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MAXIMAL REGULARITY FOR A COMPRESSIBLE FLUID MODEL OF KORTEWEG TYPE ON GENERAL DOMAINS

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ABSTRACT. This article reports the maximal regularity for a compressible fluid model of Korteweg type on general domains of the N-dimensional Euclidean space for $N \geq 2$ (e.g. the whole space; bounded domains; exterior domains; half-spaces, layers, tubes, and their perturbed domains). The detailed proof and extended results will be given in [17, 18].

1. INTRODUCTION

The motion of barotropic compressible viscous fluids is governed by

$$\begin{aligned} &\partial_t\rho + \operatorname{div}(\rho\mathbf{u}) = 0 && \text{(mass conservation)},\\ &\rho(\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}) = \operatorname{Div}(\mathbf{T} - P(\rho)\mathbf{I}) && \text{(momentum conservation)}, \end{aligned}$$

subject to initial conditions and suitable boundary conditions. Here $\rho = \rho(x,t)$ and $\mathbf{u} = \mathbf{u}(x,t) = (u_1(x,t),\ldots,u_N(x,t))^{\mathsf{T}}$[1] denote, respectively, the density field of the fluid and the velocity field of the fluid at $x \in \Omega$ and $t > 0$, where Ω is a domain of $\mathbf{R}^N$ for $N \geq 2$; $P : [0,\infty) \to \mathbf{R}$ is a given function describing the pressure field of the fluid; $\mathbf{T}$ is a stress tensor specified below, while $\mathbf{I}$ is the $N\times N$ identity matrix.

In this paper, we consider a compressible fluid model of Korteweg type, which means that the stress tensor has the following form: $\mathbf{T} = \mathbf{S}(\mathbf{u}) + \mathbf{K}(\rho)$ with

$$\begin{aligned} \mathbf{S}(\mathbf{u}) &= \mu\mathbf{D}(\mathbf{u}) + (\nu-\mu)\operatorname{div}\mathbf{u}\mathbf{I},\\ \mathbf{K}(\rho) &= \frac{\kappa}{2}\left(\Delta\rho^2 - |\nabla\rho|^2\right)\mathbf{I} - \kappa\nabla\rho\otimes\nabla\rho, \end{aligned}$$

where μ, ν denote viscosity coefficients and κ denotes a capillary coefficient. Note that $\mathbf{D}(\mathbf{u})$ is the doubled strain tensor, i.e. $\mathbf{D}(\mathbf{u}) = (D_{ij}(\mathbf{u}))$ with $D_{ij}(\mathbf{u}) = \partial_i u_j + \partial_j u_i$ for $\partial_j = \partial/\partial x_j$, and $\mathbf{a}\otimes\mathbf{b} = (a_i b_j)$ for any N-vectors $\mathbf{a} = (a_1,\ldots,a_N)^{\mathsf{T}}$, $\mathbf{b} = (b_1,\ldots,b_N)^{\mathsf{T}}$. Here $\mathbf{K}(\rho)$ is called *the Korteweg tensor*. In 1901, Korteweg formulated a constitutive equation for $\mathbf{T}$ that included density gradients (cf. also [5, Subsection 2.6]) in order to model fluid capillarity effects. Later on, Dunn and Serrin [3] derived rigorously $\mathbf{K}(\rho)$ as stated above in view of rational mechanics by introducing the thermomechanics of interstitial working.

This paper is concerned with the maximal regularity for a time-dependent linear system arising from the compressible fluid model of Korteweg type as follows:

$$\text{(1.1)}\quad \begin{cases} \partial_t\rho + \gamma_1\operatorname{div}\mathbf{u} = d & \text{in } \Omega,\ t>0,\\ \partial_t\mathbf{u} - \gamma_3^{-1}\operatorname{Div}(\mu\mathbf{D}(\mathbf{u}) + (\nu-\mu)\operatorname{div}\mathbf{u}\mathbf{I} + \gamma_2\kappa\Delta\rho\mathbf{I}) = \mathbf{f} & \text{in } \Omega,\ t>0,\\ \mathbf{n}\cdot\nabla\rho = g, \quad \mathbf{u} = 0 & \text{on } S,\ t>0,\\ (\rho,\mathbf{u})|_{t=0} = (\rho_0,\mathbf{u}_0) & \text{in } \Omega. \end{cases}$$

[1] $\mathbf{M}^{\mathsf{T}}$ denotes the transposed $\mathbf{M}$.

Here S is the boundary of Ω and $\mathbf{n}$ is the outward unit normal vector to S; the coefficients γ_i ($i = 1, 2, 3$), μ, ν, and κ are given functions with respect to $x \in \mathbf{R}^N$; $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^N a_i b_i$ for any N-vectors $\mathbf{a} = (a_1, \ldots, a_N)^\mathsf{T}$, $\mathbf{b} = (b_1, \ldots, b_N)^\mathsf{T}$; the right members d, $\mathbf{f}$, g, ρ_0, and $\mathbf{u}_0$ are given data. Here and subsequently, we use the following notation for differentiations: Let $u = u(x)$, $\mathbf{v} = (v_1(x), \ldots, v_N(x))^\mathsf{T}$, and $\mathbf{M} = (M_{ij}(x))$ be a scalar-, a vector-, and an $N \times N$ matrix-valued function defined on a domain of $\mathbf{R}^N$, and then

$$\nabla u = (\partial_1 u, \ldots, \partial_N u)^\mathsf{T}, \quad \Delta u = \sum_{i=1}^N \partial_i^2 u, \quad \Delta \mathbf{v} = (\Delta v_1, \ldots, \Delta v_N)^\mathsf{T},$$

$$\operatorname{div} \mathbf{v} = \sum_{i=1}^N \partial_i v_i, \quad \nabla \mathbf{v} = \{\partial_i v_j \mid i, j = 1, \ldots, N\},$$

$$\nabla^2 \mathbf{v} = \{\partial_i \partial_j v_k \mid i, j, k = 1, \ldots, N\}, \quad \operatorname{Div} \mathbf{M} = \left(\sum_{j=1}^N \partial_j M_{1j}, \ldots, \sum_{j=1}^N \partial_j M_{Nj}\right)^\mathsf{T}.$$

Kotschote [10] proved an optimal regularity for (1.1) with coefficients depending also on the time variable t. Roughly speaking, he proved in [10] that for a suitable exponent $p \in (1, \infty)$ the system (1.1) admits a unique solution $(\rho, \mathbf{u})$ on $J = (0, T)$, $T > 0$, with

$$\rho \in H_p^{3/2}(J, L_p(\Omega)) \cap L_p(J, H_p^3(\Omega)), \quad \mathbf{u} \in H_p^1(J, L_p(\Omega)^N) \cap L_p(J, H_p^2(\Omega)^N),$$

if and only if the data d, $\mathbf{f}$, g, ρ_0, and $\mathbf{u}_0$ satisfy the compatibility conditions and the following regularity conditions:

$$d \in H_p^{1/2}(J, L_p(\Omega)) \cap L_p(J, H_p^1(\Omega)), \quad \mathbf{f} \in L_p(J, L_p(\Omega)^N),$$
$$g \in H_p^1(J, L_p(\Omega)) \cap L_p(J, H_p^2(\Omega)), \quad (\rho_0, \mathbf{u}_0) \in B_{p,p}^{3-2/p}(\Omega) \times B_{p,p}^{2-2/p}(\Omega)^N.$$

On the other hand, the present paper relaxes the regularity of ρ with respect to the time variable t under the assumption that d only belongs to $L_p(J, H_p^1(\Omega))$ and extends the function spaces of solutions and date to an L_p-in-time and L_q-in-space setting (cf. Theorem 2.3 below for more details).

Concerning other boundary conditions, we refer to Kotschote [10, 11, 12, 13]. There are also several results, for the whole space case, such as Hattori and Li [8, 9], Danchin and Desjardins [1], Haspot [6, 7].

2. Notation and main results

This section first introduces the notation and function spaces, and then main results of this paper are stated.

2.1. **Notation.** Let $\mathbf{N}$ be the set of all natural numbers and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$, and let $\mathbf{R}$, $\mathbf{C}$ be respectively the set of all real numbers and the set of all complex numbers.

Let $q \in [1, \infty]$ and G be a domain of $\mathbf{R}^N$. Then $L_q(G)$ and $H_q^m(G)$, $m \in \mathbf{N}$, denote the usual $\mathbf{K}$-valued Lebesgue spaces on G and the usual $\mathbf{K}$-valued Sobolev spaces on G, respectively, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$. We set $H_q^0(G) = L_q(G)$ and denote the norm of $H_q^n(G)$, $n \in \mathbf{N}_0$, by $\| \cdot \|_{H_q^n(G)}$. In addition, $B_{q,p}^s(G)$ is the Besov spaces on G for further exponents $s > 0$ and $p \in (1, \infty)$. For a Banach space X and $\mathbf{R}_+ = (0, \infty)$, we denote respectively the X-valued Lebesgue spaces on $\mathbf{R}_+$

and the X-valued Sobolev spaces on $\mathbf{R}_+$ by $L_p(\mathbf{R}_+, X)$ and $H^1_p(\mathbf{R}_+, X)$, while we write the norm of $L_p(\mathbf{R}_+, X)$ as $\|\cdot\|_{L_p(\mathbf{R}_+,X)}$. One sets for $\delta > 0$

$$\begin{aligned}
L_{p,\delta}(\mathbf{R}_+, X) &= \{f \in L_{p,\mathrm{loc}}(\mathbf{R}_+, X) \mid e^{-\delta t} f(t) \in L_p(\mathbf{R}_+, X)\},\\
H^1_{p,\delta}(\mathbf{R}_+, X) &= \{f \in H^1_{p,\mathrm{loc}}(\mathbf{R}_+, X) \mid e^{-\delta t}\partial_t^k f(t) \in L_p(\mathbf{R}_+, X),\ k = 0, 1\},\\
{}_0H^1_{p,\delta}(\mathbf{R}_+, X) &= \{f \in H^1_{p,\delta}(\mathbf{R}_+, X) \mid f|_{t=0} = 0 \text{ in } X\},
\end{aligned}$$

and also

$$\begin{aligned}
H^{2,1}_{q,p,\delta}(G \times \mathbf{R}_+) &= H^1_{p,\delta}(\mathbf{R}_+, L_q(G)^N) \cap L_{p,\delta}(\mathbf{R}_+, H^2_q(G)^N),\\
{}_0H^{2,1}_{q,p,\delta}(G \times \mathbf{R}_+) &= {}_0H^1_{p,\delta}(\mathbf{R}_+, L_q(G)) \cap L_{p,\delta}(\mathbf{R}_+, H^2_q(G)).
\end{aligned}$$

Let X, Y be Banach spaces. Then $\mathcal{L}(X, Y)$ is the Banach space of all bounded linear operators from X to Y, and $\mathcal{L}(X)$ is the abbreviation of $\mathcal{L}(X, X)$. For a subset U of $\mathbf{C}$, $\mathrm{Hol}\,(U, \mathcal{L}(X, Y))$ stands for the set of all $\mathcal{L}(X, Y)$-valued holomorphic functions defined on U.

