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What circular sector is to be cut out of a circle so that the remaining surface maximizes the volume of a symmetric upright cone with the same lateral surface? Geogebra vs. Mathematica

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Abstract

The main trust of this investigation is to apply Geogebra and Mathematica to calculate the central angle of a circular sector that is to be cut out of a circle of a given radius so that the remaining surface area maximizes the volume of a circular symmetric upright cone with the same lateral surface area. Solving the proposed problem on two different parallel tracks applying these Computer Algebra Systems reveals the pros and cons of these two popular CASs. Utilizing the dynamic geometry of the former and its embedded CAS is compared to the sophisticated algebraic manipulation power of the latter. The investigation utilizes two and three dimensional visualization animations making the investigation comprehensive.

1 Motivation and Goals

Within the last decade Computer Algebra Systems especially Mathematica[ ][ ][ ] and Geogebra[ ] have deeply influenced structuring undergraduate and graduate mathematics, science and engineering curriculums across the globe. Applying their superb two and three-dimensional graphics adds new useful features to the analysis of the challenging classic geometry problems opening fresh avenues exploring new challenging issues. In addition to their graphics these software include symbolic manipulators capable of performing symbolic calculations. The author is an avid user of Mathematica and a recent user of the Geogebra. The main objective of this article is to compare their pros and cons analyzing a common geometry problem.

This note is composed of three sections and one appendix. In addition to Motivation and Goals in Sect 2 we describe the problem on hand and include its detailed solution. The appendix embodies Mathematica and Geogebra codes and their associated output. Sect 3 is the conclusion and closing remarks.

2 The problem and its solution

As is explained in the abstract the problem at hand is "to calculate the central angle of a circular sector that is to be cut out of a circle of a given radius so that the remaining surface area maximizes the volume of a circular symmetric upright cone with the same lateral surface area." Figure 1 shows a disk with a cutout sector. For the sake of simplicity the radius of the disk is set to unit. The area of the sector with the central angle is shown with the colorless wedge.

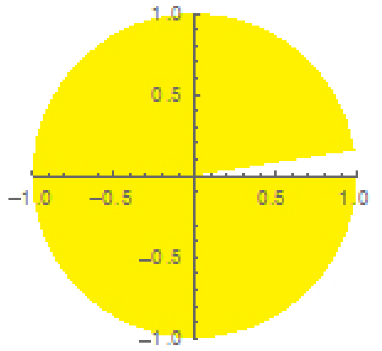


Figure 1. A unit disk with a cut out circular sector.

Area of the sector is $a = \frac{1}{2}R^2\theta$. For $\theta = 2\pi$ this yields the expected $a = \pi R^2$. Extracting the surface area a gives

$$\Delta A = \pi R^2 - a = \pi R^2 \left(1 - \frac{\theta}{2\pi}\right)$$

Utilizing this area we form an upright cone with a circular base radius of r, height h and side length $\ell = R$, respectively, shown in Figure 2.

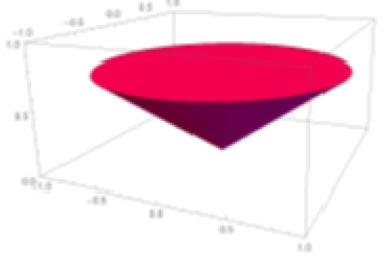


Figure 2. An upright cone with specs detailed in the text.

Lateral area of the cone is

$$S = \pi r \ell \equiv \pi r R = \Delta A = \pi R^2 \left(1 - \frac{\theta}{2\pi}\right)$$

Simplifying yields $r = R\left(1 - \frac{\theta}{2\pi}\right)$, i.e. $r = r(\theta)$. Volume of the cone is $V = \frac{1}{3}(\pi r(\theta))^2 h$, where $h = \sqrt{R^2 - r(\theta)^2}$. Substituting for $r(\theta)$ and simplifying gives

$$V = \left(\frac{1}{6}R^3\right)\left(1 - \frac{\theta}{2\pi}\right)^2 \sqrt{\theta(4\pi - \theta)}$$

In other words

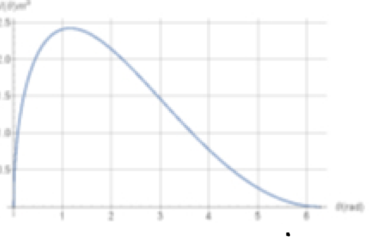
$$V(\theta) = \left(1 - \frac{\theta}{2\pi}\right)^2 \sqrt{\theta(4\pi - \theta)} \tag{1}$$

Having this final result at hand it is a trivial task to find the angle maximizing the volume. However, as pointed out in the abstract and the first Section, the challenge is utilizing two and three-dimensional graphics of Mathematica and Geogebra animating the process. In short first we claim the angle maximizing the volume is 66.06°. This comes about by solving $V'(\theta) = 0$ where $V(\theta)$ is given by (1). Mathematica code yielding this result follows.

$\mathrm{V}[\theta\_] := \left(1 - \frac{\theta}{2\pi}\right)^2 \sqrt{\theta(4\pi - \theta)}$

It is a helpful procedure to plot the function that we intend differentiating; this gives an objective view of the value of the variable optimizing the function.

