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Wall effect on the motion of a rigid body immersed in a free molecular gas

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1 Introduction

This paper is a summary of [2] and discusses the wall effect on the motion of a rigid body in a free molecular gas.

Consider a linear motion of a rigid body in a free molecular gas driven by a constant force E. The equation describing the gas is the Vlasov equation and the motion of the rigid body is governed by Newton's equations of motion. The boundary condition for the Vlasov equation is the specular boundary condition. The shape of the rigid body is a cylinder and it moves in the direction of its axis.

If the gas fills the whole space $\mathbb{R}^d$, the velocity $V(t)$ of the rigid body approaches the terminal velocity $V_\infty = V_\infty(E)$ with the algebraic rate $V_\infty - V(t) \approx Ct^{-(d+2)}$ [1]. On the other hand, if the gas fills the half space $\mathbb{R}^d_+$ — the rigid body moves in the direction perpendicular to the boundary $\partial\mathbb{R}^d_+$, which can be regarded as a plane wall — then $V(t)$ obeys the power law $V_\infty - V(t) \approx Ct^{-(d-1)}$ [2].[1] The terminal velocity V_∞ is unchanged but the rate becomes slower.

In both cases of $\mathbb{R}^d$ and $\mathbb{R}^d_+$, a molecular process called *recollision* is responsible for the algebraic approach. In the $\mathbb{R}^d_+$ case, recollisions involving the wall $\partial\mathbb{R}^d_+$ changes the asymptotic behaviour.

[1] $d \geq 2$ is assumed in this case; $d = 1$ is not excluded in [1].

2 Motion of a Rigid Body in a Free Molecular Gas

In this section, I will explain the equations governing the rigid body and the surrounding gas and explain how they are related to each other.

Consider a rigid body moving in a gas. Its shape is assumed to be a cylinder. And it moves in the direction of its axis without rotation. Denote the spatial variable by $x = (x_1, x_\perp) \in \mathbb{R} \times \mathbb{R}^{d-1}$, where the x_1-axis is taken as the axis of the cylinder. I assume $d \geq 2$. The gas fills either the whole space $\mathbb{R}^d$ [1] or the right half space $\mathbb{R}^d_+ = \{x_1 > 0\}$ [2]. The boundary $\partial\mathbb{R}^d_+ = \{x_1 = 0\}$ represents a wall placed behind the motion of the rigid body. Write the velocity of the rigid body by $V(t)$ and let $X(t) = L + \int_0^t V(s)\, ds$. Then the cylinder occupies the region

$$C(t) = \{X(t) - 1/2 \leq x_1 \leq X(t) + 1/2,\ |x_\perp| \leq 1\}. \tag{2.1}$$

In the case of $\mathbb{R}^d_+$, L is the initial distance between the wall and the rigid body. The radius and hight of the cylinder are taken as unity. Let $\Omega(t) \subset \mathbb{R}^d$ be the region occupied by the gas: $\Omega(t) = \mathbb{R}^d \backslash C(t)$ or $\mathbb{R}^d_+ \backslash C(t)$.

Before explaining the equations governing $V(t)$, I shall explain the type of gas considered in this paper: a free molecular gas.

A free molecular gas consists of molecules moving freely without mutual interaction. Because there's no interaction, there's no reason the gas stays near local thermal equilibrium. This means that macroscopic descriptions — like the Navier–Stokes equations — are not adequate; instead, mesoscopic descriptions — kinetic equations like the Boltzmann or Vlasov equations — are more appropriate.

In the kinetic theory of gases, the state of a gas is described by the velocity distribution function $f = f(x, \xi, t)$. $x \in \Omega(t)$ is a position in the gas; $\xi = (\xi_1, \xi_\perp) \in \mathbb{R} \times \mathbb{R}^{d-1}$ is a velocity of molecules; $t > 0$ is time. f is the density of the molecules in the phase space $\Omega(t) \times \mathbb{R}^d$: If I want to know how many molecules having velocity ξ are there at position x at time t, $f(x, \xi, t)$ gives that.

