数理解析研究所講究録
第2070巻 2018年 85-106

On some free boundary problems for the Navier-Stokes equations in unbounded domains

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This note deals with the global wellposedness of some free boundary problems for the Navier-Stokes equations in unbounded domains. A technical issue is how to combine the maximal regularity estimates for the highest order terms with the decay estimates for the lower order terms of solutions to the linearized equations. It does not seem to be popular enough compared with the Navier-Stokes equations with non-slip boundary conditions. In the later case, the Navier-Stokes equations are just parabolic ones, and so the decay properties for the Stokes equations with non-slip boundary conditions are enough. On the other hand, in the free boundary conditions case, after transforming an unknown time dependent domain to some known fixed one, the problem becomes a system of quasilinear parabolic equations with nonlinear boundary conditions, and so some combinations of the maximal regularity estimates for the highest order terms with the decay estimates for the lower order terms are necessary to prove the global well-posedness at least for small initial data, which is not well-known compared with the non-slip boundary conditions. In this note, I would like to show some combinations, which does not seem to be optimal/best possible, but is enough to prove the global well-posedness for small initial data. Notice that in the bounded domain case, exponentially stable maximal regularity estimates are obtained, and so it is not necessary to consider combinations mentioned above.

1 One Phase Problem in an Exterior Domain

1.1 Problem and Global in Time Unique Existence Theorem

We first consider the following free boundary problem for the Navier-Stokes equations:

$$\begin{cases} \partial_t\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I}) = 0, \quad \mathrm{div}\,\mathbf{v} = 0 & \text{for } x \in \Omega_t,\ 0 < t < T, \\ (\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I})\mathbf{n}_t = 0, \quad V_n = \mathbf{v}\cdot\mathbf{n}_t = 0 & \text{for } x \in \Gamma_t,\ 0 < t < T, \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{for } x \in \Omega_0. \end{cases} \tag{1}$$

Unknowns are the domain Ω_t with the boundary Γ_t, and the functions $\mathbf{v}(x,t) = (v_1,\ldots,v_N)^\top$, where $M^\top$ denotes the transposed M, and $\mathfrak{p}(x,t)$, $x \in \Omega_t$. The domain $\Omega_0 = \Omega$ is an exterior domain in the N-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$) with the C^2 boundary Γ. By $\mathbf{n}_t = (n_{t1},\ldots,n_{tN})^\top$ we mean the exterior normal to Γ_t, and V_n is the velocity of the evolution of Γ_t in the normal direction. $\mathbf{D}(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^\top$ is the doubled rate-of-strain tensor whose $(i,j)^{\rm th}$ component is $\partial_iv_j + \partial_jv_i$, $\partial_i = \partial/\partial x_i$. μ is a positive constant describing the viscosity coefficient. We assume that the mass density is one. Moreover, $\mathbf{I}$ is the $N\times N$ identity matrix, and

$$\mathbf{v}\cdot\mathbf{n}_t = \sum_{j=1}^N v_jn_{tj}, \quad \nabla\mathfrak{p} = (\partial_1\mathfrak{p},\ldots,\partial_N\mathfrak{p})^\top, \quad \partial_t\mathbf{v} = (\frac{\partial v_1}{\partial t},\ldots,\frac{\partial v_N}{\partial t})^\top, \quad \mathrm{div}\,\mathbf{v} = \sum_{j=1}^N\frac{\partial v_j}{\partial x_j},$$

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Partially supported by Top Global University Project, JSPS Grant-in-aid for Scientific Research (A) 17H0109, and JSPS program of the Japanese-Germann Graduate Program

$$\operatorname{Div}\mathbf{K} = (\sum_{j=1}^N \frac{\partial K_{1j}}{\partial x_j}, \ldots, \sum_{j=1}^N \frac{\partial K_{Nj}}{\partial x_j})^\top, \quad (\mathbf{v}\cdot\nabla)\mathbf{v} = (\sum_{j=1}^N v_j \frac{\partial v_1}{\partial x_j}, \ldots, \sum_{j=1}^N v_j \frac{\partial v_N}{\partial x_j})^\top,$$

where $\mathbf{K}$ is the $N\times N$ matrix of functions whose $(i,j)^{\text{th}}$ component is K_{ij}.

If we use the Hanazawa transform to transform Ω_t to some fixed domain, we have to require $W_q^{3-1/q}$ regularity of the height function ρ representing Γ_t. However, this regularity is obtained by surface tension, that is the Laplace-Beltrami operator on Γ_t. In this section, we consider the case where the surface tension is not taken into account. Thus, we can not obtain $W_q^{3-1/q}$ regularity of the height function, and so the Hanzawa transform can not be used in the present case. Another method is to use the Lagrange transform. However, we can not expect the exponential decay for the solutions of the Stokes equations with free boundary conditions, because Ω is an unbounded domain. We will have only polynomial decay properties of solutions of the Stokes equations with free boundary conditions, which is not sufficient to controle the term: first derivatives of $\int_0^t \mathbf{u}(y,s)\,ds$ times the second derivatives of $\mathbf{u}$, where $\mathbf{u}$ is the velocity field in the Lagrange coordinates $\{y\}$.

To overcome this difficulty, the idea here is to use the Lagrange transform only near the boundary. Let R be a positive number for which $\mathcal{O} = \mathbb{R}^N \setminus \Omega \subset B_{R/2}$. Here and in the following, B_L denotes the ball with radius L. Let $\kappa \in C_0^\infty(B_{2R})$ equal one in B_R. Let $\mathbf{u}(y,t)$ be the velocity field in the Lagrange coordinates $\{y\}$. We consider a partial Lagrange transform:

$$x = X_{\mathbf{u}}(y,t) = y + \int_0^t \kappa(y)\mathbf{u}(y,s)\,ds. \tag{2}$$

Assume that

$$\int_0^T \|\kappa(\cdot)\mathbf{u}(\cdot,s)\|_{H^1_\infty(\Omega)}\,ds \le \delta. \tag{3}$$

As symbols, here and in the following, for any domain G in $\mathbb{R}^N$, $L_q(G)$, $H_q^m(G)$, and $B^s_{q,p}(G)$ denote the standard Lebesgue, Sobolev, and Besov spaces on G, and $\|\cdot\|_{L_q(G)}$, $\|\cdot\|_{H_q^m(G)}$, and $\|\cdot\|_{B^s_{q,p}(G)}$ denote their respective norms.

In the assumpion (3), $\delta > 0$ is a small number that will be chosen in such a way that several conditions hold. For example, we choose $0<\delta<1/2$, so that the map $x = X_{\mathbf{u}}(y,t)$ is injective for each $t \in (0,T)$. Let

$$\Psi(y,t) = \int_0^t \kappa(y)\mathbf{u}(y,s)\,ds = (\Psi_1(y,t), \ldots, \Psi_N(y,t))^\top.$$

Let y $= X_{\mathbf{u}}^{-1}(x,t)$ be the inverse of the transformation: $x = X_{\mathbf{u}}(y,t)$ given in (2). Setting

$$\Omega_t = \{x = X_{\mathbf{u}}(y,t) \mid y \in \Omega\}, \quad \Gamma_t = \{x = X_{\mathbf{u}}(y,t) \mid y \in \Gamma\},$$

$\mathbf{v}(x,t) = \mathbf{u}(X_{\mathbf{u}}^{-1}(x,t),t)$, and $\mathfrak{p}(x,t) = \mathfrak{q}(X_{\mathbf{u}}^{-1}(x,t),t)$, we then see that Eq. (1) is transformed to

$$\begin{cases} \partial_t\mathbf{u} - \operatorname{Div}(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}) = \mathbf{f}(\mathbf{u}), & \text{in } \Omega\times(0,T), \\ \operatorname{div}\mathbf{u} = g(\mathbf{u}) = \operatorname{div}\mathbf{g}(\mathbf{u}) & \text{in } \Omega\times(0,T), \\ (\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I})\mathbf{n} = \mathbf{h}(\mathbf{u}) & \text{on } \Gamma\times(0,T), \\ \mathbf{v}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega. \end{cases} \tag{4}$$

Here, $\mathbf{u}_0 = \mathbf{v}_0$, $\mathbf{n}$ is the unit outer normal to Γ, and $\mathbf{f}(\mathbf{u})$, $g(\mathbf{u})$, $\mathbf{g}(\mathbf{u})$ and $\mathbf{h}(\mathbf{u})$ are nonlinear terms, the exact formulas of which will be given below.

As symbols, we use bold lowercase letters to denote N-vectors and bold capital letters to denote $N\times N$ matrices. For an N vector $\mathbf{a}$, $\mathbf{a}_i$ denotes the i^{th} component of $\mathbf{a}$ and for an $N\times N$ matrix $\mathbf{A}$, $\mathbf{A}_{ij}$ denotes the $(i,j)^{\text{th}}$ component of $\mathbf{A}$, and moreover, the $N\times N$ matrix whose $(i,j)^{\text{th}}$ component is K_{ij} is written as (K_{ij}). For any two $N\times N$ matrices $\mathbf{A}$ and $\mathbf{B}$, $\mathbf{A}:\mathbf{B}$ is defined by $\mathbf{A}:\mathbf{B} = \sum_{i,j=1}^N \mathbf{A}_{ij}\mathbf{B}_{ij}$. For any N-vector of functions, $\mathbf{w} = (w_1, \ldots, w_N)^\top$, $\nabla\mathbf{w}$ is the $N\times N$ matrix of functions with $(\nabla\mathbf{w})_{ij} = \partial_j w_i$, that is

$$\nabla\mathbf{w} = \begin{pmatrix} \partial_1 w_1 & \partial_2 w_1 & \cdots & \partial_N w_1 \\ \partial_1 w_2 & \partial_2 w_2 & \cdots & \partial_N w_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 w_N & \partial_2 w_N & \cdots & \partial_N w_N \end{pmatrix}.$$

Let $\partial x/\partial y$ be the Jacobi matrix of the transformation (2). We choose $\delta > 0$ so small that the inverse matrix of $\partial x/\partial y$ exists, and therefore there exists an $N \times N$ matrix $\mathbf{V}_0(\mathbf{k})$ of C^∞ functions defined on $|\mathbf{k}| < \delta$ such that $\mathbf{V}_0(0) = 0$ and

$$\left(\frac{\partial x}{\partial y}\right)^{-1} = \mathbf{I} + \mathbf{V}_0(\nabla\Psi(y,t)). \tag{5}$$

Here and in the following, $\mathbf{k} = (k_{ij})$ and k_{ij} are the variables corresponding to $\partial_i \Psi_j = \int_0^t \partial_i(\kappa \mathbf{u}_j)\,ds$. Let $V_{0ij}(\mathbf{k}) = \mathbf{V}_0(\mathbf{k})_{ij}$ and $\nabla_z = (\partial/\partial z_1, \ldots, \partial/\partial z_N)^\top$ for $z = x$ and y. We then have

$$\nabla_x = (\mathbf{I} + \mathbf{V}_0(\mathbf{k}))\nabla_y, \quad \frac{\partial}{\partial x_i} = \sum_{j=1}^N (\delta_{ij} + V_{0ij}(\mathbf{k}))\frac{\partial}{\partial y_j}. \tag{6}$$

Thus, $\mathbf{D}(\mathbf{v}) = \mathbf{D}(\mathbf{u}) + \mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla\mathbf{u}$ with

$$\mathbf{D}(\mathbf{u})_{ij} = \frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i}, \quad (\mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla\mathbf{u})_{ij} = \sum_{k=1}^N \Big(V_{0jk}(\mathbf{k})\frac{\partial u_i}{\partial y_k} + V_{0ik}(\mathbf{k})\frac{\partial u_j}{\partial y_k}\Big). \tag{7}$$

We next consider $\operatorname{div}\mathbf{v}$. By (6), we have

$$\operatorname{div}_x \mathbf{v} = \sum_{j=1}^N \frac{\partial v_j}{\partial x_j} = \sum_{j,k=1}^N (\delta_{jk} + V_{0jk}(\mathbf{k}))\frac{\partial u_j}{\partial y_k} = \operatorname{div}_y \mathbf{u} + \mathbf{V}_0(\mathbf{k}) : \nabla\mathbf{u}.$$

Let J be the Jacobian of the transformation (2). Choosing $\delta > 0$ small enough, we may assume that $J = J(\mathbf{k}) = 1 + J_0(\mathbf{k})$, where $J_0(\mathbf{k})$ is a C^∞ function defined for $|\mathbf{k}| < \sigma$ such that $J_0(0) = 0$. To obtain another representation formula of $\operatorname{div}_x\mathbf{v}$, we use the inner product $(\cdot,\cdot)_{\Omega_t}$. As symbols, here and in the following for any domain $G \subset \mathbb{R}^N$ and its boundary ∂G, $(\mathbf{u},\mathbf{v})_G$ and $(\mathbf{u},\mathbf{v})_{\partial G}$ denote inner products on G and ∂G, respectively, that is

$$(\mathbf{u},\mathbf{v})_G = \int_G \mathbf{u}\cdot\mathbf{v}\,dx, \quad (\mathbf{u},\mathbf{v})_{\partial G} = \int_{\partial G} \mathbf{u}\cdot\mathbf{v}\,ds,$$

where ds denotes the surface element on the boundary ∂G.