At this point, we introduce an assumption for the coefficients.

Assumption 2.1. *The coefficients $\gamma_i = \gamma_i(x)$ $(i = 1, 2, 3)$, $\mu = \mu(x)$, $\nu = \nu(x)$, and $\kappa = \kappa(x)$ are real valued uniformly continuous functions, defined on $\mathbf{R}^N$, which satisfy the following conditions:*

(1) *Let $i = 1, 2, 3$. There exist positive constants $\underline{\gamma_i}$, $\overline{\gamma_i}$, $\underline{\mu}$, $\overline{\mu}$, $\underline{\nu}$, $\overline{\nu}$, $\underline{\kappa}$, and $\overline{\kappa}$ such that for any $x \in \mathbf{R}^N$*

$$\underline{\gamma_i} \le \gamma_i(x) \le \overline{\gamma_i}, \quad \underline{\mu} \le \mu(x) \le \overline{\mu}, \quad \underline{\nu} \le \nu(x) \le \overline{\nu}, \quad \underline{\kappa} \le \kappa(x) \le \overline{\kappa}.$$

(2) *For any $x \in \mathbf{R}^N$,*

$$\left(\frac{\mu(x) + \nu(x)}{2\gamma_1(x)\gamma_2(x)\kappa(x)}\right)^2 - \frac{\gamma_3(x)}{\gamma_1(x)\gamma_2(x)\kappa(x)} \neq 0, \quad \kappa(x) \neq \frac{\mu(x)\nu(x)}{\gamma_1(x)\gamma_2(x)\gamma_3(x)}.$$

The definition of our general domains is given by

Definition 2.2. *Let $1 < r < \infty$ and G be a domain of $\mathbf{R}^N$ with boundary ∂G. We say that G is a uniform $W^{3-1/r}_r$ domain, if there exist positive constants α, β, and K such that for any $x_0 = (x_{01}, \ldots, x_{0N}) \in \partial G$ there are a coordinate number j and a $W^{3-1/r}_r$ function $h(x')$ $(x' = (x_1, \ldots, \widehat{x}_j, \ldots, x_N))$ defined on $B'_\alpha(x'_0)$, with $x'_0 = (x_{01}, \ldots, \widehat{x}_{0j}, \ldots, x_{0N})$ and $\|h\|_{W^{3-1/r}_r(B_{\alpha'}(x'_0))} \le K$, such that*

$$\begin{aligned}
G \cap B_\beta(x_0) &= \{x \in \mathbf{R}^N \mid x_j > h(x'), x' \in B'_\alpha(x'_0)\} \cap B_\beta(x_0),\\
\partial G \cap B_\beta(x_0) &= \{x \in \mathbf{R}^N \mid x_j = h(x'), x' \in B'_\alpha(x'_0)\} \cap B_\beta(x_0).
\end{aligned}$$

2.2. **Maximal regularity.** The maximal regularity for (1.1) is stated as follows:

Theorem 2.3. *Let $p, q \in (1, \infty)$ with $2/p + 1/q \neq 2$, and let $r \in (N, \infty)$ with* $\max(q, q') \le r$ *for $q' = q/(q-1)$. Assume that*

(a) *γ_i $(i = 1, 2, 3)$, μ, ν, and κ satisfy Assumption* 2.1*;*
(b) *$\nabla a \in L_r(\mathbf{R}^N)$ for $a \in \{\gamma_1, \gamma_2, \mu, \nu, \kappa\}$;*
(c) *Ω is a uniform $W^{3-1/r}_r$ domain;*

Then there is a constant $\delta_0 \ge 1$ such that the following assertions hold true.

(1) *For right members*

$$d \in L_{p,\delta_0}(\mathbf{R}_+, H^1_q(\Omega)), \quad \mathbf{f} \in L_{p,\delta_0}(\mathbf{R}_+, L_q(\Omega)^N), \quad g \in {}_0H^{2,1}_{q,p,\delta_0}(\Omega \times \mathbf{R}_+)$$

and for initial data $(\rho_0, \mathbf{u}_0) \in D_{q,p}(\Omega)$ *with*

$$D_{q,p}(\Omega) = \begin{cases} B^{3-2/p}_{q,p}(\Omega) \times B^{2-2/p}_{q,p}(\Omega)^N & \text{when } 2/p + 1/q > 2, \\ \{(\rho_0, \mathbf{u}_0) \in B^{3-2/p}_{q,p}(\Omega) \times B^{2-2/p}_{q,p}(\Omega)^N \mid \mathbf{n} \cdot \nabla \rho_0 = 0,\ \mathbf{u}_0 = 0 \text{ on } S\} \\ \text{when } 2/p + 1/q < 2, \end{cases}$$

the system (1.1) *admits a unique solution* $(\rho, \mathbf{u})$, *with*

$$(\rho, \mathbf{u}) \in (H^1_{p,\delta_0}(\mathbf{R}_+, H^1_q(\Omega)) \cap L_{p,\delta_0}(\mathbf{R}_+, H^3_q(\Omega))) \times H^{2,1}_{q,p,\delta_0}(\Omega \times \mathbf{R}_+),$$

$$\lim_{t \to 0+} \|(\rho, \mathbf{u}) - (\rho_0, \mathbf{u}_0)\|_{B^{3-2/p}_{q,p}(\Omega) \times B^{2-2/p}_{q,p}(\Omega)^N} = 0.$$

(2) *The solution* $(\rho, \mathbf{u})$ *satisfies the estimate:*

$$\begin{aligned} &\|e^{-\delta_0 t}\partial_t \rho\|_{L_p(\mathbf{R}_+, H^1_q(\Omega))} + \|e^{-\delta_0 t}\rho\|_{L_p(\mathbf{R}_+, H^3_q(\Omega))} \\ &\quad + \|e^{-\delta_0 t}\partial_t \mathbf{u}\|_{L_p(\mathbf{R}_+, L_q(\Omega)^N)} + \|e^{-\delta_0 t}\mathbf{u}\|_{L_p(\mathbf{R}_+, H^2_q(\Omega)^N)} \\ &\le C\Big(\|(\rho_0, \mathbf{u}_0)\|_{B^{3-2/p}_{q,p}(\Omega) \times B^{2-2/p}_{q,p}(\Omega)^N} + \|e^{-\delta_0 t} d\|_{L_p(\mathbf{R}_+, H^1_q(\Omega))} \\ &\quad + \|e^{-\delta_0 t}\mathbf{f}\|_{L_p(\mathbf{R}_+, L_q(\Omega)^N)} + \|e^{-\delta_0 t}\partial_t g\|_{L_p(\mathbf{R}_+, L_q(\Omega))} + \|e^{-\delta_0 t} g\|_{L_p(\mathbf{R}_+, H^2_q(\Omega))}\Big) \end{aligned}$$

for some positive constant C *depending on* N, p, q, r, *and* δ_0.

2.3. **$\mathcal{R}$-bounded solution operator families.** To show Theorem 2.3, we consider the following generalized resolvent problem:

$$\begin{cases} \lambda \rho + \gamma_1 \operatorname{div} \mathbf{u} = d & \text{in } \Omega, \\ \lambda \mathbf{u} - \gamma_3^{-1} \operatorname{Div}(\mu \mathbf{D}(\mathbf{u}) + (\nu - \mu) \operatorname{div} \mathbf{u} \mathbf{I} + \gamma_2 \kappa \Delta \rho \mathbf{I}) = \mathbf{f} & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla \rho = g, \quad \mathbf{u} = 0 & \text{on } S. \end{cases} \tag{2.1}$$

Here λ is the resolvent parameter varying in

$$\Sigma_{\varepsilon,\gamma} = \{\lambda \in \mathbf{C} \mid |\arg \lambda| < \pi - \varepsilon, |\lambda| > \gamma\} \quad (\varepsilon \in (0, \pi/2),\ \gamma \ge 0).$$

One recalls the definition of the $\mathcal{R}$-boundedness of operator families at this point.

Definition 2.4 ($\mathcal{R}$-boundedness). *Let* X *and* Y *be Banach spaces. A family of operators* $\mathcal{T} \subset \mathcal{L}(X, Y)$ *is called* $\mathcal{R}$*-bounded on* $\mathcal{L}(X, Y)$, *if there exist constants* $p \in [1, \infty)$ *and* $C > 0$ *such that the following assertion holds:*

For each natural number m, $\{T_j\}_{j=1}^m \subset \mathcal{T}$, $\{f_j\}_{j=1}^m \subset X$ *and for all sequences* $\{r_j(u)\}_{j=1}^m$ *of independent, symmetric,* $\{-1, 1\}$*-valued random variables on* $[0, 1]$, *there holds the inequality*

$$\left(\int_0^1 \Big\|\sum_{j=1}^m r_j(u) T_j f_j\Big\|_Y^p du\right)^{1/p} \le C \left(\int_0^1 \Big\|\sum_{j=1}^m r_j(u) f_j\Big\|_X^p du\right)^{1/p}.$$

The smallest such C *is called* $\mathcal{R}$*-bound of* $\mathcal{T}$ *on* $\mathcal{L}(X, Y)$ *and denoted by* $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Remark 2.5. The constant C in Definition 2.4 depends on p. It is known that $\mathcal{T}$ is $\mathcal{R}$-bounded for any $p \in [1, \infty)$, provided that $\mathcal{T}$ is $\mathcal{R}$-bounded for some $p \in [1, \infty)$. This fact follows from Kahane's inequality (cf. e.g. [14, Theorem 2.4]).

