho_2\, d\mathscr{H}^n,\quad I_3 \le (s-t)^{-\frac{3}{4}}\int_{\Gamma_t} v\rho_2\, d\mathscr{H}^n. \tag{14}$$

I_2 and I_4 are estimated as

$$\begin{aligned}
I_2 &\le \frac{4\pi}{(4\pi(s-t))^{1+\frac{n}{2}}}e^{-\frac{1}{4\sqrt{s-t}}}\int_{\Gamma_t\cap\operatorname{spt}\rho_2} v(|\tilde{x}-y|+|z-w|)\, d\mathscr{H}^n\\
&\le C_9\int_{\Gamma_t\cap\operatorname{spt}\rho_2} v(|\tilde{x}-y|+|z-w|)\, d\mathscr{H}^n,\\
I_4 &\le \frac{(4\pi)^2}{(4\pi(s-t))^{2+\frac{n}{2}}}e^{-\frac{1}{4\sqrt{s-t}}}\int_{\Gamma_t\cap\operatorname{spt}\rho_2} v(|\tilde{x}-y|+|z-w|)^3\, d\mathscr{H}^n\\
&\le C_9\int_{\Gamma_t\cap\operatorname{spt}\rho_2} v(|\tilde{x}-y|+|z-w|)^3\, d\mathscr{H}^n
\end{aligned} \tag{15}$$

for some constant $C_9>0$ depending only on n. Using (14), (15), $|\tilde{x}-y|+|z-w|\le R$ when $(x,z)\in\operatorname{spt}\rho_2$, and $D(\rho_1+\rho_2)\cdot\boldsymbol{\nu}\big|_{\partial\Omega}\equiv 0$, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\Gamma_t} v(\rho_1+\rho_2)\, d\mathscr{H}^n \le{}& \int_{\Gamma_t}(\rho_1+\rho_2)\left(\partial_t v-\Delta_{\Gamma_t}v-\left(dv\cdot\frac{du}{v}\right)\frac{\partial_t u}{v}\right) d\mathscr{H}^n\\
&+\frac{1}{4}\int_{\Gamma_t}(\rho_1+\rho_2)v(\boldsymbol{F}\cdot\boldsymbol{n})^2\, d\mathscr{H}^n\\
&+(C_6+C_8)\mathscr{H}^n(\Gamma_t)+2(s-t)^{-\frac{3}{4}}\int_{\Gamma_t} v\rho_2\, d\mathscr{H}^n\\
&+2C_{10}\int_{\Gamma_t\cap\operatorname{spt}\rho_2} v\, d\mathscr{H}^n+\int_{\partial\Gamma_t}(\rho_1+\rho_2)(D_{\Gamma_t}v\cdot\boldsymbol{\nu})\, d\mathscr{H}^{n-1}
\end{aligned}$$

where $C_{10}=C_9(R+R^3)$. Using (8) and (9), we obtain (10) with $C_3=C_6+C_8$, $C_4=2$, and $C_5=2C_{10}$. □

4. **Gradient estimates**

We deduce the integral estimates for the transport terms.

Lemma 9. *Let $\boldsymbol{F} \in L^p_x L^q_t(\Gamma_t)$ with $1 - \frac{n}{p} - \frac{2}{q} > 0$, u be a classical solution of* (MCF), *and $v := \sqrt{1+|du|^2}$. Then there is a constant $C_{11} > 0$ depending only on n, p, q, and T_0 such that*

$$\begin{aligned} &\int_0^s dt \int_{\Gamma_t} (\rho_1+\rho_2) du \cdot d(\boldsymbol{F}\cdot\boldsymbol{n})\, d\mathscr{H}^n + \frac{1}{4}\int_0^s dt \int_{\Gamma_t} v(\rho_1+\rho_2)(\boldsymbol{F}\cdot\boldsymbol{n})^2\, d\mathscr{H}^n \\ &\le \frac{1}{2}\int_0^s dt \int_{\Gamma_t} (\rho_1+\rho_2)|A_t|^2 v\, d\mathscr{H}^n + \int_0^s dt \int_{\Gamma_t} (\rho_1+\rho_2)\frac{|D_{\Gamma_t} v|^2}{v}\, d\mathscr{H}^n \\ &+ C_{11}\|v\|^3_\infty (1 + \|\boldsymbol{F}\|_{L^p_x L^q_t(\Gamma_t)})^2 \end{aligned} \tag{16}$$

for $0 < s < T_0$.