The velocity distribution function of a free molecular gas obeys the Vlasov equation

$$\partial_t f + \xi \cdot \nabla_x f = 0 \tag{2.2}$$

for $x \in \Omega(t)$, $\xi \in \mathbb{R}^d$ and $t > 0$. I assume that the gas is initially in thermal equilibrium:

$$f(x, \xi, 0) = f_0(\xi) := \pi^{-d/2} \exp(-|\xi|^2). \tag{2.3}$$

The density and temperature are taken as unity. I also assume that f satisfies the specular boundary condition described as follows. Let $\mathbf{e}_1$ be the unit vector parallel to the x_1-axis and denote by $\mathbf{n} = \mathbf{n}(x, t)$ the unit normal to $S(t) = \partial\Omega(t)$ pointing towards the gas region. Note that $S(t) = \partial C(t)$ or $\partial C(t) \cup \partial\mathbb{R}^d_+$. Then the specular boundary condition is

$$f(x, \xi, t) = f(x, \xi - 2[(\xi - V(t)\mathbf{e}_1) \cdot \mathbf{n}]\mathbf{n}, t) \tag{2.4}$$

for $x \in S(t)$, $\xi \in \mathbb{R}^d$ with $(\xi - V(t)\mathbf{e}_1) \cdot \mathbf{n} > 0$ and $t > 0$. This physically means that the molecules have elastic collisions at $S(t)$.

Now I move on to explain the equations governing $V(t)$.

The gas drag $D(t)$ reduces the velocity of the rigid body. It has the expression

$$D(t) = \int_{\partial C(t)} dS \int_{\mathbb{R}^d} \xi_1 (\xi - V(t)\mathbf{e}_1) \cdot \mathbf{n} f \, d\xi. \tag{2.5}$$

This is derived by considering the net momentum flux at $\partial C(t)$. Define $I^{\pm}(t)$ by

$$I^{\pm}(t) = \{x \in C(t) \mid x_1 = X(t) \pm 1/2\} \times \{\xi \in \mathbb{R}^d \mid \xi_1 \lessgtr V(t)\}. \tag{2.6}$$

$(x, \xi) \in I^{\pm}(t)$ is a position and velocity of molecules which are about to collide with the rigid body. Then the drag $D(t)$ is written as

$$D(t) = 2 \left(\int_{I^+(t)} (\xi_1 - V(t))^2 f \, d\xi dS - \int_{I^-(t)} (\xi_1 - V(t))^2 f \, d\xi dS \right) \tag{2.7}$$

by using boundary condition (2.4).

Apply a constant external force E to make the rigid body move. Then $V(t)$ obeys the Newton's equations of motion

$$\frac{dV(t)}{dt} = E - D(t), \quad V(0) = V_0, \tag{2.8}$$

where V_0 is the initial velocity. The mass of the cylinder is taken as unity.

Two equations — the Vlasov equation (2.2) and the Newton's equations of motion (2.8) — are coupled in both ways: Boundary condition (2.4) requires the knowledge of $X(t)$ and $V(t)$; the formula (2.5) for drag requires f.

3 Theorems on the Asymptotic Behaviour

Now, the problem is to solve these equations and determine the asymptotic behaviour of $V(t)$. Intuitively, it will approach the terminal velocity $V_\infty(E)$ determined by the external force E. But is it really so? How fast is the convergence? These questions are answered in the theorems stated below. And it will be shown that the asymptotic behaviour is different in two cases: the motion in $\mathbb{R}^d$ or in $\mathbb{R}^d_+$.

Before stating the theorems and proving them, I need some preliminary consideration on $V_\infty(E)$.

Define a function $D_0 \colon \mathbb{R} \to \mathbb{R}$ by

$$D_0(U) := 2 \left(\int_{I^+(t)} (\xi_1 - U)^2 f_0 \, d\xi dS - \int_{I^-(t)} (\xi_1 - U)^2 f_0 \, d\xi dS \right) \tag{3.1}$$

$$= C_d \left(\int_{-\infty}^{U} (u - U)^2 e^{-u^2} \, du - \int_{U}^{\infty} (u - U)^2 e^{-u^2} \, du \right). \tag{3.2}$$

Note that $C_d = 2\pi^{(d-2)/2}/\Gamma((d+1)/2)$, where $\Gamma(z)$ is the gamma function. The following lemma gives some properties of D_0. Its proof is easy [1].

Lemma 3.1. $D_0\colon \mathbb{R} \to \mathbb{R}$ is convex on $[0,\infty)$; it is odd, smooth and uniformly increasing on $\mathbb{R}$. In particular, it is bijective.