Let G be a domain of $\mathbf{R}^N$. For the right member $(d, \mathbf{f}, g)$ of (2.1), we set

$$\text{(2.2)} \qquad \mathcal{X}_q(G) = H^1_q(G) \times L_q(G)^N \times H^2_q(G), \quad \mathcal{X}^1_q(G) = H^1_q(G) \times L_q(G)^N.$$

Let $\mathbf{F} = (d, \mathbf{f}, g) \in \mathcal{X}_q(G)$ and $\mathbf{F}^1 = (d, \mathbf{f}) \in \mathcal{X}^1_q(G)$, and then the symbols $\mathfrak{X}_q(G)$, $\mathcal{F}_\lambda \mathbf{F}$ and the symbols $\mathfrak{X}^i_q(G)$, $\mathcal{F}^i_\lambda \mathbf{F}^i$ $(i = 0, 1)$ are defined as follows:

$$\text{(2.3)} \quad \begin{aligned} &\mathfrak{X}_q(G) = H^1_q(G) \times L_q(G)^{N+N^2+N+1}, \quad \mathcal{F}_\lambda \mathbf{F} = (d, \mathbf{f}, \nabla^2 g, \lambda^{1/2}\nabla g, \lambda g); \\ &\mathfrak{X}^0_q(G) = L_q(G)^{N+1+N+N^2+N+1}, \quad \mathcal{F}^0_\lambda \mathbf{F} = (\nabla d, \lambda^{1/2} d, \mathbf{f}, \nabla^2 g, \lambda^{1/2}\nabla g, \lambda g); \\ &\mathfrak{X}^1_q(G) = L_q(G)^{N+1+N}, \quad \mathcal{F}^1_\lambda \mathbf{F}^1 = (\nabla d, \lambda^{1/2} d, \mathbf{f}). \end{aligned}$$

One also sets for solutions of (2.1)

$$\text{(2.4)} \quad \begin{aligned} &\mathfrak{A}_q(G) = L_q(G)^{N^3+N^2} \times H^1_q(G), \quad && \mathcal{S}_\lambda \rho = (\nabla^3 \rho, \lambda^{1/2}\nabla^2 \rho, \lambda \rho); \\ &\mathfrak{A}^0_q(G) = L_q(G)^{N^3+N^2+N+1}, && \mathcal{S}^0_\lambda \rho = (\nabla^3 \rho, \lambda^{1/2}\nabla^2 \rho, \lambda\nabla\rho, \lambda^{3/2}\rho); \\ &\mathfrak{B}_q(G) = L_q(G)^{N^3+N^2+N}, && \mathcal{T}_\lambda \mathbf{u} = (\nabla^2 \mathbf{u}, \lambda^{1/2}\nabla \mathbf{u}, \lambda \mathbf{u}). \end{aligned}$$

Now we state the existence of $\mathcal{R}$-bounded solution operator families for (2.1).

Theorem 2.6. *Let $q \in (1, \infty)$ and $r \in (N, \infty)$, and let $\max(q, q') \le r$ for $q' = q/(q-1)$. Assume that*

(a) *γ_i $(i = 1, 2, 3)$, μ, ν, and κ satisfy Assumption 2.1;*
(b) *$\nabla a \in L_r(\mathbf{R}^N)$ for $a \in \{\gamma_1, \gamma_2, \mu, \nu, \kappa\}$;*
(c) *Ω is a uniform $W^{3-1/r}_r$ domain.*

Then there exists $\varepsilon_0 \in (0, \pi/2)$ such that for any $\varepsilon \in (\varepsilon_0, \pi/2)$ there is a constant $\lambda_0 \ge 1$ such that the following assertions hold true.

(1) *For any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ there are operators $\mathcal{A}(\lambda)$ and $\mathcal{B}(\lambda)$, with*

$$\mathcal{A}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathfrak{X}_q(\Omega), H^3_q(\Omega))),$$
$$\mathcal{B}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathfrak{X}_q(\Omega), H^2_q(\Omega)^N)),$$

such that, for $\mathbf{F} = (d, \mathbf{f}, g) \in \mathcal{X}_q(\Omega)$, $(\rho, \mathbf{u}) = (\mathcal{A}(\lambda)\mathcal{F}_\lambda \mathbf{F}, \mathcal{B}(\lambda)\mathcal{F}_\lambda \mathbf{F})$ is a unique solution to the system (2.1).

(2) *There is a positive constant C, depending on N, q, r, ε, ε_0, and λ_0, such that for $n = 0, 1$*

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q(\Omega), \mathfrak{A}_q(\Omega))} \left(\left\{ \left(\lambda \frac{d}{d\lambda} \right)^n (\mathcal{S}_\lambda \mathcal{A}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0} \right\} \right) \le C,$$
$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q(\Omega), \mathfrak{B}_q(\Omega))} \left(\left\{ \left(\lambda \frac{d}{d\lambda} \right)^n (\mathcal{T}_\lambda \mathcal{B}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0} \right\} \right) \le C.$$

Remark 2.7. One can prove Theorem 2.3 by combining Theorem 2.6 with the operator-valued Fourier multiplier theorem due to Weis [21, Theorem 3.4] and the theory of analytic semigroups (cf. e.g. [16, 19]). From this viewpoint, we give an outline of the proof of Theorem 2.6 in the following sections.

3. Whole space problems

This section is concerned with whole space problems as follows:

$$\text{(3.1)} \qquad \begin{cases} \lambda \rho + \gamma_1 \operatorname{div} \mathbf{u} = d & \text{in } \mathbf{R}^N, \\ \lambda \mathbf{u} - \gamma_3^{-1} \operatorname{Div}(\mu \mathbf{D}(\mathbf{u}) + (\nu - \mu) \operatorname{div} \mathbf{u} \mathbf{I} + \gamma_2 \kappa \Delta \rho \mathbf{I}) = \mathbf{f} & \text{in } \mathbf{R}^N, \end{cases}$$

$$
(3.2)\qquad \begin{cases} \lambda\rho + \operatorname{div}\mathbf{u} = d & \text{in } \mathbf{R}^N, \\ \lambda\mathbf{u} - \mu_*\Delta\mathbf{u} - \nu_*\nabla\operatorname{div}\mathbf{u} - \kappa_*\Delta\nabla\rho = \mathbf{f} & \text{in } \mathbf{R}^N, \end{cases}
$$

where μ_*, ν_*, and κ_* are positive constants. Concerning these systems, one has the following two theorems (cf. [17, 18] for the details).

Theorem 3.1. *Let $q \in (1,\infty)$ and $r \in (N,\infty)$ with $\max(q,q') \le r$ for $q' = q/(q-1)$, and let $\mathcal{X}^1_q(\mathbf{R}^N)$ be given in (2.2) for $G = \mathbf{R}^N$. Assume that the assumptions (a), (b) of Theorem 2.6 hold. Then there exists a constant $\varepsilon_* \in (0,\pi/2)$ such that for any $\varepsilon \in (\varepsilon_*,\pi/2)$ there exists a constant $\lambda_* \ge 1$ such that the following assertions hold true.*

(1) *For any $\lambda \in \Sigma_{\varepsilon,\lambda_*}$ there are operators $\Phi(\lambda)$, $\Psi(\lambda)$, with*

$$
\Phi(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_*}, \mathcal{L}(\mathcal{X}^1_q(\mathbf{R}^N), H^3_q(\mathbf{R}^N))),
$$
$$
\Psi(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_*}, \mathcal{L}(\mathcal{X}^1_q(\mathbf{R}^N), H^2_q(\mathbf{R}^N)^N)),
$$

such that, for $\mathbf{F}^1 = (d,\mathbf{f}) \in \mathcal{X}^1_q(\mathbf{R}^N)$, $(\rho,\mathbf{u}) = (\Phi(\lambda)\mathbf{F}^1, \Psi(\lambda)\mathbf{F}^1)$ is a unique solution to the system (3.1) .

(2) *There exists a positive constant C, depending on N, q, r, ε, ε_*, and λ_*, such that for $n = 0,1$*

$$
\mathcal{R}_{\mathcal{L}(\mathcal{X}^1_q(\mathbf{R}^N),\mathfrak{A}_q(\mathbf{R}^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{S}_\lambda\Phi(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_*}\right\}\right) \le C,
$$
$$
\mathcal{R}_{\mathcal{L}(\mathcal{X}^1_q(\mathbf{R}^N),\mathfrak{B}_q(\mathbf{R}^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda\Psi(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_*}\right\}\right) \le C,
$$

where $\mathfrak{A}(\mathbf{R}^N)$, $\mathfrak{B}(\mathbf{R}^N)$, $\mathcal{S}_\lambda$, and $\mathcal{T}_\lambda$ are given in (2.4) for $G = \mathbf{R}^N$.

Theorem 3.2. *Let $q \in (1,\infty)$, and let $\mathcal{X}^1_q(\mathbf{R}^N)$, $\mathfrak{X}^1_q(\mathbf{R}^N)$, and $\mathcal{F}^1_\lambda$ be given in (2.2) and (2.3) for $G = \mathbf{R}^N$. Assume that μ_*, ν_*, and κ_* are positive constants. Then there exists a constant $\varepsilon_1 \in (0,\pi/2)$ such that for any $\varepsilon \in (\varepsilon_1,\pi/2)$ the following assertions hold true.*

(1) *For any $\lambda \in \Sigma_{\varepsilon,0}$ there are operators $\mathcal{A}^1(\lambda)$, $\mathcal{B}^1(\lambda)$, with*

$$
\mathcal{A}^1(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,0}, \mathcal{L}(\mathfrak{X}^1_q(\mathbf{R}^N), H^3_q(\mathbf{R}^N))),
$$
$$
\mathcal{B}^1(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,0}, \mathcal{L}(\mathfrak{X}^1_q(\mathbf{R}^N), H^2_q(\mathbf{R}^N)^N)),
$$

such that, for $\mathbf{F}^1 = (d,\mathbf{f}) \in \mathcal{X}^1_q(\mathbf{R}^N)$, $(\rho,\mathbf{u}) = (\mathcal{A}^1(\lambda)\mathcal{F}^1_\lambda\mathbf{F}^1, \mathcal{B}^1(\lambda)\mathcal{F}^1_\lambda\mathbf{F}^1)$ is a unique solution to the system (3.2) .