For any test function $\varphi \in C_0^\infty(\Omega_t)$, we set $\psi(y) = \varphi(x)$. We then have

$$\begin{aligned}(\operatorname{div}_x\mathbf{v},\varphi)_{\Omega_t} &= -(\mathbf{v},\nabla\varphi)_{\Omega_t} = -(J\mathbf{u},(\mathbf{I}+\mathbf{V}_0)\nabla_y\psi)_\Omega = (\operatorname{div}((\mathbf{I}+\mathbf{V}_0^\top)J\mathbf{u}),\psi)_\Omega \\ &= (J^{-1}\operatorname{div}((\mathbf{I}+\mathbf{V}_0^\top)J\mathbf{u}),\varphi)_{\Omega_t}.\end{aligned}$$

Summing up, we have obtained

$$\operatorname{div}_x\mathbf{v} = \operatorname{div}_y\mathbf{u} + \mathbf{V}_0(\mathbf{k}) : \nabla\mathbf{u} = J^{-1}(\operatorname{div}_y\mathbf{u} + \operatorname{div}_y(J\mathbf{V}_0(\mathbf{k})^\top\mathbf{u})), \tag{8}$$

and so

$$J\operatorname{div}_y\mathbf{u} + J\mathbf{V}_0(\mathbf{k}) : \nabla\mathbf{u} = \operatorname{div}_y\mathbf{u} + \operatorname{div}_y(J\mathbf{V}_0(\mathbf{k})^\top\mathbf{u}).$$

In particular,

$$J_0\operatorname{div}\mathbf{u} + J\mathbf{V}_0(\mathbf{k}) : \nabla\mathbf{u} = \operatorname{div}(J\mathbf{V}_0(\mathbf{k})^\top\mathbf{u}). \tag{9}$$

To derive the transformation of the momentum equation in (1), we first observe that

$$\sum_{j=1}^N \frac{\partial}{\partial x_j}(\mu\mathbf{D}(\mathbf{v})_{ij} - \mathfrak{p}\delta_{ij}) = \sum_{j,k=1}^N \mu(\delta_{jk} + V_{0jk})\frac{\partial}{\partial y_k}(\mathbf{D}(\mathbf{u})_{ij} + (\mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla\mathbf{u})_{ij}) - \sum_{j=1}^N(\delta_{ij} + V_{0ij})\frac{\partial\mathfrak{q}}{\partial y_j}. \tag{10}$$

where we have used (7). Since

$$\frac{\partial}{\partial t}[v_i(y + \Psi(y,t),t)] = \frac{\partial v_i}{\partial t}(x,t) + \sum_{j=1}^N \kappa(y)u_j(y,t)\frac{\partial v_i}{\partial x_j}(x,t),$$

we have

$$\frac{\partial v_i}{\partial t} = \frac{\partial u_i}{\partial t} - \sum_{j,k=1}^{N} \kappa u_j(\delta_{jk} + V_{0jk})\frac{\partial u_i}{\partial y_k},$$

and therefore,

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{N} v_j \frac{\partial v_i}{\partial x_j} = \frac{\partial u_i}{\partial t} + \sum_{j,k=1}^{N} (1-\kappa)u_j(\delta_{jk} + V_{0jk}(\mathbf{k}))\frac{\partial u_i}{\partial y_k}. \tag{11}$$

Putting (10) and (11) together gives

$$\begin{aligned} 0 = \frac{\partial u_i}{\partial t} &+ \sum_{j,k=1}^{N} (1-\kappa)u_j(\delta_{jk} + V_{0jk}(\mathbf{k}))\frac{\partial u_i}{\partial y_k} \\ &- \mu \sum_{j,k=1}^{N} (\delta_{ij} + V_{0jk}(\mathbf{k}))\frac{\partial}{\partial y_k}(\mathbf{D}(\mathbf{u})_{ij} + (\mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla\mathbf{u})_{ij}) - \sum_{j=1}^{N}(\delta_{ij} + V_{0ij}(\mathbf{k}))\frac{\partial \mathfrak{q}}{\partial y_j}. \end{aligned}$$

Since $(\mathbf{I} + \nabla\Psi)(\mathbf{I} + \mathbf{V}_0) = (\partial x/\partial y)(\partial y/\partial x) = \mathbf{I}$,

$$\sum_{i=1}^{N}(\delta_{mi} + \partial_m \Psi_i)(\delta_{ij} + V_{0ij}(\mathbf{k})) = \delta_{mj},$$

and so we have

$$\begin{aligned} &\sum_{i=1}^{N}(\delta_{mi} + \partial_m\Psi_i)\Big(\frac{\partial u_i}{\partial t} + \sum_{j,k=1}^{N} (1-\kappa)u_j(\delta_{jk} + V_{0jk}(\mathbf{k}))\frac{\partial u_i}{\partial y_k}\Big) \\ &- \mu \sum_{i,j,k=1}^{N} (\delta_{mi} + \partial_m\Psi_i)(\delta_{jk} + V_{0jk}(\mathbf{k}))\frac{\partial}{\partial y_k}(\mathbf{D}(\mathbf{u})_{ij} + (\mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla\mathbf{u})_{ij}) - \frac{\partial \mathfrak{q}}{\partial y_m} = 0. \end{aligned}$$

Thus, changing i to ℓ and m to i, we define an N-vector of functions $\mathbf{f}(\mathbf{u})$ by

$$\begin{aligned} \mathbf{f}(\mathbf{u})_i = &- \sum_{j,k=1}^{N} (1-\kappa)u_j(\delta_{jk} + V_{0jk}(\mathbf{k}))\frac{\partial u_i}{\partial y_k} \\ &- \sum_{\ell=1}^{N} \partial_i\Psi_\ell\Big(\frac{\partial u_\ell}{\partial t} + \sum_{j,k=1}^{N} (1-\kappa)u_j(\delta_{jk} + V_{0jk}(\mathbf{k}))\frac{\partial u_\ell}{\partial y_k}\Big) \\ &+ \mu\Big(\sum_{j=1}^{N} \frac{\partial}{\partial y_j}(\mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla\mathbf{u})_{ij} + \sum_{j,k=1}^{N} V_{0jk}(\mathbf{k})\frac{\partial}{\partial y_k}(\mathbf{D}(\mathbf{u})_{ij} + (\mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla\mathbf{u})_{ij}) \\ &+ \sum_{j,k,\ell=1}^{N} \partial_i\Psi_\ell(\delta_{jk} + V_{0jk}(\mathbf{k}))\frac{\partial}{\partial y_k}(\mathbf{D}(\mathbf{u})_{\ell j} + (\mathcal{D}_{\mathbf{D}}(\mathbf{k})\nabla\mathbf{u})_{\ell j})\Big). \end{aligned} \tag{12}$$

And also, from (8) and (9) we have

$$g(\mathbf{u}) = -(J_0(\mathbf{k})\mathrm{div}\,\mathbf{u} + (1 + J_0(\mathbf{k}))\mathbf{V}_0(\mathbf{k}) : \nabla\mathbf{u}), \quad \mathbf{g}(\mathbf{u}) = -(1 + J_0(\mathbf{k}))\mathbf{V}_0(\mathbf{k})^\top\mathbf{u}. \tag{13}$$

Recall that

$$\Gamma_t = \{x = y + \Psi(y,t) + \xi(t) \mid y \in \Gamma\} \quad (t \in (0,T)).$$

Since

$$0 = \mathbf{n}\cdot dy = \mathbf{n}\cdot(\frac{\partial y}{\partial x}dx) = \mathbf{n}\cdot((\mathbf{I} + \mathbf{V}_0(\mathbf{k}))\,dx) = ((\mathbf{I} + \mathbf{V}_0(\mathbf{k})^\top)\mathbf{n})\cdot dx$$

on Γ, we have

$$\mathbf{n}_t = \frac{(\mathbf{I} + \mathbf{V}_0(\mathbf{k})^\top)\mathbf{n}}{|(\mathbf{I} + \mathbf{V}_0(\mathbf{k})^\top)\mathbf{n}|}. \tag{14}$$

Putting (7) and (14) together gives

$$\begin{aligned} 0 &= (\mu\mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I})\mathbf{n}_t|(\mathbf{I} + \mathbf{V}_0(\mathbf{k})^\top)\mathbf{n}| \\ &= \mu(\mathbf{D}(\mathbf{u}) + \mathcal{D}_\mathbf{D}(\mathbf{k})\nabla\mathbf{u})(\mathbf{I} + \mathbf{V}_0(\mathbf{k})^\top)\mathbf{n} - \mathfrak{q}(\mathbf{I} + \mathbf{V}_0(\mathbf{k})^\top)\mathbf{n}. \end{aligned}$$

Since $(\mathbf{I} + (\nabla\Psi)^\top)(\mathbf{I} + \mathbf{V}_0(\mathbf{k})^\top) = \mathbf{I}$, we have

$$(\mathbf{I} + (\nabla\Psi)^\top)\mu(\mathbf{D}(\mathbf{u}) + \mathcal{D}_\mathbf{D}(\mathbf{k})\nabla\mathbf{u})(\mathbf{I} + \mathbf{V}_0(\mathbf{k})^\top)\mathbf{n} - \mathfrak{q}\mathbf{n} = 0.$$

Thus,

$$\begin{aligned} \mathbf{h}(\mathbf{u}) = -\mu\{&\mathbf{D}(\mathbf{u})\mathbf{V}_0(\mathbf{k})^\top\mathbf{n} + (\mathcal{D}_\mathbf{D}(\mathbf{k})\nabla\mathbf{u})(\mathbf{I} + \mathbf{V}_0(\mathbf{k})^\top)\mathbf{n} \\ &+ (\nabla\Psi)^\top(\mathbf{D}(\mathbf{u}) + \mathcal{D}_\mathbf{D}(\mathbf{k})\nabla\mathbf{u})(\mathbf{I} + \mathbf{V}_0(\mathbf{k})^\top)\mathbf{n}\}. \end{aligned} \tag{15}$$

The main result of this section is the following theorem that showes the unique existence theorem of global in time solutions of Eq.(4) and asymptotics as $t \to \infty$.