Suppose that $V(t) \equiv U$ is a stationary solution in the case of $\mathbb{R}^d$.[2] Substituting $V(t) \equiv U$ into equation (2.8) — besides the initial condition — gives $E = D(t)$ because $dV(t)/dt = 0$. And by the method of characteristics, the solution to equations (2.2), (2.3) and (2.4) satisfy $f(x,\xi,t) \equiv f_0(\xi)$ on $I_\pm(t)$. Hence $D(t) \equiv D_0(U)$ by the definition of $D_0(U)$. Consequently, U must satisfy $E = D_0(U)$, which uniquely determines U by Lemma 3.1. Denote this U by $V_\infty(E) = D_0^{-1}(E)$. I take E so that $V_\infty(E) = 1$ for simplicity.

As the following theorems state, $V_\infty(E)$ is the terminal velocity in both cases of $\mathbb{R}^d$ and $\mathbb{R}^d_+$; the speed of approach, however, is different.

Theorem 3.1 (Caprino, Marchioro and Pulvirenti [1]). If the gas fills the whole space $\mathbb{R}^d$, then the asymptotic behaviour is $1-V(t) \approx Ct^{-(d+2)}$. More precisely, if $\gamma = 1 - V_0$ is positive and sufficiently small, then there exists a solution (f,V) — which at the present time has not been proved to be unique — satisfying

$$1 - V(t) \le \gamma e^{-D_0'(1/2)t} + \gamma^3 \frac{A_+}{(1+t)^{d+2}}, \tag{3.3}$$

$$1 - V(t) \ge \gamma e^{-D_0'(1)t} + \gamma^4 \frac{A_-}{t^{d+2}} \mathbf{1}_{\{t \ge \bar{t}\}}, \tag{3.4}$$

where $A_\pm$ and $\bar{t}$ are positive constants depending only on d. Although uniqueness is not known, the asymptotic behaviour is at least unique: Any solution (f,V) satisfies the above inequalities.

Theorem 3.2 (Koike [2]). If the gas fills the right half space $\mathbb{R}^d_+$, then the asymptotic behaviour is $1 - V(t) \approx Ct^{-(d-1)}$. More precisely, if $\gamma = 1 - V_0$ is positive and sufficiently small, and $L = X(0)$ is sufficiently large, then there exists a solution (f,V) satisfying

$$1 - V(t) \le \gamma e^{-D_0'(1/2)t} + \gamma^3 \frac{A_+}{(1+t)^{d+2}} + L^{-(d-1)} \frac{B+}{(1+t/L)^{d-1}}, \tag{3.5}$$

$$1 - V(t) \ge \gamma e^{-D_0'(1)t} + \frac{B_-}{t^{d-1}} \mathbf{1}_{\{t \ge L\}} \tag{3.6}$$

where A_+ and $B_\pm$ are positive constants depending only on d. And any solution (f,V) satisfies the above inequalities.

Remark 3.1. The lower bound in Theorem 3.2 can be improved to

$$1 - V(t) \ge \gamma e^{-D_0'(1)t} + \gamma^4 \frac{A_-}{t^{d+2}} \mathbf{1}_{\{\bar{c}\gamma L^{d-1} \ge t \ge \bar{t}\}} + \frac{B_-}{t^{d-1}} \mathbf{1}_{\{t \ge L\}} \tag{3.7}$$

[2]There is no stationary solution in the case of $\mathbb{R}^d_+$.

if γL^{d+1} is sufficiently large [2]. Here, $\bar{c}$ is a positive constant depending only on d. The estimates in Theorem 3.1 are obtained by taking the limit $L \to \infty$ in inequalities (3.5) and (3.7).

In summary, the wall effect changes the asymptotic behaviour from $t^{-(d+2)}$ to $t^{-(d-1)}$ — a slower approach to the terminal velocity.

4 Proof of Theorem 3.2

I will partly prove Theorem 3.2 in this section: The existence of a solution (f, V) satisfying inequality (3.5); the proof of inequality (3.6) and that any solution (f, V) satisfies the bounds in Theorem 3.2 is not given. See [2] for these.

4.1 Construction of a Solution by Schauder's Fixed Point Theorem

This section explains how a solution (f, V) satisfying inequality (3.5) is constructed — assuming that certain bounds hold. These bounds are later proved in section 4.2.

First, let the motion of the rigid body be given by an arbitrary known function $W\colon [0,\infty) \to (0,1)$ satisfying $W(0) = 1 - \gamma = V_0$. Let $X_W(t) = L + \int_0^t W(s)\,ds$ and define $C_W(t)$ by replacing $X(t)$ in equation (2.1) by $X_W(t)$. Furthermore, let $S_W(t) = \partial C_W(t) \cup \partial \mathbb{R}^d_+$. And define $I^\pm_W(t)$ by replacing $C(t)$, $X(t)$ and $V(t)$ in equation (2.6) by $C_W(t)$, $X_W(t)$ and $W(t)$.