(2) *There exists a positive constant C, depending on at most N, q, ε, ε_1, μ_*, ν_*, and κ_*, such that for $n = 0,1$*

$$
\mathcal{R}_{\mathcal{L}(\mathfrak{X}^1_q(\mathbf{R}^N),\mathfrak{A}^0_q(\mathbf{R}^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{S}^0_\lambda\mathcal{A}^1(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,0}\right\}\right) \le C,
$$
$$
\mathcal{R}_{\mathcal{L}(\mathfrak{X}^1_q(\mathbf{R}^N),\mathfrak{B}_q(\mathbf{R}^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda\mathcal{B}^1(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,0}\right\}\right) \le C,
$$

where $\mathfrak{A}^0(\mathbf{R}^N)$, $\mathfrak{B}(\mathbf{R}^N)$, $\mathcal{S}^0_\lambda$, and $\mathcal{T}_\lambda$ are given in (2.4) for $G = \mathbf{R}^N$.

In the last part of this section, we introduce some fundamental properties of the $\mathcal{R}$-boundedness that are used in the following sections (cf. [2, Proposition 3.4]).

Proposition 3.3. *Let X, Y, and Z be Banach spaces. Then the following assertions hold true.*

(1) *Let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families on $\mathcal{L}(X,Y)$. Then $\mathcal{T}+\mathcal{S}=\{T+S \mid T\in\mathcal{T}, S\in\mathcal{S}\}$ is also $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, and also $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}+\mathcal{S})\leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})+\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S})$.*

(2) *Let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families on $\mathcal{L}(X,Y)$ and on $\mathcal{L}(Y,Z)$, respectively. Then $\mathcal{S}\mathcal{T}=\{ST \mid S\in\mathcal{S}, T\in\mathcal{T}\}$ is also $\mathcal{R}$-bounded on $\mathcal{L}(X,Z)$, and also $\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{S}\mathcal{T})\leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(Y,Z)}(\mathcal{S})$.*

4. Half-space problem

This section is concerned with the following half-space problem:

$$\begin{cases} \lambda\rho+\operatorname{div}\mathbf{u}=d & \text{in } \mathbf{R}^N_+,\\ \lambda\mathbf{u}-\mu_*\Delta\mathbf{u}-\nu_*\nabla\operatorname{div}\mathbf{u}-\kappa_*\Delta\nabla\rho=\mathbf{f} & \text{in } \mathbf{R}^N_+,\\ \mathbf{n}_0\cdot\nabla\rho=g, \quad \mathbf{u}=0 & \text{on } \mathbf{R}^N_0, \end{cases} \tag{4.1}$$

where $\mathbf{n}_0=(0,\dots,0,-1)^{\mathsf{T}}$. Set

$$\mathcal{X}^2_q(\mathbf{R}^N_+)=\mathcal{X}_q(\mathbf{R}^N_+), \quad \mathfrak{X}^2_q(\mathbf{R}^N_+)=\mathfrak{X}^0_q(\mathbf{R}^N_+),$$
$$\mathcal{F}^2_\lambda\mathbf{F}^2=\mathcal{F}^0_\lambda\mathbf{F}^2 \quad (\mathbf{F}^2=(d,\mathbf{f},g)\in\mathcal{X}^2_q(\mathbf{R}^N_+))$$

for $\mathcal{X}_q(\mathbf{R}^N_+)$, $\mathfrak{X}^0_q(\mathbf{R}^N_+)$, and $\mathcal{F}^0_\lambda$ given in (2.2) and (2.3) with $G=\mathbf{R}^N_+$. The aim of this section is to prove

Theorem 4.1. *Let $q\in(1,\infty)$, and let $\mathcal{X}^2_q(\mathbf{R}^N_+)$, $\mathfrak{X}^2_q(\mathbf{R}^N_+)$, and $\mathcal{F}^2_\lambda$ be as above. Assume that μ_*, ν_*, and κ_* are positive constants satisfying*

$$\eta_*:=\left(\frac{\mu_*+\nu_*}{2\kappa_*}\right)^2-\frac{1}{\kappa_*}\neq 0, \quad \kappa_*\neq\mu_*\nu_*. \tag{4.2}$$

Then there exists a constant $\varepsilon_2\in(\varepsilon_1,\pi/2)$, where ε_1 is the same constant as in Theorem 3.1, such that for any $\varepsilon\in(\varepsilon_2,\pi/2)$ the following assertions hold true.

(1) *For any $\lambda\in\Sigma_{\varepsilon,0}$ there are operators $\mathcal{A}^2(\lambda)$, $\mathcal{B}^2(\lambda)$, with*

$$\mathcal{A}^2(\lambda)\in\operatorname{Hol}(\Sigma_{\varepsilon,0},\mathcal{L}(\mathfrak{X}^2_q(\mathbf{R}^N_+),H^3_q(\mathbf{R}^N_+))),$$
$$\mathcal{B}^2(\lambda)\in\operatorname{Hol}(\Sigma_{\varepsilon,0},\mathcal{L}(\mathfrak{X}^2_q(\mathbf{R}^N_+),H^2_q(\mathbf{R}^N_+)^N)),$$

such that, for $\mathbf{F}^2=(d,\mathbf{f},g)\in\mathcal{X}^2_q(\mathbf{R}^N_+)$, $(\rho,\mathbf{u})=(\mathcal{A}^2(\lambda)\mathcal{F}^2_\lambda\mathbf{F}^2,\mathcal{B}^2(\lambda)\mathcal{F}^2_\lambda\mathbf{F}^2)$ is a unique solution to the system (4.1).

(2) *There exists a positive constant C, depending on at most N, q, ε, ε_1, ε_2, μ_*, ν_*, and κ_*, such that for $n=0,1$*

$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}^2_q(\mathbf{R}^N_+),\mathfrak{A}^0_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n(\mathcal{S}^0_\lambda\mathcal{A}^2(\lambda)) \mid \lambda\in\Sigma_{\varepsilon,0}\right\}\right)\leq C,$$
$$\mathcal{R}_{\mathcal{L}(\mathfrak{X}^2_q(\mathbf{R}^N_+),\mathfrak{B}_q(\mathbf{R}^N_+))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n(\mathcal{T}_\lambda\mathcal{B}^2(\lambda)) \mid \lambda\in\Sigma_{\varepsilon,0}\right\}\right)\leq C,$$

where $\mathfrak{A}^0_q(\mathbf{R}^N_+)$, $\mathfrak{B}_q(\mathbf{R}^N_+)$, $\mathcal{S}^0_\lambda$, and $\mathcal{T}_\lambda$ are given in (2.4) for $G=\mathbf{R}^N_+$.

Remark 4.2. The uniqueness of solutions for (4.1) follows from the existence of solutions for a dual problem (cf. e.g. [17]), so that we only discusses the existence of $\mathcal{A}^2(\lambda)$, $\mathcal{B}^2(\lambda)$ in what follows.

4.1. **Reduction to** $(d, \mathbf{f}) = (0,0)$. To show Theorem 4.1, we reduce the system (4.1) to the case where $(d, \mathbf{f}) = (0,0)$ in this subsection.

For $f = f(x)$ with $x = (x', x_N) = (x_1, \ldots, x_{N-1}, x_N) \in \mathbf{R}^N_+$, let $E^e f$ and $E^o f$ be the even extension of f and the odd extension of f, respectively, i.e.

$$E^e f = (E^e f)(x) = \begin{cases} f(x', x_N) & (x_N > 0), \\ f(x', -x_N) & (x_N < 0), \end{cases}$$

$$E^o f = (E^o f)(x) = \begin{cases} f(x', x_N) & (x_N > 0), \\ -f(x', -x_N) & (x_N < 0). \end{cases}$$

One then notes that $E^e \in \mathcal{L}(H^1_q(\mathbf{R}^N_+), H^1_q(\mathbf{R}^N))$. In addition, setting for $\mathbf{f} = (f_1, \ldots, f_N)^{\mathsf{T}}$ defined on $\mathbf{R}^N_+$

$$\mathbf{E}\mathbf{f} = (E^e f_1, \ldots, E^e f_{N-1}, E^o f_N)^{\mathsf{T}},$$

we see that $\mathbf{E} \in \mathcal{L}(L_q(\mathbf{R}^N_+)^N, L_q(\mathbf{R}^N)^N)$.

Let $\mathcal{A}^1(\lambda)$ and $\mathcal{B}^1(\lambda)$ be the operators constructed in Theorem 3.2, and set for $(d, \mathbf{f}) \in H^1_q(\mathbf{R}^N_+) \times L_q(\mathbf{R}^N_+)^N$

$$R = \mathcal{A}^1(\lambda)\mathcal{F}^1_\lambda(E^e d, \mathbf{E}\mathbf{f}), \quad \mathbf{U} = \mathcal{B}^1(\lambda)\mathcal{F}^1_\lambda(E^e d, \mathbf{E}\mathbf{f}).$$

Furthermore, let $S = S(x', x_N)$ and $\mathbf{V} = \mathbf{V}(x', x_N)$ be defined as

$$S = R(x', -x_N), \quad \mathbf{V} = (U_1(x', -x_N), \ldots, U_{N-1}(x', -x_N), -U_N(x', -x_N))^{\mathsf{T}}.$$

Here and subsequently, U_J and V_J denote respectively the Jth component of $\mathbf{U}$ and the Jth component of $\mathbf{V}$ for $J = 1, \ldots, N$. It then holds that

$$\begin{aligned} &(\lambda S + \operatorname{div} \mathbf{V})(x', x_N) \\ &= (\lambda R + \operatorname{div} \mathbf{U})(x', -x_N) = (E^e d)(x', -x_N) = (E^e d)(x', x_N) \end{aligned}$$

and that for $j = 1, \ldots, N-1$

$$\begin{aligned} (\lambda V_j - \mu_* \Delta V_j - \nu_* \partial_j \operatorname{div} \mathbf{V} - \kappa_* \Delta \partial_j S)(x', x_N) &= (E^e f_j)(x', x_N), \\ (\lambda V_N - \mu_* \Delta V_N - \nu_* \partial_N \operatorname{div} \mathbf{V} - \kappa_* \Delta \partial_N S)(x', x_N) &= (E^o f_N)(x', x_N). \end{aligned}$$

Thus, by the uniqueness of solutions to (3.2), we have $\mathbf{U}(x', x_N) = \mathbf{V}(x', x_N)$. Setting $x_N = 0$ in the last identity implies $U_N(x', 0) = 0$.