Theorem 1. *Let $N \geq 3$ and let q_1 and q_2 be exponents such that $max(N, \frac{2N}{N-2}) < q_2 < \infty$ and $1/q_1 = 1/q_2 + 1/N$. Let b and p be numbers defined by*

$$b = \frac{3N}{2q_2} + \frac{1}{2}, \quad p = \frac{2q_2(1+\sigma)}{q_2 - N} \tag{16}$$

with some very small positive number σ. Then, there exists an $\epsilon > 0$ such that if initial data $\mathbf{u}_0 \in B^{2(1-1/p)}_{q_2,p}(\Omega)^N \cap B^{2(1-1/p)}_{q_1/2,p}(\Omega)^N$ satisfies the compatibility condition:

$$\operatorname{div}\mathbf{u}_0 = 0 \quad \text{in } \Omega, \quad \mathbf{D}(\mathbf{u}_0)\mathbf{n} - <\mathbf{D}(\mathbf{u}_0)\mathbf{n}, \mathbf{n}> \mathbf{n} = 0 \quad \text{on } \Gamma, \tag{17}$$

and the smallness condition:

$$\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q_2,p}} + \|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q_1/2,p}} \leq \epsilon, \tag{18}$$

then Eq. (4) admits unique solutions $\mathbf{u}$ and $\mathfrak{q}$ with

$$\mathbf{u} \in L_p((0,\infty), H^2_{q_2}(\Omega)^N) \cap H^1_p((0,\infty), L_{q_2}(\Omega)^N), \quad \mathfrak{q} \in L_p((0,\infty), H^1_{q_2}(\Omega) + \hat{H}^1_{q_2,0}(\Omega)),$$

possessing the estimate $[\mathbf{u}]_\infty \leq C\epsilon$ with

$$\begin{aligned} [\mathbf{u}]_T = \Big\{&\int_0^T (< s >^b \|\mathbf{u}(\cdot,s)\|_{H^1_\infty(\Omega)})^p\,ds \\ &+ \int_0^T (< s >^{(b-\frac{N}{2q_1})} \|\mathbf{u}(\cdot,s)\|_{H^1_{q_1}(\Omega)})^p\,ds + (\sup_{0<s<T} < s >^{\frac{N}{2q_1}} \|\mathbf{u}(\cdot,s)\|_{L_{q_1}(\Omega)})^p \\ &+ \int_0^T (< s >^{(b-\frac{N}{2q_2})} (\|\mathbf{u}(\cdot,s)\|_{H^2_{q_2}(\Omega)} + \|\partial_t\mathbf{u}(\cdot,s)\|_{L_{q_2}(\Omega)}))^p\,ds\Big\}^{1/p}. \end{aligned}$$

Here, $< s >= (1+s^2)^{\frac{1}{2}}$ and C is a constant that is independent of ϵ.

Remark 2. Let $p' = p/(p-1)$, that is $1/p' = 1 - 1/p$. And then,

$$\frac{1}{p'} = \frac{(1+2\sigma)q_2 + N}{2q_2(1+\sigma)}.$$

We choose $\sigma > 0$ small enough in such a way that the following relations hold:

$$\begin{aligned} &1 < q_1 < 2, \ \frac{N}{q_1} > b > \frac{1}{p'}, \ \Big(\frac{N}{q_1} - b\Big)p > 1, \ \Big(b - \frac{N}{2q_2}\Big)p > 1, \ b \geq \frac{N}{2q_1}, \\ &b \geq \frac{N}{q_2}, \ \Big(\frac{N}{2q_2} + \frac{1}{2}\Big)p' < 1, \ bp' > 1, \ \Big(b - \frac{N}{2q_2}\Big)p' > 1, \ \frac{N}{q_2} + \frac{2}{p} < 1. \end{aligned} \tag{19}$$

Remark 3. The exponent q_2 is used to control the nonlinear terms, and so q_2 is chosen in such a way that $N < q_2 < \infty$. Let

$$\frac{1}{q_1} = \frac{1}{N} + \frac{1}{q_2}, \quad \frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2}. \tag{20}$$

We require that $q_1 > 2$ and $q_3 > 1$ in the proof of Theorem 1, so that $q_2 > \frac{2N}{N-2}$. Thus, we have assumed that

$$\max(N, \frac{2N}{N-2}) < q_2 < \infty.$$

Remark 4. If we choose $\delta > 0$ in (3), then $x = X_{\mathbf{u}}(y,t)$ becomes a diffeomorphism with suitable regularity from Ω onto Ω_t, and so the original problem (1) is globally well-posed.

Further Notation

We further use the following symbols throughout the paper. For any N-vectors $\mathbf{a}$ and $\mathbf{b}$, let

$$\mathbf{a} \cdot \mathbf{b} = <\mathbf{a}, \mathbf{b}> = \sum_{j=1}^N \mathbf{a}_j \mathbf{b}_j, \quad \mathbf{a}_\tau = \mathbf{a} - <\mathbf{a}, \mathbf{n}> \mathbf{n}.$$

Given $1 < q < \infty$, let $q' = q/(q-1)$. For a Banach space X with norm $\|\cdot\|_X$, let $X^d = \{(f_1, \dots, f_d) \mid f_i \in X \ (i = 1, \dots, d)\}$, and write the norm of X^d as simply $\|\cdot\|_X$, which is defined by $\|f\|_X = \sum_{j=1}^d \|f_j\|_X$ for $f = (f_1, \dots, f_d) \in X^d$. Let

$$\hat H^1_{q,0}(\Omega) = \{\theta \in L_{q,\mathrm{loc}}(\Omega) \mid \nabla\theta \in L_q(\Omega)^N, \theta|_\Gamma = 0\}, \quad H^1_{q,0}(\Omega) = \{u \in H^1_q(\Omega) \mid u|_\Gamma = 0\}.$$

For $1 \le p \le \infty$, $L_p((a,b), X)$ and $H^m_p((a,b), X)$ denote the standard Lebesgue and Sobolev spaces of X-valued functions defined on an interval (a,b), and $\|\cdot\|_{L_p((a,b),X)}$, $\|\cdot\|_{H^m_p((a,b),X)}$ denote their respective norms. For $\theta \in (0,1)$, $H^\theta_p(\mathbb{R}, X)$ denotes the standard X-valued Bessel potential space defined by

$$H^\theta_p(\mathbb{R}, X) = \{f \in L_p(\mathbb{R}, X) \mid \|f\|_{H^\theta_p(\mathbb{R},X)} < \infty\},$$
$$\|f\|_{H^\theta_p(\mathbb{R},X)} = \Big(\int_{\mathbb{R}} \|\mathcal{F}^{-1}[(1+\tau^2)^{\theta/2}\mathcal{F}[f](\tau)](t)\|_X^p \, dt\Big)^{1/p},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively. Let $C_0^\infty(G)$ be the set of all C^∞ functions whose supports are compact and contained in G. The letter C denotes a generic constant and $C_{a,b,c,\dots}$ denotes that the constant $C_{a,b,c,\dots}$ depends on a, b, c, $\cdots$. The value of C and $C_{a,b,c,\dots}$ may change from line to line.

1.2 Maximal L_p-L_q regularity theorem and local well-posedness

In this subsection, we state the maximal L_p-L_q regularity of solutions to the Stokes equations with free boundary condition:

$$\begin{cases} \partial_t \mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}) = \mathbf{f}, \quad \mathrm{div}\,\mathbf{u} = g = \mathrm{div}\,\mathbf{g} & \text{in } \Omega \times (0,T), \\ (\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I})\mathbf{n} = \mathbf{h} & \text{on } \Gamma \times (0,T), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega. \end{cases} \tag{21}$$

We start with the following proposition which was proved in Shibata [9].

Proposition 5. *Let $1 < q < \infty$. If $\mathbf{u} \in H^1_q(\Omega)$ satisfies $\mathrm{div}\,\mathbf{u} = 0$ in Ω, then $\mathbf{u} \in J_q(\Omega)$.*

We next consider the weak Dirichlet problem:

$$(\nabla u, \nabla\varphi)_\Omega = (\mathbf{f}, \nabla\varphi)_\Omega \quad \text{for any } \varphi \in \hat H^1_{q',0}(\Omega). \tag{22}$$

Then, we know the following fact.

Proposition 6. *Let $1 < q < \infty$ and let Ω be an exterior domain in $\mathbb{R}^N$ ($N \geq 2$) with C^2 boundary. Then, the weak Dirichlet problem is uniquely solvable. Namely, for any $\mathbf{f} \in L_q(\Omega)^N$, problem (22) admits a unique solution $u \in \hat{H}^1_{q,0}(\Omega)$ possessing the estimate: $\|\nabla u\|_{L_q(\Omega)} \leq C\|\mathbf{f}\|_{L_q(\Omega)}$.*

Remark 7. (1) This proposition was proved by Pruess and Simonett [6, Section 7.4] and by Shibata [9, Theorem 18] independently.
(2) Let $\Omega = \mathbb{R}^N \setminus S_1$ and $\Gamma = S_1$, where S_1 denotes the unit sphere in $\mathbb{R}^N$. Let

$$f(x) = \begin{cases} \ln|x| & N = 2, \\ |x|^{-(N-2)} - 1 & N \geq 3. \end{cases}$$

Then, $f(x)$ satisfies the strong Dirichlet problem: $\Delta f = 0$ in Ω and $f|_\Gamma = 0$. Moreover, $f \in H^1_{q,0}(\Omega)$ provided that $q > N/(N-1)$. However, f does not satisfy the weak Dirichlet problem:

$$(\nabla f, \nabla\varphi)_\Omega = 0 \quad \text{for any } \varphi \in \hat{H}^1_{q',0}(\Omega).$$

In fact, $C_0^\infty(\Omega)$ is not dense in $\hat{H}^1_{q',0}(\Omega)$ when $1 < q' < N$. The detailed is discussed in Shibata [9, Appendix A].

Since the weak Dirichlet problem is uniquely solvable, by the result obtained in Shibata [8], we have the following theorem (cf. also Shibata [9]).

Theorem 8. *Let $1 < p, q < \infty$ with $2/p + 1/q \neq 1$ and $0 < T < \infty$. Then, there exists a constant γ_0 such that the following assertion holds: Let*

$$\begin{aligned} &\mathbf{u}_0 \in B^{2(1-1/p)}_{q,p}(\Omega)^N, \ \ \mathbf{f} \in L_p((0,T), L_q(\Omega)^N), \ \ e^{-\gamma t}g \in L_p(\mathbb{R}, H^1_q(\Omega)) \cap H^{1/2}_p(\mathbb{R}, L_q(\Omega)), \\ &e^{-\gamma t}\mathbf{g} \in H^1_p(\mathbb{R}, L_q(\Omega)^N), \ \ e^{-\gamma t}\mathbf{h} \in H^{1/2}_p(\mathbb{R}, L_q(\Omega)^N) \cap L_p(\mathbb{R}, H^1_q(\Omega)^N) \end{aligned} \tag{23}$$

for any $\gamma \geq \gamma_0$, which satisfy the compatibility condition:

$$\operatorname{div}\mathbf{u}_0 = g|_{t=0} \quad \text{in } \Omega \tag{24}$$

and, in addition,

$$(\mu\mathbf{D}(\mathbf{u}_0)\mathbf{n} - \mathbf{h}|_{t=0})_\tau = 0 \quad \text{on } \Gamma \tag{25}$$

provided that $2/p + 1/q < 1$. Then, problem (21) admits unique solutions $\mathbf{u}$ and $\mathfrak{p}$ with

$$\begin{aligned} &\mathbf{u} \in L_p((0,T), H^2_q(\Omega)^N) \cap H^1_p((0,T), L_q(\Omega)^N), \\ &\mathfrak{p} \in L_p((0,T), H^1_q(\Omega) + \hat{H}^1_{q,0}(\Omega)) \end{aligned} \tag{26}$$

satisfying the estimates

$$\begin{aligned} \|\mathbf{u}\|_{L_p((0,T),H^2_q(\Omega))} + \|\partial_t\mathbf{u}\|_{L_p((0,T),L_q(\Omega))} \leq C_\gamma e^{\gamma T}\big[\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|\mathbf{f}\|_{L_p((0,T),L_q(\Omega)} \\ + \|e^{-\gamma t}(g,\mathbf{h})\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \|e^{-\gamma t}(g,\mathbf{h})\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} + \|e^{-\gamma t}\mathbf{g}\|_{H^1_p(\mathbb{R},L_q(\Omega))}\big] \end{aligned} \tag{27}$$

for any $\gamma \geq \gamma_0$ and for some positive constant C depending on γ_0 but independent of $\gamma \geq \gamma_0$.

Remark 9. In the case where $2/p + 1/q < 1$, $\mathbf{D}(\mathbf{u}_0) \in B^{1-2/p}_{q,p}(\Omega)$ and $1 - 2/p > 1/q$, and so $\mathbf{D}(\mathbf{u}_0)|_\Gamma$ exists. However, $1/p < 1/2$ and $\mathbf{h} \in H^{1/2}_p(\mathbb{R}, L_q(\Omega)^N)$ implies that $\mathbf{h}$ is continuous with respect to $t \in \mathbb{R}$ in the $L_q(\Omega)$ topology, and so $\mathbf{h}|_{t=0}$ exists as an element in $L_q(\Omega)$, but we do not know whether the trace of $\mathbf{h}|_{t=0}$ to Γ exists. Thus, we implicitly assume the existence of the trace of $\mathbf{h}|_{t=0}$ to Γ in (25).