Proof. We focus on the integral estimates of $(\rho_1+\rho_2)du \cdot d(\boldsymbol{F}\cdot\boldsymbol{n})$. For simplicity, set $\bar\rho := \rho_1 + \rho_2$. Then

$$\begin{aligned} \int_{\Gamma_t} \bar\rho(du\cdot d(\boldsymbol{F}\cdot\boldsymbol{n}))\, d\mathscr{H}^n &= \int_\Omega \bar\rho(du\cdot d(\boldsymbol{F}\cdot\boldsymbol{n}))v\, dx \\ &= -\int_\Omega (\bar\rho\Delta u v + (du\cdot d\bar\rho)v + \bar\rho(du\cdot dv))(\boldsymbol{F}\cdot\boldsymbol{n})\, dx \\ &= -\int_{\Gamma_t} \Big(\bar\rho\Delta u + (du\cdot d\bar\rho) + \bar\rho\Big(\frac{du}{v}\cdot dv\Big)\Big)(\boldsymbol{F}\cdot\boldsymbol{n})\, d\mathscr{H}^n. \end{aligned}$$

Here

$$h = -\operatorname{div}\left(\frac{du}{v}\right) = -\frac{1}{v}\Delta u + \frac{1}{v^2}(du\cdot dv);$$

hence,

$$\begin{aligned} \int_{\Gamma_t} \bar\rho du\cdot d(\boldsymbol{F}\cdot\boldsymbol{n})\, d\mathscr{H}^n &= \int_{\Gamma_t} \bar\rho v h(\boldsymbol{F}\cdot\boldsymbol{n})\, d\mathscr{H}^n \\ &\quad - 2\int_{\Gamma_t} \bar\rho\Big(\frac{du}{v}\cdot dv\Big)(\boldsymbol{F}\cdot\boldsymbol{n})\, d\mathscr{H}^n \\ &\quad - \int_{\Gamma_t} (du\cdot d(\bar\rho(x,u,t)))(\boldsymbol{F}\cdot\boldsymbol{n})\, d\mathscr{H}^n \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

We focus on an integral estimate of I_3. Because

$$|du\cdot d(\bar\rho(x,u,t))| = |du\cdot d\bar\rho + |du|^2\bar\rho_{x_{n+1}}| \le v^2|D\bar\rho|,$$

we obtain

$$|I_3| \le \int_{\Gamma_t} v^2|D\bar\rho||\boldsymbol{F}\cdot\boldsymbol{n}|\, d\mathscr{H}^n.$$

Then using the Hölder inequality,

$$\int_0^s |I_3|\, dt \le \|v\|^2_\infty \left(\int_0^s dt \left(\int_{\Gamma_t} |D\bar\rho|^{p'}\, d\mathscr{H}^n\right)^{\frac{q'}{p'}}\right)^{\frac{1}{q'}} \|\boldsymbol{F}\|_{L^p_x L^q_t(\Gamma_t)}, \tag{17}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$. Using the convexity of Ω, $|\tilde x - y| \ge |x-y|$; hence,

$$|D\bar\rho| \le C_{12}\frac{1}{(s-t)^{\frac{1}{2}+\frac{n}{2}}}\exp\left(-\frac{|x-y|^2+|z-w|^2}{8(s-t)}\right),$$

where $C_{12} > 0$ is some constant depending only on n. Thus we have

$$\int_{\Gamma_t} |D\bar{\rho}|^{p'}\, d\mathscr{H}^n \le C_{12}^{p'}(s-t)^{-\frac{p'}{2}-\frac{np'}{2}+\frac{n}{2}};$$

or

$$\left(\int_0^s dt \left(\int_{\Gamma_t} |D\bar{\rho}|^{p'}\, d\mathscr{H}^n\right)^{\frac{q'}{p'}}\right)^{\frac{1}{q'}} \le C_{12} T_0^{\frac{1}{2}(1-\frac{n}{p}-\frac{2}{q})} < \infty.$$

Therefore, using (17) we obtain

$$\int_0^s |I_3|\, dt \le C_{12} T_0^{\frac{1}{2}-\frac{n}{2p}-\frac{1}{q}} \|v\|_\infty^2 \|\boldsymbol{F}\|_{L_x^p L_t^q(\Gamma_t)}. \tag{18}$$