Let $\rho = R + \widetilde{\rho}$ and $\mathbf{u} = \mathbf{U} + \widetilde{\mathbf{u}}$ in (4.1). We then achieve, by $U_N = 0$ on $\mathbf{R}^N_0$ mentioned above, the following reduced system:

$$\text{(4.3)} \qquad \begin{cases} \lambda \widetilde{\rho} + \operatorname{div} \widetilde{\mathbf{u}} = 0 & \text{in } \mathbf{R}^N_+, \\ \lambda \widetilde{\mathbf{u}} - \mu_* \Delta \widetilde{\mathbf{u}} - \nu_* \nabla \operatorname{div} \widetilde{\mathbf{u}} - \kappa_* \Delta \nabla \widetilde{\rho} = 0 & \text{in } \mathbf{R}^N_+, \\ \mathbf{n}_0 \cdot \nabla \widetilde{\rho} = \widetilde{g}, \quad \widetilde{u}_j = \widetilde{l}_j, \quad \widetilde{u}_N = 0 & \text{on } \mathbf{R}^N_0, \end{cases}$$

for $\widetilde{g}$ and $\widetilde{l}_j$ $(j = 1, \ldots, N-1)$ given by

$$\text{(4.4)} \qquad \widetilde{g} = g - \mathbf{n}_0 \cdot \nabla \mathcal{A}^1(\lambda)\mathcal{F}^1_\lambda(E^e d, \mathbf{E}\mathbf{f}), \quad \widetilde{l}_j = -(\mathcal{B}^1(\lambda)\mathcal{F}^1_\lambda(E^e d, \mathbf{E}\mathbf{f}))_j,$$

where $(\mathbf{v})_j$ denotes the jth component of $\mathbf{v}$.

The reduced versions of $\mathcal{X}^2_q(\mathbf{R}^N_+)$, $\mathfrak{X}^2_q(\mathbf{R}^N_+)$, and $\mathcal{F}^2_\lambda$ are respectively denoted by $\widetilde{\mathcal{X}}^2_q(\mathbf{R}^N_+)$, $\widetilde{\mathfrak{X}}^2_q(\mathbf{R}^N_+)$, and $\widetilde{\mathcal{F}}^2_\lambda$, that is, one sets $\widetilde{\mathcal{X}}^2_q(\mathbf{R}^N_+) = H^2_q(\mathbf{R}^N_+)^N$ and sets for $\widetilde{\mathbf{F}}^2 = (\widetilde{g}, \widetilde{l}_1, \ldots, \widetilde{l}_{N-1}) \in \widetilde{\mathcal{X}}^2_q(\mathbf{R}^N_+)$

$$\widetilde{\mathcal{F}}^2_\lambda \widetilde{\mathbf{F}}^2 = (\nabla^2 \widetilde{\mathbf{F}}^2, \lambda^{1/2} \nabla \widetilde{\mathbf{F}}^2, \lambda \widetilde{\mathbf{F}}^2) \in \widetilde{\mathfrak{X}}^2_q(\mathbf{R}^N_+), \quad \widetilde{\mathfrak{X}}^2_q(\mathbf{R}^N_+) = L_q(\mathbf{R}^N_+)^{N^3+N^2+N}.$$

Concerning the system (4.3), we prove

Theorem 4.3. *Let $q \in (1, \infty)$, and let $\widetilde{\mathcal{X}}_q^2(\mathbf{R}_+^N)$, $\widetilde{\mathfrak{X}}_q^2(\mathbf{R}_+^N)$, and $\widetilde{\mathcal{F}}_\lambda^2$ be as above. Assume that μ_*, ν_*, and κ_* are positive constants satisfying* (4.2). *Then there is an $\widetilde{\varepsilon}_2 \in (0, \pi/2)$ such that for any $\varepsilon \in (\widetilde{\varepsilon}_2, \pi/2)$ the following assertions hold true.*

(1) *For any $\lambda \in \Sigma_{\varepsilon,0}$ there are operator families $\widetilde{\mathcal{A}}^2(\lambda)$, $\widetilde{\mathcal{B}}^2(\lambda)$, with*

$$\widetilde{\mathcal{A}}^2(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,0}, \mathcal{L}(\widetilde{\mathfrak{X}}_q^2(\mathbf{R}_+^N), H_q^3(\mathbf{R}_+^N))),$$
$$\widetilde{\mathcal{B}}^2(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,0}, \mathcal{L}(\widetilde{\mathfrak{X}}_q^2(\mathbf{R}_+^N), H_q^2(\mathbf{R}_+^N)^N)),$$

such that $(\widetilde{\rho}, \widetilde{\mathbf{u}}) = (\widetilde{\mathcal{A}}^2(\lambda)\widetilde{\mathcal{F}}_\lambda^2\widetilde{\mathbf{F}}^2, \widetilde{\mathcal{B}}^2(\lambda)\widetilde{\mathcal{F}}_\lambda^2\widetilde{\mathbf{F}}^2)$ is a unique solution to the system (4.3) *for $\widetilde{\mathbf{F}}^2 = (\widetilde{g}, \widetilde{l}_1, \ldots, \widetilde{l}_{N-1}) \in \widetilde{\mathcal{X}}_q^2(\mathbf{R}_+^N)$.*

(2) *There exists a positive constant C, depending on at most N, q, ε, $\widetilde{\varepsilon}_2$ μ_*, ν_*, and κ_*, such that for $n = 0, 1$*

$$\mathcal{R}_{\mathcal{L}(\widetilde{\mathfrak{X}}_q^2(\mathbf{R}_+^N), \mathfrak{A}_q^0(\mathbf{R}_+^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{S}_\lambda^0 \widetilde{\mathcal{A}}^2(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,0}\right\}\right) \le C,$$
$$\mathcal{R}_{\mathcal{L}(\widetilde{\mathfrak{X}}_q^2(\mathbf{R}_+^N), \mathfrak{B}_q(\mathbf{R}_+^N))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda \widetilde{\mathcal{B}}^2(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,0}\right\}\right) \le C,$$

where $\mathfrak{A}_q^0(\mathbf{R}_+^N)$, $\mathfrak{B}_q(\mathbf{R}_+^N)$, $\mathcal{S}_\lambda^0$, and $\mathcal{T}_\lambda$ are given in (2.4) *for $G = \mathbf{R}_+^N$.*

If we prove Theorem 4.3, then we have Theorem 4.1 immediately with the following observation: Noting $\nabla E^e d = \mathbf{E}\nabla d$, we see that

$$\mathcal{F}_\lambda^1(E^e d, \mathbf{E}\mathbf{f}) = (\mathbf{E}\nabla d, E^e(\lambda^{1/2}d), \mathbf{E}\mathbf{f}).$$

In view of this relation and (4.4), one sets for $\mathbf{F}^2 = (d, \mathbf{f}, g) \in \mathcal{X}_q^2(\mathbf{R}_+^N)$

$$\mathcal{A}^2(\lambda)\mathcal{F}_\lambda^2\mathbf{F}^2 = \mathcal{A}^1(\lambda)(\mathbf{E}\nabla d, E^e(\lambda^{1/2}d), \mathbf{E}\mathbf{f}) + \widetilde{\mathcal{A}}^2(\lambda)\widetilde{\mathcal{F}}_\lambda^2\widetilde{\mathbf{F}}^2,$$
$$\mathcal{B}^2(\lambda)\mathcal{F}_\lambda^2\mathbf{F}^2 = \mathcal{B}^1(\lambda)(\mathbf{E}\nabla d, E^e(\lambda^{1/2}d), \mathbf{E}\mathbf{f}) + \widetilde{\mathcal{B}}^2(\lambda)\widetilde{\mathcal{F}}_\lambda^2\widetilde{\mathbf{F}}^2,$$

where $\widetilde{\mathbf{F}}^2$ is given by

$$\begin{aligned}\widetilde{\mathbf{F}}^2 = \big(&g - \mathbf{n}_0 \cdot \nabla\mathcal{A}^1(\lambda)(\mathbf{E}\nabla d, E^e(\lambda^{1/2}d), \mathbf{E}\mathbf{f}),\\ &- (\mathcal{B}^1(\lambda)(\mathbf{E}\nabla d, E^e(\lambda^{1/2}d), \mathbf{E}\mathbf{f}))_1, \ldots,\\ &- (\mathcal{B}^1(\lambda)(\mathbf{E}\nabla d, E^e(\lambda^{1/2}d), \mathbf{E}\mathbf{f}))_{N-1}\big)^{\mathsf{T}}.\end{aligned}$$

It is then clear that $(\rho, \mathbf{u}) = (\mathcal{A}^2(\lambda)\mathcal{F}_\lambda^2\mathbf{F}^2, \mathcal{B}^2(\lambda)\mathcal{F}_\lambda^2\mathbf{F}^2)$ is a solution to the system (4.1), and also $\mathcal{A}^2(\lambda)$, $\mathcal{B}^2(\lambda)$ satisfy the required inequalities of Theorem 4.1 (2) by Proposition 3.3 and Theorems 3.2, 4.3. This completes the proof of Theorem 4.1, so that it suffices to show Theorem 4.3 in the following subsections.

4.2. **$\mathcal{R}$-bounded solution operator families for** (4.3). This subsection constructs $\mathcal{R}$-bounded solution operator families for the system (4.3).

One firsts computes the representation formulas of solutions of (4.3). Here and subsequently, we denote $\widetilde{\rho}$, $\widetilde{\mathbf{u}} = (\widetilde{u}_1, \ldots, \widetilde{u}_N)^{\mathsf{T}}$, $\widetilde{g}$, and $\widetilde{l}_j$ ($j = 1, \ldots, N-1$) by ρ, $\mathbf{u} = (u_1, \ldots, u_N)^{\mathsf{T}}$, g, and l_j, respectively, for notational simplicity.