Using Theorem 8 and Banach's fixed point theorem, Shibata [9] proved the following local in time unique existence theorem for Eq. (1).

Theorem 10. *Let $2 < p < \infty$, $N < q < \infty$ and $S > 0$. Let Ω be an exterior domain in $\mathbb{R}^N$ $(N \geq 2)$ whose boundary Γ is a C^2 compact hypersurface. Assume that $2/p + N/q < 1$. Then, there exists a time $T > 0$ depending on S such that if initial data $\mathbf{u}_0 \in B^{2(1-1/p)}_{q,p}(\Omega)^N$ satisfies $\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} \leq S$ and the compatibility condition:*

$$\operatorname{div} \mathbf{u}_0 = 0 \text{ in } \Omega, \ \ (\mathbf{D}(\mathbf{u}_0)\mathbf{n})_\tau = 0 \text{ on } \Gamma, \tag{28}$$

then problem (4) *admits a unique solution* $(\mathbf{u}, \mathfrak{q})$ *with*

$$\mathbf{u} \in L_p((0,T), H^2_q(\Omega)^N) \cap H^1_p((0,T), L_q(\Omega)^N), \quad \mathfrak{q} \in L_p((0,T), H^1_q(\Omega) + \hat{H}^1_{q,0}(\Omega))$$

possessing the estimate:

$$\|\mathbf{u}\|_{L_p((0,T),H^2_q(\Omega))} + \|\partial_t \mathbf{u}\|_{L_p((0,T),L_q(\Omega))} \leq CS, \quad \int_0^T \|\kappa(\cdot)\mathbf{v}(\cdot,s)\|_{H^1_\infty(\Omega)}\,ds \leq \delta$$

for some constant $C > 0$ independent of T and S. Here, δ is the constant appearing in (3).

1.3 A new formulation of Eq. (4)

Let $T > 0$ and let

$$\mathbf{u} \in H^1_p((0,T), L_q(\Omega)^N) \cap L_p((0,T), H^2_q(\Omega)^N), \quad \mathfrak{q} \in L_p((0,T), H^1_q(\Omega) + \hat{H}^1_{q,0}(\Omega)) \tag{29}$$

be solutions of Eq. (4) satisfying the condition (3). We then prove the global in time unique existence theorem by prolonging this local solution to any time interval. To prolong $\mathbf{u}$ beyond $(0,T)$, we need some decay estimates of $\mathbf{u}$. For this purpose, we rewrite Eq. (4) in order that the nonlinear terms have suitable decay properties.

In the following, we repeat the argument in Subsec: 1.1. Let

$$a_{ij}(t) = \delta_{ij} + \tilde{a}_{ij}(t), \ \ J(t) = 1 + \tilde{J}(t), \ \ \ell_{ij}(t) = \delta_{ij} + \tilde{\ell}_{ij}(t) \tag{30}$$

with

$$\begin{aligned} &\tilde{a}_{ij}(t) = V_{0ij}\Big(\int_0^t \nabla(\kappa(y)\mathbf{v}(y,s))\,ds\Big), \quad \tilde{J}(t) = J_0\Big(\int_0^t \nabla(\kappa(y)\mathbf{v}(y,s))\,ds\Big), \\ &\tilde{\ell}_{ij}(t) = m_{ij}\Big(\int_0^t \nabla(\kappa(y)\mathbf{v}(y,s))\,ds\Big) := \int_0^t \frac{\partial}{\partial y_j}(\kappa(y)u_i(y,t))\,ds. \end{aligned} \tag{31}$$

By (6) and (14), we have

$$\begin{aligned} &\frac{\partial}{\partial x_j} = \sum_{j=1}^N a_{ji}(t)\frac{\partial}{\partial y_j}, \quad n_{ti} = d(t)\sum_{j=1}^N a_{ji}(t)n_j, \\ &\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} = \sum_{k=1}^N \Big(a_{kj}(t)\frac{\partial u_i}{\partial y_k} + a_{ki}(t)\frac{\partial u_j}{\partial y_k}\Big) = D_{ij}(\mathbf{u}) + \tilde{D}_{ij}(t)\nabla\mathbf{u} \end{aligned} \tag{32}$$

where $d(t) = |\mathbf{A}(t)^\top\mathbf{n}| = |(\mathbf{I} + \mathbf{V}_0(\mathbf{k})^\top)\mathbf{n}|$, and

$$D_{ij}(\mathbf{u}) = \frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i}, \quad \tilde{D}_{ij}(t)\nabla\mathbf{u} = \sum_{k=1}^N \Big(\tilde{a}_{kj}(t)\frac{\partial u_i}{\partial y_k} + \tilde{a}_{ki}(t)\frac{\partial u_j}{\partial y_k}\Big).$$

Moreover, by (8) we have

$$\operatorname{div}\mathbf{v} = \sum_{j,k=1}^N J(t)a_{kj}(t)\frac{\partial u_j}{\partial y_k} = \sum_{j,k=1}^N \frac{\partial}{\partial y_k}(J(t)a_{kj}(t)u_j). \tag{33}$$

And then, Eq. (4) is written as follows:

$$\left\{\begin{aligned}
&\sum_{i=1}^{N}\ell_{is}(t)(\partial_t u_i + (1-\kappa)\sum_{j,k=1}^{N}u_j a_{kj}(t)\frac{\partial u_i}{\partial y_k})\\
&\quad -\mu\sum_{i,j.k=1}^{N}\ell_{is}(t)a_{kj}(t)\frac{\partial}{\partial y_k}D_{ij,t}(\mathbf{u}) - \frac{\partial \mathfrak{q}}{\partial y_s} = 0 && \text{in } \Omega\times(0,T),\\
&\sum_{j,k=1}^{N}J(t)a_{kj}(t)\frac{\partial u_j}{\partial y_k} = \sum_{j,k=1}^{N}\frac{\partial}{\partial y_k}(J(t)a_{kj}(t)u_j) = 0 && \text{in } \Omega\times(0,T),\\
&\mu\sum_{i,j,k=1}^{N}\ell_{is}(t)a_{kj}(t)D_{ij,t}(\mathbf{u})n_k - \mathfrak{q}n_s = 0 && \text{on } \Gamma\times(0,T),\\
&\mathbf{u}|_{t=0} = \mathbf{u}_0 && \text{in } \Omega,
\end{aligned}\right. \tag{34}$$

where s runs from 1 through N. Here, we have used the fact that $(\ell_{ij}) = \mathbf{A}^{-1}$.

In order to get some decay properties of the nonlinear terms, we write

$$\int_0^t \nabla(\kappa(y)\mathbf{u}(y,s))\,ds = \int_0^T \nabla(\kappa(y)\mathbf{u}(y,s))\,ds - \int_t^T \nabla(\kappa(y)\mathbf{u}(y,s))\,ds.$$

In (31), by the Taylor formula we write

$$\begin{aligned}
a_{ij}(t) &= a_{ij}(T) + \mathcal{A}_{ij}(t), & \ell_{ij}(t) &= \ell_{ij}(T) + \mathcal{L}_{ij}(t),\\
D_{ij,t}(\mathbf{u}) &= D_{ij,T}(\mathbf{u}) + \mathcal{D}_{ij}(t)\nabla\mathbf{u}, & J(t) &= J(T) + \mathcal{J}(t)
\end{aligned} \tag{35}$$

with

$$\begin{aligned}
\mathcal{A}_{ij}(t) &= -\int_0^1 V'_{0ij}\Big(\int_0^T \nabla(\kappa(y)\mathbf{u}(y,s))\,ds - \theta\int_t^T \nabla(\kappa(y)\mathbf{u}(y,s))\,ds\Big)\,d\theta\int_t^T \nabla(\kappa(y)\mathbf{u}(y,s))\,ds\\
\mathcal{L}_{ij}(t) &= -\int_t^T \frac{\partial}{\partial y_j}(\kappa(y)u_i(y,s))\,ds, \quad \mathcal{D}_{ij}(t)\nabla\mathbf{u} = \sum_{k=1}^{N}(\mathcal{A}_{kj}(t)\frac{\partial u_i}{\partial y_k} + \mathcal{A}_{ki}(t)\frac{\partial u_j}{\partial y_k}),\\
\mathcal{J}(t) &= -\int_0^1 J_0'\Big(\int_0^T \nabla(\kappa(y)\mathbf{u}(y,s))\,ds - \theta\int_t^T \nabla(\kappa(y)\mathbf{u}(y,s))\,ds\Big)\,d\theta\int_t^T \nabla(\kappa(y)\mathbf{u}(y,s))\,ds,
\end{aligned}$$

where V'_{0ij} and J'_0 are derivatives of V_{0ij} and J_0 with respect to $\mathbf{k}$. By the relation:

$$\sum_{s=1}^{N}\ell_{is}(T)a_{sm}(T) = \delta_{si}, \tag{36}$$

the first equation in (34) is rewritten as follows:

$$\partial_t u_m - \sum_{j,k=1}^{N}a_{kj}(T)\frac{\partial}{\partial y_k}(\mu D_{mj,T}(\mathbf{u}) - \delta_{mj}\mathfrak{q}) = \tilde{f}_m(\mathbf{u})$$

with

$$\begin{aligned}
\tilde{f}_m(\mathbf{u}) &= -\sum_{s=1}^{N}a_{sm}(T)\{\sum_{i=1}^{N}\mathcal{L}_{is}(t)\partial_t u_i + \sum_{i,j,k=1}^{N}(1-\kappa)\ell_{is}(t)a_{kj}(t)u_j\frac{\partial u_i}{\partial y_k}\}\\
&\quad + \mu\sum_{s=1}^{N}a_{sm}(T)\{\sum_{i,j,k=1}^{N}\ell_{is}(T)a_{kj}(T)\frac{\partial}{\partial y_k}(\mathcal{D}_{ij}(t)\nabla\mathbf{u}) + \sum_{i,j,k=1}^{N}\ell_{is}(T)\mathcal{A}_{kj}(t)\frac{\partial}{\partial y_k}D_{ij,t}(\mathbf{u})\\
&\quad + \sum_{i,j,k=1}^{N}\mathcal{L}_{is}(t)a_{kj}(t)\frac{\partial}{\partial y_k}D_{ij,t}(\mathbf{u})\}.
\end{aligned} \tag{37}$$

The second equation in (34) is rewritten as

$$\widetilde{\operatorname{div}}\,\mathbf{u} = \tilde{g}(\mathbf{u}) = \operatorname{div}\tilde{\mathbf{g}}(\mathbf{u})$$

with

$$\begin{aligned}
\widetilde{\operatorname{div}}\,\mathbf{u} &= \sum_{j,k=1}^{N} J(T)a_{kj}(T)\frac{\partial u_j}{\partial y_k} = \sum_{j,k=1}^{N} \frac{\partial}{\partial y_k}(J(T)a_{kj}(T)u_j), \\
\tilde{g}(\mathbf{u}) &= \sum_{j,k=1}^{N} (J(T)\mathcal{A}_{kj}(t) + \mathcal{J}(t)a_{kj}(t))\frac{\partial u_j}{\partial y_k}, \\
\tilde{g}_k(\mathbf{u}) &= \sum_{j=1}^{N}(J(T)\mathcal{A}_{kj}(t) + \mathcal{J}(t)a_{kj}(t))u_j, \quad \tilde{\mathbf{g}}(\mathbf{u}) = (\tilde{g}_1(\mathbf{u}), \ldots, \tilde{g}_N(\mathbf{u}))^\top.
\end{aligned} \tag{38}$$

And the boundary condition in (34) is rewritten as

$$\sum_{j,k=1}^{N} a_{kj}(T)(\mu D_{mj,T}(\mathbf{u}) - \delta_{mj}\mathfrak{q})n_k = \tilde{h}_m(\mathbf{u})$$

with

$$\tilde{h}_m(\mathbf{u}) = -\mu\sum_{j,k=1}^{N}(a_{kj}(T)\mathcal{D}_{mj}(t)\nabla\mathbf{u} + \mathcal{A}_{kj}(t)D_{mj,t}(\mathbf{u}))n_k - \mu\sum_{i,j,k,s=1}^{N} a_{sm}(T)\mathcal{L}_{is}(t)a_{kj}(t)D_{ij,t}(\mathbf{u})n_k. \tag{39}$$