$Plot(V[\theta], \{\theta, 0, 2\pi\}, GridLines \to Automatic, AxesLabel \to \{``\theta(rad)", V(m^3)\})$



$\mathrm{d}V = D[V[\theta], \{\theta, 1\}]$ //Simplify

$\frac{(2\pi - \theta)(4\pi^2 - 12\pi\theta + 3\theta^2)}{4\pi^2\sqrt{\theta(4\pi - \theta)}}$

$solV = Solve[dV == 0, \theta]$

$\frac{180.}{\pi}\theta /. solV$

$\{66.06\}$

3 Appendix

The main objective of this note is to compare graphics applications of Mathematica vs. Geogebra. Here we provide a version of the Mathematica code capable of animating the entire process. The code is descriptive; it embodies easy to follow commands. Commands are optimized producing the needed output. For instance the version of the code given in Figure 3 contains Manipulate as the main animation command. It follows by the Graphics Disk command. The combination of the last two sequential commands put the appearance of the disk in motion widening the central colorless area accordingly. The Cone command displays the cone and the Manipulate command put its evolution in motion. This is followed by displaying a table of the value of the angle and its corresponding conic volume. The code ends with a plot of the volume of the cone vs. angle. The author believes an expert Mathematica user would appreciate the efficiency of the given four lines compact code producing the needed numeric, 2D and 3D graphics information.

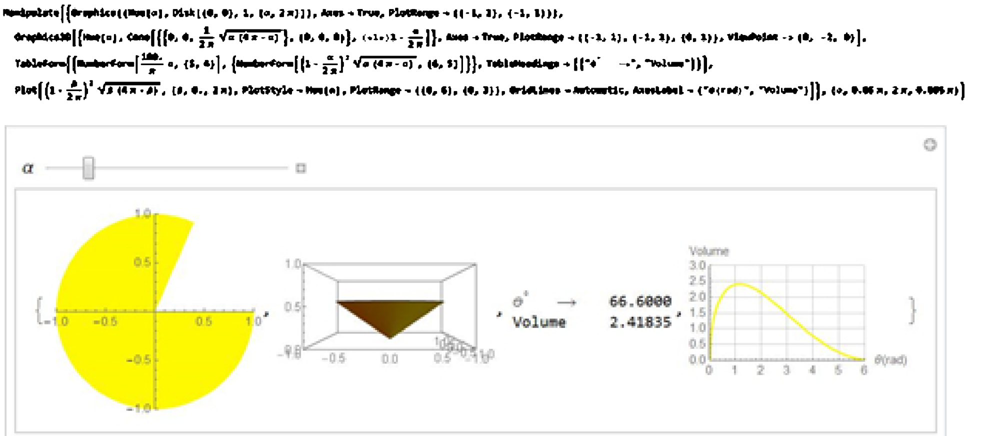


Figure 3. Output of the Mathematica code embodying all the needed information.

Figure 4 is a screen-shot of the Geogebra code and its output. The code embodies a different style of commands vs. Mathematica producing the needed information. The screen-shot is composed of one algebra screen on the left, two 2D Graphics screens and one 3D Graphics screen. The inserted slider in the bottom right graphics screen animates the plot. The colored animation buttons at the bottom of the Graphics 2 put the entire output to dynamic mode.

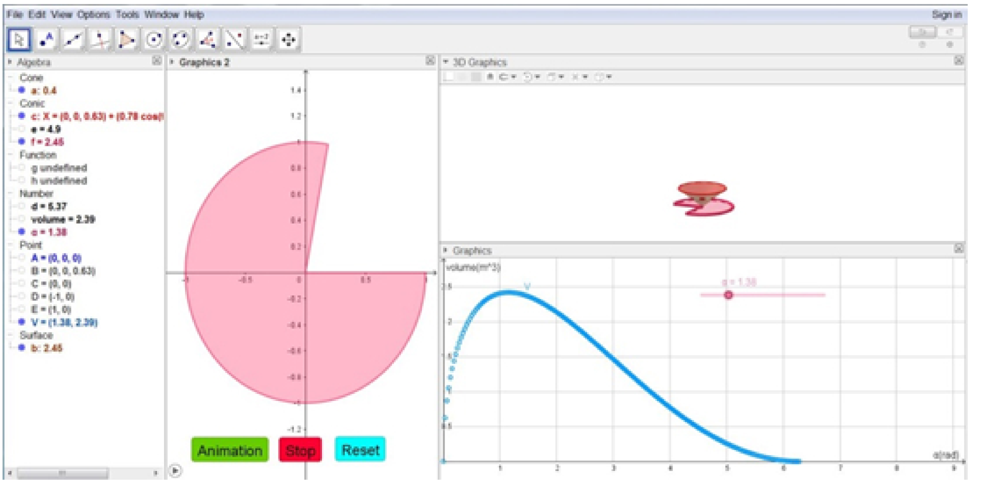


Figure 4. Screen shot of the Geogebra.

4 Conclusions

By way of example utilizing two different popular Computer Algebra Systems, Mathematica and Geogebra the author compared their respective codes conducive to comparable solutions. The author has been using Mathematica since its inception in 1985; he is a fresh user of Geobegra. As such he encountered a challenging but rewarding learning curve. These two CASs are very powerful, they are fabricated assisting pursuing scientific goals. Their interfaces are totally different; they have different effective and pleasant rewarding "flavors." Based on the author's experience for the future investigations he will adopt both.

References

[1] GeoGebra is a Dynamic Mathematics Software (DMS). It can be used as Dynamic Geometry Software (DGS) with basic features of Computer Algebra Systems (CAS).

[2] Mathematica^{TM} is a Symbolic Computation Software, V11.0, Wolfram Research Inc. 2015.

[3] Wolfram, S. (1996) Mathematica book, 3rd Edition, Cambridge University Press.

[4] Sarafian, H. (2015) Mathematica Graphics Example Book For Beginners, Scientific Research Publishing.