Next, denote by $f = f_W$ the solution to equations (2.2), (2.3) and

$$f(x,\xi,t) = f(x, \xi - 2[(\xi - W(t)\mathbf{e}_1)\cdot n]\mathbf{n}, t) \tag{4.1}$$

for $x \in S_W(t)$, $\xi \in \mathbb{R}^d$ with $(\xi - W(t)\mathbf{e}_1)\cdot n > 0$ and $t > 0$. This is constructed by the method of characteristics. I will only consider $f(x,\xi,t)$ for $(x,\xi) \in I^\pm_W(t)$ and $t > 0$ — which is all I need to estimate the drag. Define the backward-characteristics starting from $(x,\xi) \in I^\pm_W(t)$ by $x(s) = x - (t-s)\xi$ and $\xi(s) = \xi$ for $s \le t$ until $x(s)$ hits the boundary $S_W(s) = \partial C_W(s) \cup \partial\mathbb{R}^d_+$. Denote this time of *recollision* by τ_1. If $x(\tau_1) \in \partial C_W(\tau_1)$, then put $\xi'(\tau_1) = (2W(\tau_1) - \xi_1(\tau_1), \xi_\perp(\tau_1))$; if $x_1(\tau_1) = 0$, then put $\xi'(\tau_1) = (-\xi_1(\tau_1), \xi_\perp(\tau_1))$. Using $\xi'(\tau_1)$, extend the characteristics by $x(s) = x(\tau_1) - (\tau_1 - s)\xi'(\tau_1)$ and $\xi(s) = \xi'(\tau_1)$ for $s < \tau_1$ — until $x(s)$ again hits the boundary. Repeat this to define τ_n until $s = 0$ is reached; infinite recollisions does not happen except on a subset of $I^\pm_W(t)$ of measure zero by [1, Proposition A.1]. Then $f_W(x,\xi,t) = f_0(\xi_0)$, where $\xi_0 = \xi(0)$ since equations (2.2) and (4.1) imply that $f(x(s),\xi(s),s)$ is constant.

Next, define the updated velocity V_W as follows. Let

$$r^\pm_W(t) = \pm 2 \int_{I^\pm_W(t)} (\xi_1 - W(t))^2 (f_W - f_0)\, d\xi dS. \tag{4.2}$$

Define a function $K\colon (0,1) \to \mathbb{R}$ by

$$K(U) = \frac{D_0(1) - D_0(U)}{1-U}. \tag{4.3}$$

Define V_W by solving the equations

$$\frac{d}{dt}V_W(t) = K(W(t))(1 - V_W(t)) - r_W^+(t) - r_W^-(t), \quad V_W(0) = 1 - \gamma. \tag{4.4}$$

This is solved explicitly:

$$1 - V_W(t) = \gamma e^{-\int_0^t K(W(s))\,ds} + \int_0^t e^{-\int_s^t K(W(\tau))\,d\tau}(r_W^+(s) + r_W^-(s))\,ds. \tag{4.5}$$

Suppose that V is a fixed point of the map $W \mapsto V_W$. Then equations (4.4) become equations (2.8) because $E = D_0(1)$ and $D(t) = D_0(V(t)) + r_V^+(t) + r_V^-(t)$. Thus (f_V, V) satisfies the original equations (2.2), (2.3) and (2.4); and equations (2.8) with (2.5).

To show the existence of a fixed point, the function space $\mathcal{K}$ — which is the domain of the map $W \mapsto V_W$ — is defined as follows.

Definition 4.1. Let A_+, B_+ and M be positive constants. A C^1-smooth function $W\colon [0,\infty) \to (0,1)$ belongs to $\mathcal{K}(\gamma, L, A_+, B_+, M)$ if $W(0) = 1 - \gamma$,

$$1 - W(t) \le \gamma e^{-D_0'(1/2)t} + \gamma^3 \frac{A_+}{(1+t)^{d+2}} + L^{-(d-1)} \frac{B+}{(1+t/L)^{d-1}}, \tag{4.6}$$

$$1 - W(t) \ge \gamma e^{-D_0'(1)t} \tag{4.7}$$

and $|dW(t)/dt| \le M$ for $t \ge 0$.

An important part of the proof is to show that $W \in \mathcal{K}$ implies $V_W \in \mathcal{K}$.