By (33),

$$\sum_{j,k=1}^{N} a_{kj}(T)\frac{\partial}{\partial y_k}(\mu D_{mj,T}(\mathbf{u}) - \delta_{mj}\mathfrak{q}) = J(T)^{-1}\sum_{j,k=1}^{N}\frac{\partial}{\partial y_k}(J(T)a_{kj}(T)(\mu D_{mj,T}(\mathbf{u}) - \delta_{mj}\mathfrak{q})).$$

Thus, letting

$$\begin{gathered}
S_{mk}(\mathbf{u},\mathfrak{q}) = \sum_{j=1}^{N} J(T)a_{kj}(T)(D_{mj,T}(\mathbf{u}) - \delta_{mj}\mathfrak{q}), \quad \tilde{\mathbf{S}}(\mathbf{u},\mathfrak{q}) = (S_{ij}(\mathbf{u},\mathfrak{q})), \\
\tilde{\mathbf{f}}(\mathbf{u}) = (\tilde{f}_1(\mathbf{u}), \ldots, \tilde{f}_N(\mathbf{u}))^\top, \quad \tilde{\mathbf{h}}(\mathbf{u}) = (\tilde{h}_1(\mathbf{u}), \ldots, \tilde{h}_N(\mathbf{u}))^\top,
\end{gathered}$$

and using (33), we see that $\mathbf{u}$ and $\mathfrak{q}$ satisfy the following equations:

$$\begin{cases}
\partial_t\mathbf{u} - J(T)^{-1}\operatorname{Div}\tilde{\mathbf{S}}(\mathbf{u},\mathfrak{q}) = \tilde{\mathbf{f}}(\mathbf{u}) & \text{in } \Omega\times(0,T), \\
\widetilde{\operatorname{div}}\,\mathbf{u} = \tilde{g}(\mathbf{u}) = \operatorname{div}\tilde{\mathbf{g}}(\mathbf{u}) & \text{in } \Omega\times(0,T), \\
\tilde{\mathbf{S}}(\mathbf{u},\mathfrak{q})\mathbf{n} = J(T)\tilde{\mathbf{h}}(\mathbf{u}) & \text{on } \Gamma\times(0,T), \\
\mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega.
\end{cases} \tag{40}$$

This is the new formula of the equations which local in time solutions $\mathbf{u}$ and $\mathfrak{q}$ of Eq. (4) satisfy. We call Eq. (40) slightly perturbed Stokes equations.

1.4 Slightly perturbed Stokes equations

In this subsection we summarize some results obtained by Shibata [10] concerning the slightly perturbed Stokes equations. Let r be an exponent such that $N < r < \infty$. Let $a_{ij}(T)$, $\tilde{a}_{ij}(T)$, $J(T)$ and $\tilde{J}(T)$ be functions defined in (30). We assume that

$$\|(\tilde{a}_{ij}(T), \tilde{J}(T))\|_{L_\infty(\Omega)} + \|\nabla(\tilde{a}_{ij}(T), \tilde{J}(T))\|_{L_r(\Omega)} \le \sigma \tag{41}$$

with some small constant $\sigma > 0$. In the following, we write $a_{ij}(T)$, $\tilde{a}_{ij}(T)$, $J(T)$ and $\tilde{J}(T)$ simply by a_{ij}, $\tilde{a}_{ij}$, J and $\tilde{J}$, respectively. And also, we write $\mathbf{A}(T) = (a_{ij}(T))$ by $\mathbf{A}$. We want to state the maximal L_p-L_q regularity and some decay properties of solutions of Eq. (40). To define the solenoidal space for (40), we introduce the weak Dirichlet problem:

$$(\tilde{\nabla} u, J\tilde{\nabla}\varphi)_\Omega = (\mathbf{f}, J\tilde{\nabla}\varphi)_\Omega \quad \text{for any } \varphi \in \hat{H}^1_{q',0}(\Omega). \tag{42}$$

Here,

$$\tilde{\nabla}\varphi = \Big(\sum_{k=1}^N a_{k1}\frac{\partial\varphi}{\partial x_k}, \ldots, \sum_{k=1}^N a_{kN}\frac{\partial\varphi}{\partial x_k}\Big)^\top = \mathbf{A}^\top\nabla\varphi.$$

Since $\tilde{a}_{ij}$ and $\tilde{J}$ vanish outside of B_{2R}, $\widetilde{\operatorname{div}}\,\mathbf{u} = \operatorname{div}\mathbf{u}$ and $\tilde{\nabla}\varphi = \nabla\varphi$ in $\mathbb{R}^N \setminus B_{2R}$. Thus, by Proposition 6 and (41) with small $\sigma > 0$, we have the following result.

Proposition 11. *Let $1 < q \le r$. Then, for any $\mathbf{f} \in L_q(\Omega)^N$ problem (42) admits a unique solution $u \in \hat{H}^1_{q,0}(\Omega)$ possessing the estimate: $\|\nabla u\|_{L_q(\Omega)} \le C\|\mathbf{f}\|_{L_q(\Omega)}$.*

Let $\tilde{J}(\Omega)$ be the space defined by

$$\tilde{J}(\Omega) = \{\mathbf{u} \in L_q(\Omega) \mid (\mathbf{u}, J\tilde{\nabla}\varphi)_\Omega = 0 \quad \text{for any } \varphi \in \hat{H}^1_{q',0}(\Omega)\}.$$

Given $\mathbf{f} \in L_q(\Omega)^N$, let $u \in \hat{H}^1_{q,0}(\Omega)$ be a unique solution of the weak Dirichlet problem (42), and then $\mathbf{f} - \tilde{\nabla} u \in \tilde{J}_q(\Omega)$, and so the projection $\tilde{P} : L_q(\Omega)^N \to \tilde{J}(\Omega)$ is defined by $\tilde{P}\mathbf{f} = \mathbf{f} - \tilde{\nabla} u$. Obviously, $\|\tilde{P}\mathbf{f}\|_{L_q(\Omega)} \le C\|\mathbf{f}\|_{L_q(\Omega)}$.

We consider the initial problem:

$$\partial_t\mathbf{u} - J(T)^{-1}\operatorname{Div}\tilde{\mathbf{S}}(\mathbf{u}, \mathfrak{q}) = 0, \quad \widetilde{\operatorname{div}}\,\mathbf{u} = 0 \quad \text{in } \Omega\times(0,T), \quad \tilde{\mathbf{S}}(\mathbf{u},\mathfrak{q})\mathbf{n}|_\Gamma = 0, \quad \mathbf{u}|_{t=0} = \mathbf{f}. \tag{43}$$

Shibata [10] proved the following theorem.

Theorem 12. *Assume that $N \ge 3$. Then, there exists a $\sigma > 0$ such that if the assumption (41) holds, then for any $q \in (1, r]$, there exists a C^0 analytic semigroup $\{T_S(t)\}_{t\ge0}$ such that for any $\mathbf{f} \in L_q(\Omega)$, a unique solution $\mathbf{u}$ of Eq. (43) is represented by $\mathbf{u} = T_S(t)\tilde{P}\mathbf{f}$.*

Moreover, for any $p \in [q, \infty]$, $\mathbf{f} \in \tilde{J}_q(\Omega)$, and $t > 0$ we have the following estimates:

$$\begin{aligned} \|T_S(t)\tilde{P}\mathbf{f}\|_{L_p(\Omega)} &\le C_{q,p}t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|\mathbf{f}\|_{L_q(\Omega)}, \\ \|\nabla T_S(t)\tilde{P}\mathbf{f}\|_{L_p(\Omega)} &\le C_{q,p}t^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\|\mathbf{f}\|_{L_q(\Omega)}. \end{aligned} \tag{44}$$

If we consider the equations:

$$\begin{cases} \partial_t\mathbf{u} - J^{-1}\operatorname{Div}\tilde{\mathbf{S}}(\mathbf{u},\mathfrak{q}) = \mathbf{f}, \quad \widetilde{\operatorname{div}}\,\mathbf{u} = 0 & \text{in } \Omega\times(0,T), \\ \tilde{\mathbf{S}}(\mathbf{u},\mathfrak{q})\mathbf{n} = 0 & \text{on } \Gamma, \\ \mathbf{u}|_{t=0} = 0 & \text{in } \Omega. \end{cases} \tag{45}$$

Let $\psi \in \hat{H}^1_{q,0}(\Omega)$ be a solution of the weak Dirichlet problem:

$$(\tilde{\nabla}\psi, J\tilde{\nabla}\varphi)_\Omega = (\mathbf{f}, J\tilde{\nabla}\varphi)_\Omega \quad \text{for any } \varphi \in \hat{H}^1_{q',0}(\Omega).$$

Let $\mathbf{g} = \mathbf{f} - \tilde{\nabla}\psi$, and then $\mathbf{g} \in \tilde{J}_q(\Omega)$ and

$$\|\mathbf{g}\|_{L_q(\Omega)} + \|\nabla\psi\|_{L_q(\Omega)} \le C\|\mathbf{f}\|_{L_q(\Omega)}.$$

Using this decomposition, we can rewrite Eq. (45) as

$$\begin{cases} \partial_t\mathbf{u} - J^{-1}\operatorname{Div}\tilde{\mathbf{S}}(\mathbf{u},\mathfrak{q}-\psi) = \mathbf{g}, \quad \widetilde{\operatorname{div}}\,\mathbf{u} = 0 & \text{in } \Omega\times(0,T), \\ \tilde{\mathbf{S}}(\mathbf{u},\mathfrak{q}-\psi)\mathbf{n} = 0 & \text{on } \Gamma, \\ \mathbf{u}|_{t=0} = 0 & \text{in } \Omega. \end{cases}$$

where we have used the fact that $\psi|_\Gamma = 0$. By Duhamel's principle, we have

$$\mathbf{u} = \int_0^t T_S(t-s)\mathbf{g}(s)\,ds = \int_0^t T_S(t-s)\tilde{P}\mathbf{f}(s)\,ds. \tag{46}$$

This is a solution formula of Eq. (45).

Finally, we consider the equations:

$$\begin{cases} \partial_t\mathbf{u} + \lambda_0\mathbf{u} - J(T)^{-1}\mathrm{Div}\,\tilde{\mathbf{S}}(\mathbf{u},\mathfrak{q}) = \mathbf{f} & \text{in } \Omega\times(0,T), \\ \widetilde{\mathrm{div}}\,\mathbf{u} = g = \mathrm{div}\,\mathbf{g} & \text{in } \Omega\times(0,T), \\ \tilde{\mathbf{S}}(\mathbf{u},\mathfrak{q})\mathbf{n} = \mathbf{h} & \text{on } \Gamma\times(0,T), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \Omega. \end{cases} \tag{47}$$

Let

$$\tilde{\mathbf{n}} = \frac{\mathbf{A}^\top\mathbf{n}}{|\mathbf{A}^\top\mathbf{n}|}, \quad \tilde{\mathbf{D}}(\mathbf{u}) = (D_{ij,T}(\mathbf{u})).$$

Using these symbols, the boundary conditions in (47) is written as follows:

$$(\tilde{\mathbf{D}}(\mathbf{u}) - \mathfrak{q}\mathbf{I})\tilde{\mathbf{n}} = (J(T)|\mathbf{A}^\top\mathbf{n}|)^{-1}\mathbf{h} \quad \text{on } \Gamma\times(0,T).$$

For any N-vector $\mathbf{d}$, let $\mathbf{d}_{\tilde{\tau}} = \mathbf{d} - <\mathbf{d},\tilde{\mathbf{n}}>\tilde{\mathbf{n}}$. The following theorem was proved in Shibata [10].