Let us define the partial Fourier transform with respect to $x' = (x_1, \ldots, x_{N-1})$ and its inverse transform by

$$\widehat{u} = \widehat{u}(x_N) = \widehat{u}(\xi', x_N) = \int_{\mathbf{R}^{N-1}} e^{-ix'\cdot\xi'} u(x', x_N)\, dx',$$

$$\mathcal{F}_{\xi'}^{-1}[v(\xi',x_N)](x') = \frac{1}{(2\pi)^{N-1}}\int_{\mathbf{R}^{N-1}} e^{ix'\cdot\xi'} v(\xi',x_N)\,d\xi'.$$

Set $\varphi = \operatorname{div}\mathbf{u}$. Applying the partial Fourier transform to the system (4.3) yields the ordinary differential equations:

$$\lambda\widehat{\rho} + \widehat{\varphi} = 0, \ x_N > 0, \tag{4.5}$$

$$\lambda\widehat{u}_j - \mu_*(\partial_N^2 - |\xi'|^2)\widehat{u}_j - \nu_* i\xi_j\widehat{\varphi} - \kappa_* i\xi_j(\partial_N^2 - |\xi'|^2)\widehat{\rho} = 0, \ x_N > 0, \tag{4.6}$$

$$\lambda\widehat{u}_N - \mu_*(\partial_N^2 - |\xi'|^2)\widehat{u}_N - \nu_*\partial_N\widehat{\varphi} - \kappa_*\partial_N(\partial_N^2 - |\xi'|^2)\widehat{\rho} = 0, \ x_N > 0, \tag{4.7}$$

with the boundary conditions:

$$\partial_N\widehat{\rho}(0) = -\widehat{g}(0), \tag{4.8}$$

$$\widehat{u}_j(0) = \widehat{l}_j(0), \quad \widehat{u}_N(0) = 0. \tag{4.9}$$

One inserts (4.5) into (4.6), (4.7), and (4.8), and thus

$$\lambda^2\widehat{u}_j - \lambda\mu_*(\partial_N^2 - |\xi'|^2)\widehat{u}_j - i\xi_j\{\lambda\nu_* - \kappa_*(\partial_N^2 - |\xi'|^2)\}\widehat{\varphi} = 0, \ x_N > 0, \tag{4.10}$$

$$\lambda^2\widehat{u}_N - \lambda\mu_*(\partial_N^2 - |\xi'|^2)\widehat{u}_N - \partial_N\{\lambda\nu_* - \kappa_*(\partial_N^2 - |\xi'|^2)\}\widehat{\varphi} = 0, \ x_N > 0, \tag{4.11}$$

$$\partial_N\widehat{\varphi}(0) = \lambda\widehat{g}(0). \tag{4.12}$$

Multiplying (4.10) by $i\xi_j$ and applying ∂_N to (4.11), we sum the resultant equations in order to obtain

$$\lambda^2\widehat{\varphi} - \lambda(\mu_* + \nu_*)(\partial_N^2 - |\xi'|^2)\widehat{\varphi} + \kappa_*(\partial_N^2 - |\xi'|^2)^2\widehat{\varphi} = 0, \ x_N > 0,$$

which implies that

$$P_\lambda(\partial_N)\widehat{\varphi} = 0, \quad P_\lambda(t) = \lambda^2 - \lambda(\mu_* + \nu_*)(t^2 - |\xi'|^2) + \kappa_*(t^2 - |\xi'|^2)^2. \tag{4.13}$$

Here we set

$$\omega_\lambda = \sqrt{|\xi'|^2 + \frac{\lambda}{\mu_*}}, \quad \Re\omega_\lambda > 0 \quad \text{for } \lambda \in \Sigma_{\varepsilon,0},\ \varepsilon \in (0,\pi/2). \tag{4.14}$$

Applying $P_\lambda(\partial_N)$ to (4.10) and (4.11) furnishes by (4.13)

$$(\partial_N^2 - \omega_\lambda^2)P_\lambda(\partial_N)\widehat{u}_J = 0 \quad (J = 1,\dots,N). \tag{4.15}$$

One considers the roots of $P_\lambda(t)$ at this point. Since

$$P_\lambda(t) = \kappa_*\lambda^2\left\{\frac{1}{\kappa_*} - \left(\frac{\mu_* + \nu_*}{\kappa_*}\right)\left(\frac{t^2 - |\xi'|^2}{\lambda}\right) + \left(\frac{t^2 - |\xi'|^2}{\lambda}\right)^2\right\},$$

we set $s = (t^2 - |\xi'|^2)/\lambda$ and solve the equation:

$$s^2 - \frac{\mu_* + \nu_*}{\kappa_*}s + \frac{1}{\kappa_*} = 0. \tag{4.16}$$

By the assumption $\eta_* \neq 0$ in (4.2), we have the solutions s_1, s_2 ($s_1 \neq s_2$) of (4.16) such that $s_1 = s_-$ and $s_2 = s_+$ with

$$s_\pm = \begin{cases} \dfrac{\mu_* + \nu_*}{2\kappa_*} \pm \sqrt{\eta_*} & (\eta_* > 0), \\[2mm] \dfrac{\mu_* + \nu_*}{2\kappa_*} \pm i\sqrt{|\eta_*|} & (\eta_* < 0). \end{cases}$$

Let $\alpha_* = \arg s_2 \in [0,\pi/2)$, and set for $\lambda \in \Sigma_{\varepsilon,0}$ with $\varepsilon \in (\alpha_*,\pi/2)$

$$t_1 = \sqrt{|\xi'|^2 + s_1\lambda}, \qquad t_2 = \sqrt{|\xi'|^2 + s_2\lambda}, \tag{4.17}$$

$$t_3 = -\sqrt{|\xi'|^2 + s_1\lambda}, \quad t_4 = -\sqrt{|\xi'|^2 + s_2\lambda}.$$

We then see that $t_k = t_k(\xi', \lambda)$ $(k = 1, 2, 3, 4)$ are the roots of $P_\lambda(t)$ different from each other and that $\Re t_1 > 0$, $\Re t_2 > 0$, $\Re t_3 < 0$, and $\Re t_4 < 0$.

Remark 4.4. We have in general the following situations concerning roots with positive real parts for the characteristic equation of (4.15):

(1) Case $\eta_* < 0$. It holds that $\omega_\lambda \neq t_1$, $\omega_\lambda \neq t_2$, and $t_1 \neq t_2$.
(2) Case $\eta_* = 0$. There are two cases: $\omega_\lambda \neq t_1$ and $t_1 = t_2$; $\omega_\lambda = t_1 = t_2$.
(3) Case $\eta_* > 0$. There are three cases: $\omega_\lambda \neq t_1$, $\omega_\lambda \neq t_2$, and $t_1 \neq t_2$; $\omega_\lambda = t_1$ and $t_1 \neq t_2$; $\omega_\lambda = t_2$ and $t_1 \neq t_2$.

The condition (4.2) guarantees that we have the three roots with positive real parts different from each other.

In view of (4.15) and Remark 4.4, we look for solutions $\widehat{u}_J$ of the forms:

$$\widehat{u}_J = \alpha_J e^{-\omega_\lambda x_N} + \beta_J(e^{-t_1 x_N} - e^{-\omega_\lambda x_N}) + \gamma_J(e^{-t_2 x_N} - e^{-\omega_\lambda x_N}).$$

Here and subsequently, J runs from 1 to N, while j runs from 1 to $N-1$. It then holds that

$$\partial_N \widehat{u}_J = \big(-\omega_\lambda \alpha_J + \omega_\lambda \beta_J + \omega_\lambda \gamma_J\big)e^{-\omega_\lambda x_N} - t_1\beta_J e^{-t_1 x_N} - t_2 \gamma_J e^{-t_2 x_N}, \tag{4.18}$$

$$\widehat{\varphi} = \big(i\xi'\cdot\alpha' - i\xi'\cdot\beta' - i\xi'\cdot\gamma' - \omega_\lambda\alpha_N + \omega_\lambda\beta_N + \omega_\lambda\gamma_N\big)e^{-\omega_\lambda x_N} + \big(i\xi'\cdot\beta' - t_1\beta_N\big)e^{-t_1 x_N} + \big(i\xi'\cdot\gamma' - t_2\gamma_N\big)e^{-t_2 x_N}, \tag{4.19}$$

where $i\xi'\cdot a' = \sum_{j=1}^{N-1} i\xi_j a_j$ for $a \in \{\alpha, \beta, \gamma\}$. By (4.10) and (4.11), we have

$$\mu_*\lambda(\partial_N^2 - \omega_\lambda^2)\widehat{u}_j + i\xi_j\{\nu_*\lambda - \kappa_*(\partial_N^2 - |\xi'|^2)\}\widehat{\varphi} = 0, \ x_N > 0,$$
$$\mu_*\lambda(\partial_N^2 - \omega_\lambda^2)\widehat{u}_N + \partial_N\{\nu_*\lambda - \kappa_*(\partial_N^2 - |\xi'|^2)\}\widehat{\varphi} = 0, \ x_N > 0,$$

which, combined with (4.19) and the assumption $\kappa_* \neq \mu_*\nu_*$, furnishes that

$$i\xi'\cdot\alpha' - i\xi'\cdot\beta' - i\xi'\cdot\gamma' - \omega_\lambda\alpha_N + \omega_\lambda\beta_N + \omega_\lambda\gamma_N = 0, \tag{4.20}$$

$$\mu_*\lambda\beta_j(t_1^2 - \omega_\lambda^2) + i\xi_j(i\xi'\cdot\beta' - t_1\beta_N)\{\nu_*\lambda - \kappa_*(t_1^2 - |\xi'|^2)\} = 0, \tag{4.21}$$

$$\mu_*\lambda\gamma_j(t_2^2 - \omega_\lambda^2) + i\xi_j(i\xi'\cdot\gamma' - t_2\gamma_N)\{\nu_*\lambda - \kappa_*(t_2^2 - |\xi'|^2)\} = 0, \tag{4.22}$$

$$\mu_*\lambda\beta_N(t_1^2 - \omega_\lambda^2) - t_1(i\xi'\cdot\beta' - t_1\beta_N)\{\nu_*\lambda - \kappa_*(t_1^2 - |\xi'|^2)\} = 0, \tag{4.23}$$

$$\mu_*\lambda\gamma_N(t_2^2 - \omega_\lambda^2) - t_2(i\xi'\cdot\gamma' - t_2\gamma_N)\{\nu_*\lambda - \kappa_*(t_2^2 - |\xi'|^2)\} = 0. \tag{4.24}$$