Proposition 4.1. There exist positive constants A_+, B_+ and M independent of γ and L such that $W \in \mathcal{K}(\gamma, L, A_+, B_+, M)$ implies $V_W \in \mathcal{K}(\gamma, L, A_+, B_+, M)$ if γ and L^{-1} are sufficiently small.

The following estimates of $r_W^\pm(t)$ are needed to prove this.

Proposition 4.2. If $W \in \mathcal{K}$, then

$$0 \le r_W^+(t) \le C\left[\frac{(\gamma + \gamma^3 A_+)^3}{(1+t)^{d+2}} + L^{-3(d-1)} \frac{B_+^3}{(1+t/L)^{d-1}}\right], \tag{4.8}$$

where C is a positive constant depending only on d.

Proposition 4.3. If $W \in \mathcal{K}$, then

$$0 \le r_W^-(t) \le C\left[\frac{(\gamma + \gamma^3 A_+)^3}{(1+t)^{3(d+2)}} + L^{-3(d-1)} \frac{B_+^3}{(1+t/L)^{3(d-1)}} + \frac{L^{-(d-1)}}{(1+t/L)^{d-1}}\right], \tag{4.9}$$

where C is a positive constant depending only on d.

Now I can prove Proposition 4.1 by using these estimates of $r_W^\pm(t)$ — which are later proven in section 4.2.

Proof. Let $W \in \mathcal{K}(\gamma, L, A_+, B_+, M)$. The constants A_+, B_+ and M are specified later.

First, I shall show that V_W satisfies inequality (4.7). Note that K is increasing since D_0 is convex on $[0,\infty)$ by Lemma 3.1. Thus $K(W(t)) \le \lim_{U\to 1} K(U) = D_0'(1)$. Also note that $r_W^+(t)+r_W^-(t)$ is non-negative by Propositions 4.2 and 4.3. Therefore, by equation (4.5),

$$1 - V_W(t) \ge \gamma e^{-D_0'(1)t}. \tag{4.10}$$

Next, I shall prove that V_W satisfies inequality (4.6). Note that by inequality (4.7), $W(t) \ge 1/2$ if γ and L^{-1} are sufficiently small. Since D_0 is convex, $K(W(t)) \ge D_0'(1/2)$. By Propositions 4.2 and (4.3),

$$\begin{aligned} 1 - V_W(t) \le \gamma e^{-D_0'(1/2)t} &+ C(\gamma+\gamma^3 A_+)^3 \int_0^t \frac{e^{-D_0'(1/2)(t-s)}}{(1+s)^{d+2}}\, ds \\ &+ C\left[L^{-3(d-1)}B_+^3 + L^{-(d-1)}\right] \int_0^t \frac{e^{-D_0'(1/2)(t-s)}}{(1+s/L)^{d-1}}\, ds. \end{aligned} \tag{4.11}$$

Splitting the integral at $s = t/2$ and considering them separately gives

$$1 - V_W(t) \le \gamma e^{-D_0'(1/2)t} + \bar{C}\left[\frac{(\gamma+\gamma^3 A_+)^3}{(1+t)^{d+2}} + \frac{L^{-3(d-1)}B_+^3 + L^{-(d-1)}}{(1+t/L)^{d-1}}\right], \tag{4.12}$$

where $\bar{C}$ is a positive constant depending only on d. Let $A_+ = B_+ = 2\bar{C}$. Now take γ and L^{-1} sufficiently small so that

$$1 - V_W(t) \le \gamma e^{-D_0'(1/2)t} + \gamma^3 \frac{A_+}{(1+t)^{d+2}} + L^{-(d-1)} \frac{B+}{(1+t/L)^{d-1}}. \tag{4.13}$$

Let $M = D_0'(1) + 1$. Showing $|dV_W(t)| \le M$ is easy. Note that $|r_W^+(t) + r_W^-(t)| \le 1$ if γ and L^{-1} are sufficiently small by Propositions 4.2 and (4.3). Now equation (4.4) implies

$$|dV_W(t)/dt| \le |K(W(t))(1-V_W(t))| + |r_W^+(t)+r_W^-(t)| \le D_0'(1)+1 = M. \tag{4.14}$$

These show that $V_W \in \mathcal{K}(\gamma, L, A_+, B_+, M)$. □

Now I can apply Schauder's fixed point theorem to the map $\mathcal{K} \ni W \mapsto V_W \in \mathcal{K}$. What must be proved is that $\mathcal{K}$ is a compact convex subset of $C_b([0,\infty))$ — the space of bounded continuous functions — and that the map $W \mapsto V_W$ is continuous with respect to the topology of $C_b([0,\infty))$ defined by the norm $||W|| = \sup_{0\le t<\infty} |W(t)|$. The convexity is trivial; the compactness is proved by the Arzelà–Ascoli theorem;[3] the continuity is not difficult but the proof is

[3]Since the domain $[0,\infty)$ is non-compact, uniform boundedness and uniform continuity are not enough: Also use the fact that $1 - W(t)$ decays uniformly for $W \in \mathcal{K}$.

lengthy. See [1] or [2] for this. These consideration guarantees the existence of a fixed point V.