Theorem 13. *Let $1 < p,q < \infty$ and assume that $2/p + N/q \neq 1$. Then, there exist constants $\sigma > 0$ and $\lambda_0 > 0$ such that if the assumption* (41) *holds, then the following assertion holds: Let $\mathbf{u}_0 \in B^{2(1-1/p)}_{q,p}(\Omega)^N$ be initial data for Eq.* (47) *and let $\mathbf{f}$, g, $\mathbf{g}$, d, $\mathbf{h}$ be given functions in the right side of Eq.* (47) *with*

$$\mathbf{f}\in L_p(\mathbb{R},L_q(\Omega)^N),\quad g\in H^1_p(\mathbb{R},H^1_q(\Omega))\cap H^{1/2}_p(\mathbb{R},L_q(B_R)),\quad \mathbf{g}\in H^1_p(\mathbb{R},L_q(\Omega)^N),$$
$$\mathbf{h}\in H^1_p(\mathbb{R},H^1_q(\Omega)^N)\cap H^{1/2}_p(\mathbb{R},L_q(\Omega)^N).$$

Assume that the compatibility condition: $\widetilde{\mathrm{div}}\,\mathbf{u}_0 = g|_{t=0}$ in Ω holds. In addition, the compatibility condition: $(\mu\tilde{\mathbf{D}}(\mathbf{u}_0))_{\tilde{\tau}} = (J(T)|\mathbf{A}^\top|)^{-1}\mathbf{h}|_{t=0})_{\tilde{\tau}}$ on Γ holds provided $2/p + 1/q < 1$. Then, problem (47) *admits unique solutions $\mathbf{u}$ and $\mathfrak{q}$ with*

$$\mathbf{u}\in H^1_p((0,T),L_q(\Omega)^N)\cap L_p((0,T),H^2_q(\Omega)^N),\quad \mathfrak{q}\in L_p((0,T),H^1_q(\Omega)+\hat{H}^1_{q,0}(\Omega))$$

possessing the estimate:

$$\begin{aligned} &\|\mathbf{u}\|_{L_p((0,T),H^2_q(\Omega))} + \|\partial_t\mathbf{u}\|_{L_p((0,T),L_q(\Omega))} \leq C(\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|\mathbf{f}\|_{L_p(\mathbb{R},L_q(\Omega))} \\ &\quad + \|(g,\mathbf{h})\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} + \|(g,\mathbf{h})\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \|\partial_t\mathbf{g}\|_{L_p(\mathbb{R},L_q(\Omega))}) \end{aligned} \tag{48}$$

for some constant C.

Since $\partial_t(<t>^b\,\mathbf{u}) = <t>^b\,\partial_t\mathbf{u} + b<t>^{b-1}\,\mathbf{u}$, if $\mathbf{u}$ and $\mathfrak{q}$ satisfy Eq. (47), then $<t>^b\,\mathbf{u}$ and $<t>^b\,\mathfrak{q}$ satisfy the equations:

$$\begin{aligned} \partial_t(<t>^b\,\mathbf{u}) + \lambda_0(<t>^b\,\mathbf{u}) - J(T)^{-1}\mathrm{Div}\,\tilde{\mathbf{S}}(<t>^b\,\mathbf{u},<t>^b\,\mathfrak{q})& \\ = <t>^b\,\mathbf{f} + b<t>^{b-1}\,\mathbf{u} &\quad \text{in } \Omega\times(0,T), \\ \widetilde{\mathrm{div}}\ <t>^b\,\mathbf{u} = <t>^b\,g = \mathrm{div}\,(<t>^b\,\mathbf{g}) &\quad \text{in } \Omega\times(0,T), \\ \tilde{\mathbf{S}}(<t>^b\,\mathbf{u},<t>^b\,\mathfrak{q})\mathbf{n} = <t>^b\,\mathbf{h} &\quad \text{on } \Gamma\times(0,T), \\ <t>^b\,\mathbf{u}|_{t=0} = \mathbf{u}_0 &\quad \text{in } \Omega. \end{aligned}$$

Thus, repeated use of Theorem 13 yields that

$$\begin{aligned} &\|<t>^b\,\mathbf{u}\|_{L_p((0,T),H^2_q(\Omega))} + \|<t>^b\,\partial_t\mathbf{u}\|_{L_p((0,T),L_q(\Omega))} \\ &\leq C(\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q,p}(\Omega)} + \|<t>^b\,\mathbf{f}\|_{L_p(\mathbb{R},L_q(\Omega))} + \|(<t>^b\,g,<t>^b\,\mathbf{h})\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))} \\ &\quad + \|(<t>^b\,g,<t>^b\,\mathbf{h})\|_{L_p(\mathbb{R},H^1_q(\Omega))} + \|\partial_t(<t>^b\,\mathbf{g})\|_{L_p(\mathbb{R},L_q(\Omega))}), \end{aligned} \tag{49}$$

provided that the right hand side is finite.

1.5 The estimate of the nonlinear terms

We state the estimate of $\tilde{\mathbf{f}}(\mathbf{u})$, $\tilde{g}(\mathbf{u})$, $\tilde{\mathbf{g}}(\mathbf{u})$, and $\tilde{\mathbf{h}}(\mathbf{u})$, which are defined in Subsec. 1.3. To use the estimate (49), we have to extend $\tilde{g}(\mathbf{u})$, $\tilde{\mathbf{g}}(\mathbf{u})$ and $\tilde{\mathbf{h}}(\mathbf{u})$ to $\mathbb{R}$. Notice that $\tilde{g}(\mathbf{u})$, $\tilde{\mathbf{g}}(\mathbf{u})$ and $\tilde{\mathbf{h}}(\mathbf{u})$ vanish at $t=T$. Given f defined on $(0,T)$ vanishing at $t=T$, we define $e_T[f]$ by

$$e_T[f](t)=\begin{cases}0 & \text{for } t>T,\\ f(t) & \text{for } 0<t<T,\\ f(-t) & \text{for } -T<t<0,\\ 0 & \text{for } t<-T.\end{cases}$$

Notice that $e_T[f]=f$ on $0,T)$. Then, for $q=q_1/2$ and $q=q_2$ we have

$$\begin{aligned}&\| <t>^b\tilde{\mathbf{f}}(\mathbf{u})\|_{L_p((0,T),L_q(\Omega))}+\| <t>^b(e_T[\tilde{g}(\mathbf{u})],<t>^b e_T[\tilde{J}(T)\mathbf{h}(\mathbf{u})])\|_{H^{1/2}_p(\mathbb{R},L_q(\Omega))}\\ &\quad+\|(<t>^b e_T[\tilde{g}(\mathbf{u})],<t>^b e_T[\tilde{J}(T)\mathbf{h}(\mathbf{u})])\|_{L_p(\mathbb{R},H^1_q(\Omega))}+\| <t>^b\partial_t(e_T[\mathbf{h}(\mathbf{u})])\|_{L_p(\mathbb{R},L_q(\Omega))}\\ &\quad\le C(\mathcal{I}+[\mathbf{u}]^2_T),\end{aligned}\tag{50}$$

where $\mathcal{I}=\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q_2,p}(\Omega))}+\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q_1/2,p}(\Omega)}$. The proof of (50) is given in Shibata [11].

1.6 A proof of Theorem 1

Let T be a positive number >2. Then, there exists an $\epsilon_0>0$ depending on T such that if initial data $\mathbf{u}_0$ satisfies $\|\mathbf{u}_0\|_{B^{2(1-1/p)}_{q_2,p}(\Omega)}\le\epsilon_0$, then problem (4) admits unique solutions $\mathbf{u}$ and $\mathfrak{q}$ satisfying the regularity condition (29) and the condition (3). We prolong $\mathbf{u}$ and $\mathfrak{q}$ to any time interval beyond T. For this purpose, it suffices to prove that

$$[\mathbf{u}]_T\le C(\mathcal{I}+[\mathbf{u}]^2_T)\tag{51}$$

for some constant $C>0$. Once obtaining (51), we can show that there exists a small constant $\epsilon\in(0,\epsilon_0)$ such that if $\mathcal{I}\le\epsilon$ then $[\mathbf{u}]_T\le C\epsilon$ for some constant $C>0$ independent of ϵ, and so we can prolong $\mathbf{u}$ to any time interval beyond $(0,T)$. In the following, we use the results stated in Subsec. 1.4 with $r=q_2$ and Subsec. 1.5.

As was seen in Subsec. 1.3, $\mathbf{u}$ and $\mathfrak{q}$ satisfy Eq. (40). To estimate $\mathbf{u}$, we divide $\mathbf{u}$ and $\mathfrak{q}$ into two parts as $\mathbf{u}=\mathbf{w}+\mathbf{v}$, and $\mathfrak{q}=\mathfrak{r}+\mathfrak{p}$, where $\mathbf{w}$ and $\mathfrak{r}$ are solutions of the equations:

$$\begin{cases}\partial_t\mathbf{w}+\lambda_0\mathbf{w}-J(T)^{-1}\mathrm{Div}\,\tilde{\mathbf{S}}(\mathbf{w},\mathfrak{r})=\tilde{\mathbf{f}}(\mathbf{u}) & \text{in } \Omega\times(0,T),\\ \widetilde{\mathrm{div}}\,\mathbf{w}=e_T[\tilde{g}(\mathbf{u})]=\mathrm{div}\,e_T[\tilde{\mathbf{g}}(\mathbf{u})] & \text{in } \Omega\times(0,T),\\ \tilde{\mathbf{S}}(\mathbf{w},\mathfrak{r})=e_T[J(T)\tilde{\mathbf{h}}(\mathbf{u})] & \text{on } \Gamma\times(0,T),\\ \mathbf{w}|_{t=0}=\mathbf{u}_0 & \text{in } \Omega,\end{cases}\tag{52}$$

and $\mathbf{v}$ and $\mathfrak{p}$ are solutions of the equations:

$$\begin{cases}\partial_t\mathbf{v}-J(T)^{-1}\mathrm{Div}\,\tilde{\mathbf{S}}(\mathbf{v},\mathfrak{p})=-\lambda_0\mathbf{w} & \text{in } \Omega\times(0,T),\\ \widetilde{\mathrm{div}}\,\mathbf{v}=0 & \text{in } \Omega\times(0,T),\\ \tilde{\mathbf{S}}(\mathbf{v},\mathfrak{p})=0 & \text{on } \Gamma\times(0,T),\\ \mathbf{v}|_{t=0}=0 & \text{in } \Omega.\end{cases}\tag{53}$$

Concerning the estimate of $\mathbf{w}$, applying (49) and using estimate (50), we have

$$[\mathbf{w}]_T\le C(\mathcal{I}+[\mathbf{u}]^2_T).\tag{54}$$

We next consider $\mathbf{v}$. By (46), we have

$$\mathbf{v}(\cdot,t)=-\lambda_0\int_0^t T_S(t-s)\tilde{P}\mathbf{w}(\cdot,s)\,ds.\tag{55}$$

Using the estimate (44) yields that

$$\begin{aligned}\|\nabla^j \mathbf{v}(\cdot,t)\|_{L_r(\Omega)} &\le C_{r,\tilde q_1}\int_0^{t-1}(t-s)^{-\frac{j}{2}-\frac{N}{2}\left(\frac{1}{\tilde q_1}-\frac{1}{r}\right)}\|\mathbf{w}(\cdot,s)\|_{L_{\tilde q_1}(\Omega)}\,ds\\ &\quad + C_{r,\tilde q_2}\int_{t-1}^{t}(t-s)^{-\frac{j}{2}-\frac{N}{2}\left(\frac{1}{\tilde q_2}-\frac{1}{r}\right)}\|\mathbf{w}(\cdot,s)\|_{L_{\tilde q_2}(\Omega)}\,ds\end{aligned} \tag{56}$$

for $j=0,1$, for any $t>1$ and for any indices r, $\tilde q_1$ and $\tilde q_2$ such that $1<\tilde q_1,\tilde q_2\le r\le\infty$ and $\tilde q_1$, $\tilde q_2\le q_2$, where $\nabla^0\mathbf{v}=\mathbf{v}$ and $\nabla^1\mathbf{v}=\nabla\mathbf{v}$.