As the similar argument as above, we also obtain

$$\begin{aligned} &\int_0^s (|I_1| + |I_2|)\, dt + \int_0^s dt \int_{\Gamma_t} v(\rho_1+\rho_2)(\boldsymbol{F}\cdot\boldsymbol{n})^2\, d\mathscr{H}^n \\ &\le \frac{1}{2}\int_0^s dt \int_{\Gamma_t} \bar{\rho}|A_t|^2 v\, d\mathscr{H}^n + \int_0^s dt \int_{\Gamma_t} \bar{\rho}\frac{|D_{\Gamma_t} v|^2}{v}\, d\mathscr{H}^n \\ &\quad + C_{13} T_0^{1-\frac{n}{p}-\frac{2}{q}} \|v\|_\infty^3 \|\boldsymbol{F}\|_{L_x^p L_t^q(\Gamma_t)}^2 \end{aligned} \tag{19}$$

for a positive constant $C_{13} > 0$ depending only on n, p, q. Combining (18) and (19), we obtain (16). □

Proof of Theorem 3. We only focus on the gradient estimates near the boundary $\partial\Omega$. Fix $(y,w) \in N_{R/4} \times \mathbb{R}$ and $0 < T < T_0$. For $0 < t < s < T$, using (10) and (16) and $\mathscr{H}^n(\Gamma_t) \le \|v\|_\infty |\Omega|$,

$$\begin{aligned} &\frac{d}{dt}\int_{\Gamma_t} v(\rho_1+\rho_2)\, d\mathscr{H}^n \\ &\le C_4(s-t)^{-\frac{3}{4}} \int_{\Gamma_t} v(\rho_1+\rho_2)\, d\mathscr{H}^n + C_{14}(1+\|\boldsymbol{F}\|_{L_x^p L_t^q(\Gamma_t)})^2 \int_0^{t'} \|v(\cdot,t)\|_\infty^3\, dt, \end{aligned}$$

where $C_{14} > 0$ is a positive constant depending only on n, p, q and $|\Omega|$. By the Gronwall inequality for $0 < t < s$,

$$\begin{aligned} &\exp\left(-C_4 \int_0^t (s-\tau)^{-\frac{3}{4}}\, d\tau\right) \int_{\Gamma_t} v(\rho_1((x,z),t) + \rho_2((x,z),t))\, d\mathscr{H}^n \\ &\le \int_{\Gamma_0} v(x,0)(\rho_1((x,z),0) + \rho_2((x,z),0))\, d\mathscr{H}^n \\ &\quad + C_{14}(1+\|\boldsymbol{F}\|_{L_x^p L_t^q(\Gamma_t)})^2 \int_0^t \exp\left(4C_4((s-\tau)^{\frac{1}{4}} - s^{\frac{1}{4}})\right) \|v(\cdot,\tau)\|_\infty^3\, d\tau \\ &\le 2\|v(\cdot,0)\|_\infty^2 + C_{14} T (1+\|\boldsymbol{F}\|_{L_x^p L_t^q(\Gamma_t)})^2 \sup_{0<t<T} \|v(\cdot,t)\|_\infty^3. \end{aligned}$$

As $t \to s$, we obtain

$$v(y,s) \le 2\|v(\cdot,0)\|_\infty^2 + C_{14} T (1+\|\boldsymbol{F}\|_{L_x^p L_t^q(\Gamma_t)})^2 \sup_{0<t<T} \|v(\cdot,t)\|_\infty^3.$$

Now, select (y,s) such that $v(y,s) = \sup_{0<t<T} \|v(\cdot,t)\|_\infty$. Then,

$$\sup_{0<t<T} \|v(\cdot,t)\|_\infty \leq 2\|v(\cdot,0)\|_\infty^2 + C_{14}T(1+\|\boldsymbol{F}\|_{L_x^p L_t^q(\Gamma_t)})^2 \sup_{0<t<T} \|v(\cdot,t)\|_\infty^3. \tag{20}$$

Suppose for a contradiction that there exists $C > 2$ such that

$$\sup_{0<t<T} \|v(\cdot,t)\|_\infty > C\|v(\cdot,0)\|_\infty^2$$

for all $0 < T < T_0$. Then (20) implies that

$$C_{14}\frac{T}{\|v(\cdot,0)\|_\infty^2}(1+\|\boldsymbol{F}\|_{L_x^p L_t^q(\Gamma_t)})^2 \sup_{0<t<T} \|v(\cdot,t)\|_\infty^3 \geq \frac{\sup_{0<t<T}\|v(\cdot,t)\|_\infty}{\|v(\cdot,0)\|_\infty^2} - 2 > C - 2 > 0,$$

which is contradiction as taking $T \downarrow 0$. □

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