$V(t)$ satisfies inequality (3.5) since $V \in \mathcal{K}$. An appropriate lower bound of $r_V^-(t)$ shows that V also satisfies inequality (3.6). For the proof of this and that any solution (f, V) also satisfies the bounds in Theorem 3.2, see [2].

4.2 Estimates of $r_W^\pm(t)$

This section proves Propositions 4.2 and 4.3 — which was used in the previous section without proof.

4.2.1 Estimate of $r_W^+(t)$

First, I shall show that $r_W^+(t) \geq 0$. It suffices to show that $f_W - f_0 \geq 0$ by equation (4.2). Let $(x, \xi) \in I_W^+(t)$. Note that $|\xi'(\tau_1)| = |\xi(\tau_1)|$ if $x_1(\tau_1) = 0$ and

$$(\xi_1'(\tau_1))^2 = \xi_1^2 + 4W(\tau_1)(W(\tau_1) - \xi_1) \tag{4.15}$$

if $x(\tau_1) \in \partial C_W(\tau_1)$. $W(\tau_1) - \xi_1 < 0$ because $x(s)$ is about to hit the right side of $\partial C_W(s)$. Hence $|\xi'(\tau_1)| < |\xi|$ in the latter case. Repeating this argument, $|\xi_0| \leq |\xi|$ is proved. Then $f_W - f_0 \geq 0$ since $f_W = f_0(\xi_0)$. This proves that $r_W^+(t) \geq 0$. Also note that

$$0 \leq f_W - f_0 \leq f_W = \pi^{-d/2} e^{-|\xi_0|^2} \leq \pi^{-d/2} e^{-|\xi_\perp|^2}. \tag{4.16}$$

Next, decompose $r_W^\pm(t)$ into two parts as follows. Define the subset $A_t \subset I_W^+(t)$ by

$$A_t = \{(x, \xi) \in I_W^+(t) \mid \tau_1 > 0 \text{ and } x(\tau_1) \in \partial C_W(\tau_1)\}. \tag{4.17}$$

And let

$$A_{\leq t/2} = \{(x, \xi) \in A_t \mid \tau_1 \leq t/2\}; \quad A_{>t/2} = \{(x, \xi) \in A_t \mid \tau_1 > t/2\}. \tag{4.18}$$

Using this, define I and II by

$$I = \int_{A_{\leq t/2}} (\xi_1 - W(t))^2 (f_W - f_0)\, d\xi dS; \quad II = \int_{A_{>t/2}} (\xi_1 - W(t))^2 (f_W - f_0)\, d\xi dS. \tag{4.19}$$

Then $r_W^+(t) = I + II$. This is because $(x, \xi) \notin A_t$ implies $|\xi_0| = |\xi|$ and hence $f_W = f_0$ — which means that these (x, ξ) do not contribute to the integral defining $r_W^+(t)$.

I has the following bound:

Lemma 4.1. If $W \in \mathcal{K}$, then

$$I \leq C \left[\frac{(\gamma + \gamma^3 A_+)^3}{(1+t)^{d+2}} + L^{-3(d-1)} \frac{B_+^3}{(1 + t/L)^{d-1}} \right], \tag{4.20}$$

where C is a positive constant depending only on d.

Proof. I will only prove the case of $d \geq 3$ for simplicity.

Let $(x, \xi) \in A_{\leq t/2}$. Then

$$\xi_1 = \frac{1}{t-\tau_1} \int_{\tau_1}^{t} W(s)\, ds \tag{4.21}$$

by $x(\tau_1) \in \partial C_W(\tau_1)$.