Recall that $T>2$. In what follows, we prove that

$$\Big(\int_2^T(<t>^b\|\mathbf{v}(\cdot,t)\|_{H^1_\infty(\Omega)})^p\,dt\Big)^{1/p}\le C(\mathcal{I}+[\mathbf{u}]_T^2), \tag{57}$$

$$\sup_{2\le t\le T}(<t>^{\frac{N}{2q_1}}\|\mathbf{v}(\cdot,t)\|_{L_{q_1}(\Omega)}\le C(\mathcal{I}+[\mathbf{u}]_T^2), \tag{58}$$

$$\Big(\int_2^T(<t>^{b-\frac{N}{2q_1}}\|\mathbf{v}(\cdot,t)\|_{H^1_{q_1}(\Omega)})^p\,dt\Big)^{1/p}\le C(\mathcal{I}+[\mathbf{u}]_T^2), \tag{59}$$

$$\Big(\int_2^T(<t>^{b-\frac{N}{2q_2}}\|\mathbf{v}(\cdot,t)\|_{L_{q_2}(\Omega)})^p\,dt\Big)^{1/p}\le C(\mathcal{I}+[\mathbf{u}]_T^2). \tag{60}$$

By (56) with $r=\infty$, $\tilde q_1=q_1/2$ and $\tilde q_2=q_2$,

$$\|\mathbf{v}(\cdot,t)\|_{H^1_\infty(\Omega)}\le C\int_0^t\|T_S(t-s)\tilde P\mathbf{w}(\cdot,s)\|_{H^1_\infty(\Omega)}\,ds=C(I_\infty(t)+II_\infty(t)+III_\infty(t))$$

with

$$\begin{aligned}I_\infty(t)&=\int_0^{t/2}(t-s)^{-\frac{N}{q_1}}\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)}\,ds,\\ II_\infty(t)&=\int_{t/2}^{t-1}(t-s)^{-\frac{N}{q_1}}\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)}\,ds,\\ III_\infty(t)&=\int_{t-1}^{t}(t-s)^{-\frac{N}{2q_2}-\frac12}\|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)}\,ds.\end{aligned}$$

Since

$$\begin{aligned}I_\infty(t)&\le(t/2)^{-\frac{N}{q_1}}\Big(\int_0^{t/2}<s>^{-bp'}\,ds\Big)^{1/p'}\Big(\int_0^{t/2}(<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds\Big)^{1/p}\\ &\le C(bp'-1)^{-1/p'}(\mathcal{I}+[\mathbf{u}]_T^2)t^{-\frac{N}{q_1}}\end{aligned}$$

as follows from the condition: $bp'>1$ in (19), by the condition: $(\frac{N}{q_1}-b)p>1$ in (19), we have

$$\int_2^T(<t>^bI_\infty(t))^p\,dt\le C\int_2^T<t>^{-\left(\frac{N}{q_1}-b\right)p}\,dt(\mathcal{I}+[\mathbf{u}]_T^2)^p\le C((\frac{N}{q_1}-b)p-1)^{-1}(\mathcal{I}+[\mathbf{u}]_T^2)^p.$$

By Hölder's inequality,

$$\begin{aligned}<t>^bII_\infty(t)&\le C\int_{t/2}^{t-1}(t-s)^{-\frac{N}{q_1}}<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)}\,ds\\ &\le C\Big(\int_{t/2}^{t-1}(t-s)^{-\frac{N}{q_1}}\,ds\Big)^{1/p'}\Big(\int_{t/2}^{t-1}(t-s)^{-\frac{N}{q_1}}(<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds\Big)^{1/p}\\ &\le C\Big(\frac{N}{q_1}-1\Big)^{-1/p'}\Big(\int_{t/2}^{t-1}(t-s)^{-\frac{N}{q_1}}(<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds\Big)^{1/p}\end{aligned}$$

because $N/q_1 = N/q_2 + 1 > 1$. By the change of integration order and (54),

$$\begin{aligned}\int_2^T (< t >^b II_\infty(t))^p\,dt &\le C\Big(\frac{N}{q_1} - 1\Big)^{-\frac{p}{p'}} \int_2^T dt \int_{t/2}^{t-1} (t-s)^{-\frac{N}{q_1}} (< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds \\ &\le C\Big(\frac{N}{q_1} - 1\Big)^{-\frac{p}{p'}} \int_1^{T-1} (< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds \int_{s+1}^{2s} (t-s)^{-\frac{N}{q_1}}\,dt \\ &\le C\Big(\frac{N}{q_1} - 1\Big)^{-p} (\mathcal{I} + [\mathbf{u}]_T^2)^p.\end{aligned}$$

Since $\frac{N}{2q_2} + \frac{1}{2} < 1$ as follows from $q_2 > N$, by Hölder's inequality,

$$\begin{aligned}< t >^b III_\infty(t) &\le C \int_{t-1}^t (t-s)^{-\frac{N}{2q_2} - \frac12} < s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)}\,ds \\ &\le C\Big(\int_{t-1}^t (t-s)^{-\frac{N}{2q_2} - \frac12}\,ds\Big)^{1/p'} \Big(\int_{t-1}^t (t-s)^{-\frac{N}{2q_2} - \frac12} (< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)})^p\,ds\Big)^{1/p} \\ &\le C\Big(\frac{N}{2q_2} - \frac12\Big)^{-1/p'} \Big(\int_{t-1}^t (t-s)^{-\frac{N}{2q_2} - \frac12} (< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)})^p\,ds\Big)^{1/p}.\end{aligned}$$

By the change of integration order, we have

$$\begin{aligned}\int_2^T (< t >^b III_\infty(t))^p\,dt &\le C\Big(1 - \frac{N}{2q_2}\Big)^{-\frac{p}{p'}} \int_2^T dt \int_{t-1}^t (t-s)^{-\frac{N}{2q_2} - \frac12} (< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)})^p\,ds \\ &\le C\Big(1 - \frac{N}{2q_2}\Big)^{-\frac{p}{p'}} \int_1^T (< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)})^p\,ds \int_s^{s+1} (t-s)^{-\frac{N}{2q_2} - \frac12}\,dt \\ &= C\Big(1 - \frac{N}{2q_2}\Big)^{-p} (\mathcal{I} + [\mathbf{u}]_T^2)^p.\end{aligned}$$

Summing up, we have obtained (57).

We next prove (58). By (56) with $r = q_1$, $\tilde q_1 = q_1/2$ and $\tilde q_2 = q_1$,

$$\|\mathbf{v}(\cdot,t)\|_{L_{q_1}(\Omega)} \le C(I_{q_1,\infty}(t) + II_{q_1,1}(t) + III_{q_1,1}(t))$$

with

$$\begin{aligned}I_{q_1,1}(t) &= \int_0^{t/2} (t-s)^{-\frac{N}{2q_1}} \|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)}\,ds, \\ II_{q_1,1}(t) &= \int_{t/2}^{t-1} (t-s)^{-\frac{N}{2q_1}} \|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)}\,ds, \\ III_{q_1,1}(t) &= \int_{t-1}^t \|\mathbf{w}(\cdot,s)\|_{L_{q_1}(\Omega)}\,ds.\end{aligned}$$

By (54)

$$\begin{aligned}I_{q_1,1}(t) &\le (t/2)^{-\frac{N}{2q_1}} \Big(\int_0^{t/2} < s >^{-bp'}\,ds\Big)^{1/p'} \Big(\int_0^T (< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds\Big)^{1/p} \\ &\le Ct^{-\frac{N}{2q_1}} (\mathcal{I} + [\mathbf{u}]_T^2).\end{aligned}$$

Analogously, by Hölder's inequality and (54),

$$\begin{aligned}II_{q_1,1}(t) &\le C \int_{t/2}^{t-1} (t-s)^{-\frac{N}{2q_1}} < s >^{-b} < s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)}\,ds \\ &\le C < t >^{-b} \Big(\int_{t/2}^{t-1} (t-s)^{-\frac{Np'}{2q_1}}\,ds\Big)^{1/p'} \Big(\int_0^T (< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds\Big)^{1/p}\end{aligned}$$

$$= C\Big(1 - \frac{Np'}{2q_1}\Big)^{1/p'} < t >^{-b-\frac{N}{2q_1}+\frac{1}{p'}} (\mathcal{I} + [\mathbf{u}]_T^2)$$
$$\leq C\Big(1 - \frac{Np'}{2q_1}\Big)^{1/p'} < t >^{-\frac{N}{2q_1}} (\mathcal{I} + [\mathbf{u}]_T^2),$$

because $b > \frac{1}{p'}$. Finally, by (54),

$$\begin{aligned} III_{q_1,1}(t) &\leq Ct^{-b} \int_{t-1}^{t} < s >^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}\, ds \\ &\leq Ct^{-b}\Big(\int_{t-1}^{t} ds\Big)^{1/p'} \Big(\int_0^T (< s >^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)})^p\, ds\Big)^{1/p} \\ &\leq Ct^{-b}(\mathcal{I} + [\mathbf{u}]_T^2). \end{aligned}$$

Summing up, we have obtained (58).

We next prove (59). By (56),

$$\|\mathbf{v}(\cdot, t)\|_{H^1_{q_1}(\Omega)} \leq C(I_{q_1,2}(t) + II_{q_1,2}(t) + III_{q_1,2}(t))$$

with

$$\begin{aligned} I_{q_1,2}(t) &= \int_0^{t/2} (t - s)^{-\frac{N}{2q_1}} \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}\, ds, \\ II_{q_1,2}(t) &= \int_{t/2}^{t-1} (t - s)^{-\frac{N}{2q_1}} \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}\, ds, \\ III_{q_1,2}(t) &= \int_{t-1}^{t} (t - s)^{-\frac{1}{2}} \|\mathbf{w}(\cdot, s)\|_{L_{q_1}(\Omega)}\, ds. \end{aligned}$$

By (54),

$$\begin{aligned} I_{q_1,2}(t) &\leq (t/2)^{-\frac{N}{2q_1}} \Big(\int_0^{t/2} < s >^{-bp'}\, ds\Big)^{1/p'} \Big(\int_0^{t/2} (< s >^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)})^p\, ds\Big)^{1/p} \\ &\leq Ct^{-\frac{N}{2q_1}}(\mathcal{I} + [\mathbf{u}]_T^2), \end{aligned}$$

and so, by the condition: $(\frac{N}{q_1} - b)p > 1$ in (19)

$$\Big(\int_2^T (< t >^{b-\frac{N}{2q_1}} I_{q_1,2}(t))^p\, dt\Big)^{1/p} \leq C\Big((\frac{N}{q_1} - b)p - 1\Big)^{-1/p}(\mathcal{I} + [\mathbf{u}]_T^2).$$

By Hölder's inequality,

$$\begin{aligned} < t >^{b-\frac{N}{2q_1}} II_{q_1,2}(t) &\leq C < t >^{-\frac{N}{2q_1}} \int_{t/2}^{t-1} (t - s)^{-\frac{N}{2q_1}} < s >^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)}\, ds \\ &\leq C < t >^{-\frac{N}{2q_1}} \Big(\int_{t/2}^{t-1} (t - s)^{-\frac{Np'}{2q_1}}\, ds\Big)^{1/p'} \Big(\int_0^T (< s >^b \|\mathbf{w}(\cdot, s)\|_{L_{q_1/2}(\Omega)})^p\, ds\Big)^{1/p} \\ &\leq C(1 + t)^{-\left(\frac{N}{q_1} - \frac{1}{p'}\right)}(\mathcal{I} + [\mathbf{u}]_T^2). \end{aligned}$$

Since $(\frac{N}{q_1} - \frac{1}{p'})p > 1$ as follows from $\frac{N}{q_1} = 1 + \frac{N}{q_2} > 1 = \frac{1}{p} + \frac{1}{p'}$, we have

$$\Big(\int_2^T (< t >^{b-\frac{N}{2q_1}} II_{q_1,2}(t))^p\, dt\Big)^{1/p} \leq C\Big((\frac{N}{q_1} - b)p - 1\Big)^{-1/p}(\mathcal{I} + [\mathbf{u}]_T^2).$$

Since $q_1/2 < q_1 < q_2$, we have

$$\|\mathbf{w}(\cdot, t)\|_{L_{q_1}(\Omega)} \leq \|\mathbf{w}(\cdot, t)\|_{L_{q_1/2}(\Omega)}^{\frac{q_2}{N+2q_2}} \|\mathbf{w}(\cdot, t)\|_{L_{q_2}(\Omega)}^{\frac{N+q_2}{N+2q_2}}.$$