By (4.21)-(4.24), we have

$$\mu_*\lambda(t_1^2 - \omega_\lambda^2)\left(\beta_j + \frac{i\xi_j}{t_1}\beta_N\right) = 0, \quad \mu_*\lambda(t_2^2 - \omega_\lambda^2)\left(\gamma_j + \frac{i\xi_j}{t_2}\gamma_N\right) = 0.$$

As was seen in Remark 4.4, we know that $\omega_\lambda \neq t_1$ and $\omega_\lambda \neq t_2$ under the condition (4.2), and therefore the last two identities imply

$$\beta_j = -\frac{i\xi_j}{t_1}\beta_N, \quad \gamma_j = -\frac{i\xi_j}{t_2}\gamma_N. \tag{4.25}$$

These relations, furthermore, yield

$$i\xi'\cdot\beta' - t_1\beta_N = -t_1^{-1}(t_1^2 - |\xi'|^2)\beta_N, \tag{4.26}$$

$$i\xi'\cdot\gamma' - t_2\gamma_N = -t_2^{-1}(t_2^2 - |\xi'|^2)\gamma_N. \tag{4.27}$$

On the other hand, we have by (4.19) and (4.20)

$$\widehat{\varphi} = (i\xi'\cdot\beta' - t_1\beta_N)e^{-t_1x_N} + (i\xi'\cdot\gamma' - t_2\gamma_N)e^{-t_2x_N}. \tag{4.28}$$

Next we consider the boundary conditions. By (4.9) and (4.12), we have

$$\alpha_j = \widehat{l}_j(0), \quad \alpha_N = 0, \tag{4.29}$$

$$t_1\big(i\xi'\cdot\beta' - t_1\beta_N\big) + t_2\big(i\xi'\cdot\gamma' - t_2\gamma_N\big) = -\lambda\widehat{g}(0). \tag{4.30}$$

It especially holds by the first identity of (4.29) that

$$i\xi'\cdot\alpha' = i\xi'\cdot\widehat{\mathbf{l}}'(0), \quad \widehat{\mathbf{l}}'(0) = (\widehat{l}_1(0),\dots,\widehat{l}_{N-1}(0))^{\mathsf{T}}, \tag{4.31}$$

and also by (4.26), (4.27), and (4.30)

$$(t_1^2 - |\xi'|^2)\beta_N + (t_2^2 - |\xi'|^2)\gamma_N = \lambda\widehat{g}(0). \tag{4.32}$$

We now derive simultaneous equations with respect to β_N and γ_N. By (4.25),

$$i\xi'\cdot\beta' = t_1^{-1}|\xi'|^2\beta_N, \quad i\xi'\cdot\gamma' = t_2^{-1}|\xi'|^2\gamma_N,$$

which, inserted into (4.20) together with the second identity of (4.29) and (4.31), furnishes that

$$i\xi'\cdot\widehat{\mathbf{l}}'(0) - t_1^{-1}|\xi'|^2\beta_N - t_2^{-1}|\xi'|^2\gamma_N + \omega_\lambda\beta_N + \omega_\lambda\gamma_N = 0.$$

Hence,

$$(t_1\omega_\lambda - |\xi'|^2)t_2\beta_N + (t_2\omega_\lambda - |\xi'|^2)t_1\gamma_N = -t_1t_2i\xi'\cdot\widehat{\mathbf{l}}'(0),$$

which, combined with (4.32), implies

$$\mathbf{L}\begin{pmatrix}\beta_N\\ \gamma_N\end{pmatrix} = \begin{pmatrix}\lambda\widehat{g}(0)\\ -t_1t_2i\xi'\cdot\widehat{\mathbf{l}}'(0)\end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} t_1^2 - |\xi'|^2 & t_2^2 - |\xi'|^2\\ (t_1\omega_\lambda - |\xi'|^2)t_2 & (t_2\omega_\lambda - |\xi'|^2)t_1\end{pmatrix}. \tag{4.33}$$

Finally, we solve (4.33) and the equations (4.5)-(4.8). By direct calculations,

$$\begin{aligned}\det\mathbf{L} &= t_2(t_2^2 - |\xi'|^2)(t_1\omega_\lambda - |\xi'|^2) - t_1(t_1^2 - |\xi'|^2)(t_2\omega_\lambda - |\xi'|^2)\\ &= (t_2 - t_1)\{t_1t_2\omega_\lambda(t_2 + t_1) - |\xi'|^2(t_2^2 + t_1t_2 + t_1^2 - |\xi'|^2)\}.\end{aligned}$$

Here one has

Lemma 4.5. *Assume that μ_*, ν_*, and κ_* are positive constants satisfying* (4.2). *Then* $\det\mathbf{L} \neq 0$ *for any* $(\xi',\lambda) \in \mathbf{R}^{N-1}\times(\overline{\mathbf{C}_+}\setminus\{0\})$, *where* $\overline{\mathbf{C}_+} = \{z \in \mathbf{C} \mid \Re z \geq 0\}$.

Proof. See [18] (cf. also [17]) for the proof. □

Let us write $\mathbf{L}^{-1}$ as follows:

$$\mathbf{L}^{-1} = \frac{1}{\det\mathbf{L}}\begin{pmatrix} L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix},$$

where

$$\begin{aligned} L_{11} &= t_1(t_2\omega_\lambda - |\xi'|^2), & L_{12} &= -(t_2^2 - |\xi'|^2),\\ L_{21} &= -t_2(t_1\omega_\lambda - |\xi'|^2), & L_{22} &= t_1^2 - |\xi'|^2.\end{aligned}$$

We thus see that, by solving (4.33),

$$\begin{aligned}\beta_N &= \frac{\lambda L_{11}}{\det\mathbf{L}}\widehat{g}(0) - \frac{t_1t_2L_{12}}{\det\mathbf{L}}i\xi'\cdot\widehat{\mathbf{l}}'(0),\\ \gamma_N &= \frac{\lambda L_{21}}{\det\mathbf{L}}\widehat{g}(0) - \frac{t_1t_2L_{22}}{\det\mathbf{L}}i\xi'\cdot\widehat{\mathbf{l}}'(0),\end{aligned} \tag{4.34}$$

which, combined with (4.25), gives the exact formulas of β_j, γ_j for $j = 1, \dots, N-1$. Hence we obtain

$$\begin{aligned}
\widehat{\rho}(x_N) &= \left(\frac{t_1^2 - |\xi'|^2}{\lambda t_1}\right) e^{-t_1 x_N}\beta_N + \left(\frac{t_2^2 - |\xi'|^2}{\lambda t_2}\right) e^{-t_2 x_N}\gamma_N, \\
\widehat{u}_j(x_N) &= \widehat{l}_j(0)e^{-\omega_\lambda x_N} - \frac{i\xi_j}{t_1}(e^{-t_1 x_N} - e^{-\omega_\lambda x_N})\beta_N \\
&\quad - \frac{i\xi_j}{t_2}(e^{-t_2 x_N} - e^{-\omega_\lambda x_N})\gamma_N, \\
\widehat{u}_N(x_N) &= \left(e^{-t_1 x_N} - e^{-\omega_\lambda x_N}\right)\beta_N + \left(e^{-t_2 x_N} - e^{-\omega_\lambda x_N}\right)\gamma_N,
\end{aligned}$$

where we have used (4.5), (4.26), (4.27), and (4.28) in order to derive the representation formula of ρ. Setting $\rho = \mathcal{F}_{\xi'}^{-1}[\widehat{\rho}(x_N)](x')$ and $u_J = \mathcal{F}_{\xi'}^{-1}[\widehat{u}_J(x_N)](x')$ $(J = 1, \dots, N)$, we see that ρ and $\mathbf{u} = (u_1, \dots, u_N)^{\mathsf{T}}$ solve the system (4.3).

One can construct the $\mathcal{R}$-bounded solution operator families for (4.3) by means of the representation formulas of solutions obtained above in the same manner as in [17] (cf. also [18]). This completes the proof of Theorem 4.3.

5. Proof of Theorem 2.6

Combining the standard localization technique (cf. e.g. [4], [15]) with Theorem 4.1, we have the following theorem for (2.1).

Theorem 5.1. *Let $q \in (1, \infty)$ and $r \in (N, \infty)$ with $\max(q, q') \leq r$ for $q' = q/(q-1)$, and let $\mathcal{X}_q(\Omega)$, $\mathfrak{X}_q^0(\Omega)$, and $\mathcal{F}_\lambda^0$ be given in* (2.2) *and* (2.3) *for $G = \Omega$. Assume that the assumptions* (a), (b), *and* (c) *of Theorem* 2.6 *hold. Then there exists $\widetilde{\varepsilon}_0 \in (0, \pi/2)$ such that for any $\varepsilon \in (\widetilde{\varepsilon}_0, \pi/2)$ there is a constant $\widetilde{\lambda}_0 \geq 1$ such that the following assertions hold true.*

(1) *For any $\lambda \in \Sigma_{\varepsilon, \widetilde{\lambda}_0}$ there are operators $\mathcal{A}^0(\lambda)$ and $\mathcal{B}^0(\lambda)$, with*

$$\begin{aligned}
\mathcal{A}^0(\lambda) &\in \mathrm{Hol}\,(\Sigma_{\varepsilon, \widetilde{\lambda}_0}, \mathcal{L}(\mathfrak{X}_q^0(\Omega), H_q^3(\Omega))), \\
\mathcal{B}^0(\lambda) &\in \mathrm{Hol}\,(\Sigma_{\varepsilon, \widetilde{\lambda}_0}, \mathcal{L}(\mathfrak{X}_q^0(\Omega), H_q^2(\Omega)^N)),
\end{aligned}$$

such that, for $\mathbf{F} = (d, \mathbf{f}, g) \in \mathcal{X}_q(\Omega)$, $(\rho, \mathbf{u}) = (\mathcal{A}^0(\lambda)\mathcal{F}_\lambda^0\mathbf{F}, \mathcal{B}^0(\lambda)\mathcal{F}_\lambda^0\mathbf{F})$ is a unique solution to the system (2.1).