This and inequality (4.6) give an upper bound of $W(t) - \xi_1$:

$$\begin{aligned}
W(t) - \xi_1 &= \frac{1}{t-\tau_1} \int_{\tau_1}^{t} [(1 - W(s)) - (1 - W(t))]\, ds \\
&\leq \frac{1}{t-\tau_1} \int_{\tau_1}^{t} \left[\gamma e^{-D_0'(1/2)s} + \gamma^3 \frac{A_+}{(1+s)^{d+2}} + L^{-(d-1)} \frac{B_+}{(1+s/L)^{d-1}} \right] ds \\
&= \gamma e^{-D_0'(1/2)\tau_1} \frac{1 - e^{-D_0'(1/2)(t-\tau_1)}}{D_0'(1/2)(t-\tau_1)} \\
&\quad + \frac{\gamma^3 A_+}{(d+1)(t-\tau_1)} \left[\frac{1}{(1+\tau_1)^{d+1}} - \frac{1}{(1+t)^{d+1}} \right] \\
&\quad + \frac{L^{-(d-1)} B_+}{(d-2)(t-\tau_1)} \left[\frac{L}{(1+\tau_1/L)^{d-2}} - \frac{L}{(1+t/L)^{d-2}} \right].
\end{aligned} \tag{4.22}$$

By the mean value theorem, there exists $s \in (\tau_1, t)$ such that

$$\frac{1}{(1+\tau_1)^{d+1}} - \frac{1}{(1+t)^{d+1}} = \frac{(d+1)(t-\tau_1)(1+s)^d}{(1+\tau_1)^{d+1}(1+t)^{d+1}} \tag{4.23}$$

and similarly for the last term in inequality (4.22). Using the condition $\tau_1 \leq t/2$,

$$W(t) - \xi_1 \leq C \frac{\gamma + \gamma^3 A_+}{1+t} + L^{-(d-1)} \frac{B_+}{1+t/L}. \tag{4.24}$$

Next, note that $x(\tau_1) \in \partial C_W(\tau_1)$ implies $|x_\perp - (t-\tau_1)\xi_\perp| \leq 1$. Thus

$$|\xi_\perp| \leq 4/t \tag{4.25}$$

since $\tau_1 \leq t/2$.

Inequalities (4.16), (4.24) and (4.25) give

$$\begin{aligned}
I &\leq C \left[\frac{(\gamma + \gamma^3 A_+)^3}{(1+t)^3} + L^{-3(d-1)} \frac{B_+^3}{(1+t/L)^3} \right] \int_{|\xi_\perp| \leq 4/t} e^{-|\xi_\perp|^2}\, d\xi_\perp \\
&\leq C \left[\frac{(\gamma + \gamma^3 A_+)^3}{(1+t)^{d+2}} + L^{-3(d-1)} \frac{B_+^3}{(1+t/L)^{d-1}} \right].
\end{aligned} \tag{4.26}$$

□

II has the following bound:

Lemma 4.2. If $W \in \mathcal{K}$, then

$$II \leq C\left[\frac{(\gamma+\gamma^3 A_+)^3}{(1+t)^{3(d+2)}} + L^{-3(d-1)}\frac{B_+^3}{(1+t/L)^{3(d-1)}}\right], \tag{4.27}$$

where C is a positive constant depending only on d.

Proof. Let $(x,\xi) \in A_{>t/2}$. An upper bound of $W(t)-\xi_1$ is derived similarly to inequality (4.24):

$$\begin{aligned} W(t)-\xi_1 &\leq \frac{1}{t-\tau_1}\int_{\tau_1}^{t}\left[\gamma e^{-D_0'(1/2)s} + \gamma^3\frac{A_+}{(1+s)^{d+2}} + L^{-(d-1)}\frac{B_+}{(1+s/L)^{d-1}}\right] ds \\ &\leq \gamma e^{-D_0'(1/2)\tau_1} + \gamma^3\frac{A_+}{(1+\tau_1)^{d+2}} + L^{-(d-1)}\frac{B_+}{(1+\tau_1/L)^{d-1}} \\ &\leq C\left[\frac{\gamma+\gamma^3 A_+}{(1+t)^{d+2}} + L^{-(d-1)}\frac{B_+}{(1+t/L)^{d-1}}\right] \end{aligned} \tag{4.28}$$

since $\tau_1 > t/2$. This implies inequality (4.27). □

Proposition 4.2 is an immediate consequence of Lemmas 4.1 and 4.2.