(2) *There is a positive constant C, depending on N, q, r, ε, $\widetilde{\varepsilon}_0$, and $\widetilde{\lambda}_0$, such that for $n = 0, 1$*

$$\begin{aligned}
\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^0(\Omega), \mathfrak{A}_q^0(\Omega))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{S}_\lambda^0\mathcal{A}^0(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}\right\}\right) &\leq C, \\
\mathcal{R}_{\mathcal{L}(\mathfrak{X}_q^0(\Omega), \mathfrak{B}_q(\Omega))}\left(\left\{\left(\lambda\frac{d}{d\lambda}\right)^n (\mathcal{T}_\lambda\mathcal{B}^0(\lambda)) \mid \lambda \in \Sigma_{\varepsilon, \lambda_0}\right\}\right) &\leq C,
\end{aligned}$$

where $\mathfrak{A}^0(\Omega)$, $\mathfrak{B}(\Omega)$, $\mathcal{S}_\lambda^0$, and $\mathcal{T}_\lambda$ are given in (2.4) *for $G = \Omega$.*

In Theorem 5.1, we note that

$$\mathcal{S}_\lambda^0\rho = (\lambda^{3/2}\rho, \lambda\nabla\rho, \lambda^{1/2}\nabla^2\rho, \nabla^3\rho), \quad \mathcal{F}_\lambda^0\mathbf{F} = (\nabla d, \lambda^{1/2}d, \mathbf{f}, \lambda g, \lambda^{1/2}\nabla g, \lambda g).$$

One has to replace $\lambda^{3/2}\rho$ by $\lambda\rho$ and $\lambda^{1/2}d$ by d to prove Theorem 2.6 from Theorem 5.1. In what follows, we discuss how to obtain Theorem 2.6 from Theorem 5.1.

Let $\mathbf{F} = (d, \mathbf{f}, g) \in \mathcal{X}_q(\Omega)$ and $(\rho, \mathbf{u})$ be the solutions to the system (2.1). Let $E \in \mathcal{L}(H_q^1(\Omega), H_q^1(\mathbf{R}^N))$ be an extension operator and $E_0 \in \mathcal{L}(L_q(\Omega)^N, L_q(\mathbf{R}^N)^N)$

be the zero extension operator. One sets $\varepsilon_0 = \max(\varepsilon_*, \widetilde{\varepsilon}_0)$ with constants ε_* and $\widetilde{\varepsilon}_0$ obtained respectively in Theorem 3.1 and Theorem 5.1. In addition, for $\varepsilon \in (\varepsilon_0, \pi/2)$, one sets $\lambda_0 = \max(\lambda_*, \widetilde{\lambda}_0)$ with constants λ_* and $\widetilde{\lambda}_0$ obtained respectively in Theorem 3.1 and Theorem 5.1.

For $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and for $\Phi(\lambda)$ and $\Psi(\lambda)$ of Theorem 3.1, we define

$$(R, \mathbf{U}) = (\Phi(\lambda)(Ed, E_0\mathbf{f}), \Psi(\lambda)(E_d, E_0\mathbf{f})). \tag{5.1}$$

Then $(R, \mathbf{U})$ is the solution to

$$\begin{cases} \lambda R + \gamma_1 \operatorname{div} \mathbf{U} = Ed & \text{in } \mathbf{R}^N, \\ \lambda \mathbf{U} - \gamma_3^{-1} \operatorname{Div}(\mu \mathbf{D}(\mathbf{U}) + (\nu - \mu) \operatorname{div} \mathbf{U}\mathbf{I} + \gamma_2 \kappa \Delta R \mathbf{I}) = E_0 \mathbf{f} & \text{in } \mathbf{R}^N. \end{cases}$$

In addition, setting $\rho = R + \sigma$ in (2.1), we achieve

$$\begin{cases} \lambda \sigma + \gamma_1 \operatorname{div} \mathbf{u} = \widetilde{d} & \text{in } \Omega, \\ \lambda \mathbf{u} - \gamma_3^{-1} \operatorname{Div}(\mu \mathbf{D}(\mathbf{u}) + (\nu - \mu) \operatorname{div} \mathbf{u}\mathbf{I} + \gamma_2 \kappa \Delta \sigma \mathbf{I}) = \widetilde{\mathbf{f}} & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla \sigma = \widetilde{g}, \quad \mathbf{u} = 0 & \text{on } S, \end{cases}$$

where we have set

$$\begin{aligned} &\widetilde{d} = \gamma_1 \operatorname{div} \mathbf{U}, \\ &\widetilde{\mathbf{f}} = \lambda \mathbf{U} - \gamma_3^{-1} \operatorname{Div}(\mu \mathbf{D}(\mathbf{U}) + (\nu - \mu) \operatorname{div} \mathbf{U}\mathbf{I}), \\ &\widetilde{g} = g - \mathbf{n} \cdot \nabla R = g - \sum_{j=1}^N n_j \partial_j R \quad (\mathbf{n} = (n_1, \dots, n_N)^\mathsf{T}). \end{aligned}$$

Thus the solution $(\rho, \mathbf{u})$ of (2.1) can be written as

$$\begin{aligned} &\rho = R + \sigma = \Phi(\lambda)(Ed, E_0\mathbf{f}) + \mathcal{A}^0(\lambda)\mathcal{F}^0_\lambda(\widetilde{d}, \widetilde{\mathbf{f}}, \widetilde{g}), \\ &\mathbf{u} = \mathcal{B}^0(\lambda)\mathcal{F}^0_\lambda(\widetilde{d}, \widetilde{\mathbf{f}}, \widetilde{g}). \end{aligned} \tag{5.2}$$

In the following calculations, $\mathbf{n}$ is extended to $\mathbf{R}^N$ in a suitable way (cf. [20, Appendix A]). Recall that

$$\mathcal{F}^0_\lambda(\widetilde{d}, \widetilde{\mathbf{f}}, \widetilde{g}) = (\nabla \widetilde{d}, \lambda^{1/2}\widetilde{d}, \widetilde{\mathbf{f}}, \nabla^2 \widetilde{g}, \lambda^{1/2}\nabla \widetilde{g}, \lambda \widetilde{g}) \tag{5.3}$$

and that

$$\begin{aligned} &\nabla \widetilde{d} = (\nabla \gamma_1) \operatorname{div} \mathbf{U} + \gamma_1 \nabla \operatorname{div} \mathbf{U}, \quad \lambda^{1/2}\widetilde{d} = \gamma_1 \lambda^{1/2} \operatorname{div} \mathbf{U}, \\ &\widetilde{\mathbf{f}} = \lambda \mathbf{U} - \gamma_3^{-1}\Big(\mu \Delta \mathbf{U} + \mathbf{D}(\mathbf{U})\nabla \mu + \nu \nabla \operatorname{div} \mathbf{U} + (\operatorname{div} \mathbf{U}) \nabla(\nu - \mu)\Big), \\ &\nabla^2 \widetilde{g} = \nabla^2 g - \sum_{j=1}^N \Big((\nabla^2 n_j)\partial_j R + \nabla n_j \otimes \nabla \partial_j R + \nabla \partial_j R \otimes \nabla n_j + n_j \nabla^2 \partial_j R\Big), \\ &\lambda^{1/2}\nabla \widetilde{g} = \lambda^{1/2}\nabla g - \sum_{j=1}^N \Big((\nabla n_j)\lambda^{1/2}\partial_j R + n_j \lambda^{1/2} \nabla \partial_j R\Big), \\ &\lambda \widetilde{g} = \lambda g - \sum_{j=1}^N n_j \lambda \partial_j R. \end{aligned} \tag{5.4}$$

Let $\mathsf{F} = (\mathsf{F}_1, \dots, \mathsf{F}_5) \in \mathfrak{X}_q(\Omega)$, i.e.

$$\mathsf{F}_1 \in H^1_q(\Omega), \quad \mathsf{F}_2, \mathsf{F}_4 \in L_q(\Omega)^N, \quad \mathsf{F}_3 \in L_q(\Omega)^{N^2}, \quad \mathsf{F}_5 \in L_q(\Omega).$$

In view of (5.1)-(5.4), we define for the density an operator $\mathcal{A}(\lambda)$ with

$$\mathcal{A}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathfrak{X}_q(\Omega), H_q^3(\Omega)))$$

in the following manner:

$$\begin{aligned}
&\mathcal{A}(\lambda)\mathsf{F} = \Phi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2)\\
&+\mathcal{A}^0(\lambda)\Big((\nabla\gamma_1)\operatorname{div}\Psi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2) + \gamma_1\nabla\operatorname{div}\Psi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2),\\
&\quad \gamma_1\lambda^{1/2}\operatorname{div}\Psi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2),\ \lambda\Psi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2) - \gamma_3^{-1}\mu\Delta\Psi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2)\\
&\quad - \gamma_3^{-1}\nu\nabla\operatorname{div}\Psi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2) - \gamma_3^{-1}\mathbf{D}(\Psi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2))\nabla\mu\\
&\quad - \gamma_3^{-1}\operatorname{div}\Psi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2)\nabla(\nu-\mu),\\
&\quad \mathsf{F}_3 - \sum_{j=1}^N(\nabla^2 n_j)\partial_j\Phi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2) - \sum_{j=1}^N \nabla n_j \otimes \nabla\partial_j\Phi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2)\\
&\quad - \sum_{j=1}^N \nabla\partial_j\Phi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2)\otimes\nabla n_j - \sum_{j=1}^N n_j\nabla^2\partial_j\Phi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2),\\
&\quad \mathsf{F}_4 - \sum_{j=1}^N(\nabla n_j)\lambda^{1/2}\partial_j\Phi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2) - \sum_{j=1}^N n_j\lambda^{1/2}\nabla\partial_j\Phi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2),\\
&\quad \mathsf{F}_5 - \sum_{j=1}^N n_j\lambda\partial_j\Phi(\lambda)(E\mathsf{F}_1, E_0\mathsf{F}_2)\Big).
\end{aligned}$$

Furthermore, we define an operator $\mathcal{B}(\lambda) \in \mathrm{Hol}\,(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathfrak{X}_q(\Omega), H_q^2(\Omega)^N))$ for the velocity in the same manner as $\mathcal{A}(\lambda)$. Thus $(\rho, \mathbf{u}) = (\mathcal{A}(\lambda)\mathcal{F}_\lambda\mathbf{F}, \mathcal{B}(\lambda)\mathcal{F}_\lambda\mathbf{F})$ is a solution to the system (2.1), and also $\mathcal{A}(\lambda)$, $\mathcal{B}(\lambda)$ satisfy the required estimates of Theorem 2.6 (2) by Proposition 3.3 and Theorems 3.1, 5.1 (cf. [18] for more details). This finishes the outline of the proof of Theorem 2.6.

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