4.2.2 Estimate of $r_W^-(t)$

The proof of the non-negativity of $r_W^-(t)$ is similar to that of $r_W^+(t)$. Note that

$$0 \leq f_0 - f_W \leq f_0 = \pi^{-d/2}e^{-|\xi|^2}. \tag{4.29}$$

Decompose $r_W^-(t)$ into two parts as follows:

$$\begin{aligned} r_W^-(t) &= \int_{I_W^-(t)\cap\{\xi_1\leq 1\}} (\xi_1 - W(t))^2(f_0-f_W)\, d\xi dS \\ &\quad + \int_{I_W^-(t)\cap\{\xi_1> 1\}} (\xi_1 - W(t))^2(f_0-f_W)\, d\xi dS \\ &=: III + IV. \end{aligned} \tag{4.30}$$

III has the following bound, which is easily proved by noting that $0 < \xi_1 - W(t) \leq 1 - W(t)$ and inequality (4.6).

Lemma 4.3. If $W \in \mathcal{K}$, then

$$III \leq C\left[\frac{(\gamma+\gamma^3 A_+)^3}{(1+t)^{3(d+2)}} + L^{-3(d-1)}\frac{B_+^3}{(1+t/L)^{3(d-1)}}\right], \tag{4.31}$$

where C is a positive constant depending only on d.

IV has the following bound:

Lemma 4.4. Let $W \in \mathcal{K}$. Take γ and L^{-1} sufficiently small so that inequality (4.6) implies $W(t) \geq 1/2$. Then

$$IV \leq C\frac{L^{-(d-1)}}{(1+t/L)^{d-1}}, \tag{4.32}$$

where C is a positive constant depending only on d.

Proof. Let $(x,\xi) \in I_W^-(t)$ and suppose that $\xi_1 > 1$. I can safely assume that $\tau_2 > 0$: Otherwise, there is at most one recollision, which is necessarily on $\partial\mathbb{R}^d_+$ since $\xi_1 > 1$. This implies that $|\xi_0| = |\xi|$ and hence $f_W = f_0$. These do not contribute to the integral defining $r_W^-(t)$.

Next, a bound of $|\xi_\perp|$ is derived as follows. Since the second recollision is necessarily on the left side of $\partial C_W(\tau_2)$, the following relation must hold:

$$(t-\tau_2)\xi_1 + \int_{\tau_2}^t W(s)\,ds = 2X_W(t). \tag{4.33}$$

Note that this implies

$$t - \tau_2 = \frac{2\left(L + \int_0^t W(s)\,ds\right)}{\xi_1 + \frac{1}{t-\tau_2}\int_{\tau_2}^t W(s)\,ds} \geq \frac{2(L+t/2)}{\xi_1+1} \tag{4.34}$$

because $1/2 \leq W(t) < 1$. Next, $x(\tau_2) \in \partial C_W(\tau_2)$ implies $|x_\perp - (t-\tau_2)\xi_\perp| < 1$. Hence

$$|\xi_\perp| \leq \frac{2}{t-\tau_2} \leq \frac{2(\xi_1+1)}{2L+t} \tag{4.35}$$

by inequality (4.34).

Inequalities(4.29) and (4.35) give the bound of IV:

$$\begin{aligned} IV &\leq C\int_1^\infty (\xi_1 - W(t))^2 e^{-\xi_1^2}\,d\xi_1 \int_{|\xi_\perp| \leq \frac{2(\xi_1+1)}{2L+t}} e^{-|\xi_\perp|^2}\,d\xi_\perp & (4.36)\\ &\leq C\frac{L^{-(d-1)}}{(1+t/L)^{d-1}}\int_1^\infty (\xi_1 - 1/2)^2(\xi_1+1)^{d-1}e^{-\xi_1^2}\,d\xi_1 & (4.37)\\ &\leq C\frac{L^{-(d-1)}}{(1+t/L)^{d-1}}. & (4.38)\end{aligned}$$

□

Proposition 4.3 is an immediate consequence of Lemmas 4.3 and 4.4.

5 Discussion

Now that Propositions 4.2 and 4.3 are proved, this concludes the proof of Theorem 3.2 — although I left some details to the references.

The effect of the wall $\partial\mathbb{R}^d_+$ is seen in the estimate of $r^-_W(t)$. See also the proof of its lower bound derived in [2, Proposition 8]. It is caused by molecules leaving the left side of $\partial C_W(t)$, reaching $\partial\mathbb{R}^d_+$ at time τ_1 and then coming back to the left side of $\partial C_W(\tau_2)$. See equation (4.33). An important point is that this effect dominates the rate of approach to the terminal velocity however large L is.

References

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[2] Koike, K.: Wall effect on the motion of a rigid body immersed in a free molecular flow. Kinet. Relat. Models (to appear)