Let $\alpha = \frac{q_2}{N+2q_2}$ and $\beta = \frac{N+q_2}{N+2q_2}$, and then $\alpha+\beta = 1$, and so, by (54) and Hölder's inequality

$$
\begin{aligned}
&\| < t >^b \mathbf{w}\|_{L_p((0,T),L_{q_1}(\Omega))} \\
&\le \Big(\int_0^T (< t >^b \|\mathbf{w}(\cdot,t)\|_{L_{q_1/2}(\Omega)})^{p\alpha}(< t >^b \|\mathbf{w}(\cdot,t)\|_{L_{q_2}(\Omega)})^{p\beta}\,dt\Big)^{1/p} \\
&\le \Big(\int_0^T (< t >^b \|\mathbf{w}(\cdot,t)\|_{L_{q_1/2}(\Omega)})^p\,dt\Big)^{\alpha/p}\Big(\int_0^T (< t >^b \|\mathbf{w}(\cdot,t)\|_{L_{q_2}(\Omega)})^p\,dt\Big)^{\beta/p} \\
&\le C(\mathcal{I} + [\mathbf{u}]_T^2).
\end{aligned}
\tag{61}
$$

Since

$$
\begin{aligned}
< t >^{b-\frac{N}{2q_1}} III_{q_1,2}(t) &\le \int_{t-1}^t (t-s)^{-\frac12} < s >^{b-\frac{N}{2q_1}} \|\mathbf{w}(\cdot,s)\|_{L_{q_1}(\Omega)}\,ds \\
&\le \Big(\int_{t-1}^t (t-s)^{-\frac12}\,ds\Big)^{1/p'}\Big(\int_{t-1}^t (t-s)^{-\frac12}(< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_1}(\Omega)})^p\,ds\Big)^{1/p},
\end{aligned}
$$

by the change of integration order, we have

$$
\begin{aligned}
&\int_2^T (< t >^{b-\frac{N}{2q_1}} III_{q_1,2}(t))^p\,dt \le 2^{\frac{p}{p'}}\int_2^T dt\int_{t-1}^t (t-s)^{-\frac12}(< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_1}(\Omega)})^p\,ds \\
&\le 2^{\frac{p}{p'}}\int_0^T (< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_1}(\Omega)})^p\,ds\int_s^{s+1}(t-s)^{-\frac12}\,dt = 2^p\| < t >^b \mathbf{w}\|_{L_p((0,T),L_{q_1}(\Omega))},
\end{aligned}
$$

which, combined with (61), leads to

$$
\Big(\int_2^T (< t >^{b-\frac{N}{2q_1}} III_{q_1,2}(t))^p\,dt\Big)^{1/p} \le C(\mathcal{I} + [\mathbf{u}]_T^2).
$$

Summing up, we have obtained (59).

We finally prove (60). By (56) with $r = q_2$, $\tilde q_1 = q_1/2$ and $\tilde q_2 = q_2$,

$$
\|\mathbf{v}(\cdot,t)\|_{L_{q_2}(\Omega)} \le C(I_{q_2}(t) + II_{q_2}(t) + III_{q_2}(t))
$$

with

$$
\begin{aligned}
I_{q_2}(t) &= \int_0^{t/2} (t-s)^{-\frac{N}{2}\left(\frac{2}{q_1}-\frac{1}{q_2}\right)}\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)}\,ds, \\
II_{q_2}(t) &= \int_{t/2}^{t-1} (t-s)^{-\frac{N}{2}\left(\frac{2}{q_1}-\frac{1}{q_2}\right)}\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)}\,ds, \\
III_{q_2}(t) &= \int_{t-1}^t \|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)}\,ds.
\end{aligned}
$$

By Hölder's inequality,

$$
\begin{aligned}
I_{q_2}(t) &\le (t/2)^{-\frac{N}{2}\left(\frac{2}{q_1}-\frac{1}{q_2}\right)}\Big(\int_0^{t/2} < s >^{-bp'}\,ds\Big)^{1/p'}\Big(\int_0^{t/2}(< s >^b \|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds\Big)^{1/p} \\
&\le C < t >^{-\frac{N}{2}\left(\frac{2}{q_1}-\frac{1}{q_2}\right)}(\mathcal{I} + [\mathbf{u}]_T^2)
\end{aligned}
$$

for $t \ge 2$. Since

$$
\frac{N}{2}\Big(\frac{2}{q_1}-\frac{1}{q_2}\Big) - \Big(b - \frac{N}{2q_2}\Big) = \frac{N}{q_1} - b,
$$

by the condition: $(\frac{N}{q_1} - b)p > 1$ in (19),

$$
\Big(\int_2^T (< t >^{b-\frac{N}{2q_2}} I_{q_2}(t))^p\,dt\Big)^{1/p} \le C\Big(\int_2^T t^{-\left(\frac{N}{q_1}-b\right)p}\,dt\Big)^{1/p}(\mathcal{I} + [\mathbf{u}]_T^2)
$$

$$\le C\Big(\big(\frac{N}{q_1}-b)p-1\Big)^{-1/p}(\mathcal{I}+[\mathbf{u}]^2_T).$$

Since
$$\frac{N}{2}\Big(\frac{2}{q_1}-\frac{1}{q_2}\Big)=\frac{N}{2}\Big(\frac{1}{q_2}+\frac{2}{N}\Big)=\frac{N}{2q_2}+1>1,$$
by Hölder's inequality

$$\begin{aligned}
<t>^{b-\frac{N}{2q_2}} II_{q_2}(t) &\le C\int_{t/2}^{t-1}(t-s)^{-\left(\frac{N}{2q_2}+1\right)}<s>^{b-\frac{N}{2q_2}}\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)}\,ds\\
&\le C\Big(\int_{t/2}^{t-1}(t-s)^{-\left(\frac{N}{2q_2}+1\right)}\,ds\Big)^{1/p'}\Big(\int_{t/2}^{t-1}(t-s)^{-\left(\frac{N}{2q_2}+1\right)}(<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds\Big)^{1/p}\\
&\le C\Big(\frac{N}{2q_2}\Big)^{-1/p'}\Big(\int_{t/2}^{t-1}(t-s)^{-\left(\frac{N}{2q_2}+1\right)}(<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds\Big)^{1/p},
\end{aligned}$$

and so, by the change of integration order and (54)

$$\begin{aligned}
\int_2^T(<t>^{b-\frac{N}{2q_2}} II_{q_2}(t))^p\,dt &\le C\Big(\frac{N}{2q_2}\Big)^{-p/p'}\int_2^T dt\int_{t/2}^{t-1}(t-s)^{-\left(\frac{N}{2q_2}+1\right)}(<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds\\
&\le C\Big(\frac{N}{2q_2}\Big)^{-p/p'}\int_0^T(<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_1/2}(\Omega)})^p\,ds\int_{s+1}^{2s}(t-s)^{-\left(\frac{N}{2q_2}+1\right)}\,dt \le C\Big(\frac{N}{2q_2}\Big)^{-p}(\mathcal{I}+[\mathbf{u}]^2_T)^p.
\end{aligned}$$

Analogously, by Hölder's inequality

$$\begin{aligned}
<t>^{b-\frac{N}{2q_2}} III_{q_2}(t) &\le C\int_{t-1}^{t}<s>^{b-\frac{N}{2q_2}}\|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)}\,ds,\\
&\le C\Big(\int_{t-1}^{t}ds\Big)^{1/p'}\Big(\int_{t-1}^{t}(<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)})^p\,ds\Big)^{1/p}\\
&= C\Big(\int_{t-1}^{t}(<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)})^p\,ds\Big)^{1/p},
\end{aligned}$$

and so, by the change of integration order and (54)

$$\begin{aligned}
\int_2^T(<t>^{b-\frac{N}{2q_2}} III_{q_2}(t))^p\,dt &\le C\int_2^T dt\int_{t-1}^{t}(<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)})^p\,ds\\
&\le C\int_0^T(<s>^b\|\mathbf{w}(\cdot,s)\|_{L_{q_2}(\Omega)})^p\,ds\int_s^{s+1}dt\le C(\mathcal{I}+[\mathbf{u}]^2_T)^p.
\end{aligned}$$

Summing up, we have obtained (60).

Recalling that $T\ge 2$, applying the maximal L_p-L_q regularity theorem due to Shibata [9] to Eq. (53) and using (54) give that

$$\|\mathbf{v}\|_{L_p((0,2),H^2_q(\Omega))}+\|\partial_t\mathbf{v}\|_{L_p((0,2),L_q(\Omega))}\le C_q\|\lambda_0\mathbf{w}\|_{L_p((0,2),L_q(\Omega))}\le C(\mathcal{I}+[\mathbf{u}]^2_T) \tag{62}$$

for any $q\in[q_1/2,q_2]$. Thus, by real interpolation, we have

$$\sup_{0<t<2}\|\mathbf{v}(\cdot,t)\|_{B^{2(1-1/p)}_{q,p}(\Omega)}\le C(\|\mathbf{v}\|_{L_p((0,2),H^2_q(\Omega))}+\|\partial_t\mathbf{v}\|_{L_p((0,2),L_q(\Omega))})\le C_q(\mathcal{I}+[\mathbf{u}]^2_T) \tag{63}$$

for any $q\in[q_1/2,q_2]$. Combining (57), (58), (59), (60), (62), (63) and the Sobolev imbedding theorem, we have

$$\begin{aligned}
&\|<t>^b\mathbf{v}\|_{L_p((0,T),H^1_\infty(\Omega))}+\|<t>^{\frac{N}{2q_1}}\mathbf{v}\|_{L_\infty((0,T),L_{q_1}(\Omega))}\\
&\quad+\|<t>^{b-\frac{N}{2q_1}}\mathbf{v}\|_{L_p((0,T),H^1_{q_1}(\Omega))}+\|<t>^{b-\frac{N}{2q_2}}\mathbf{v}\|_{L_p((0,T),L_{q_2}(\Omega))}\le C(\mathcal{I}+[\mathbf{u}]^2_T).
\end{aligned}\tag{64}$$

From (53), $\mathbf{v}$ satisfies the equations:

$$\begin{cases} \partial_t \mathbf{v} + \lambda_0 \mathbf{v} - J(T)^{-1} \mathrm{Div}\, \tilde{\mathbf{S}}(\mathbf{v}, \mathfrak{p}) = -\lambda_0 \mathbf{w} + \lambda_0 \mathbf{v} & \text{in } \Omega \times (0,T), \\ \widetilde{\mathrm{div}}\, \mathbf{v} = 0 & \text{in } \Omega \times (0,T), \\ \tilde{\mathbf{S}}(\mathbf{v}, \mathfrak{p}) = 0 & \text{on } \Gamma \times (0,T), \\ \mathbf{v}|_{t=0} = 0 & \text{in } \Omega, \end{cases}$$

and so by (49) we have

$$\begin{aligned} &\| < t >^{b - \frac{N}{2q_2}} \mathbf{v} \|_{L_p((0,T), H^2_{q_2}(\Omega))} + \| < t >^{b - \frac{N}{2q_2}} \partial_t \mathbf{v} \|_{L_p((0,T), L_{q_2}(\Omega))} \\ &\leq C \| < t >^{b - \frac{N}{2q_2}} (\mathbf{v}, \mathbf{w}) \|_{L_p((0,T), L_{q_2}(\Omega))}, \end{aligned}$$

which, combined with (64), leads to

$$[\mathbf{v}]_T \leq C(\mathcal{I} + [\mathbf{u}]_T^2). \tag{65}$$

Since $\mathbf{u} = \mathbf{w} + \mathbf{v}$, by (54) and (65), we see that $\mathbf{u}$ satisfies the inequality (51), which completes the proof of Theorem 1.

2 Two Phase Problem

Let Ω_+ be a bounded domain in $\mathbb{R}^N$ and Γ its boundary that is a smooth compact hypersurface. Let $\Omega_- = \mathbb{R}^N \setminus \overline{\Omega_+}$ and two different incompressible viscous fluids occupy $\Omega_\pm$, respectively. Let $\Omega_{\pm t}$ and Γ_t be the time evolution of $\Omega_\pm$ and Γ for $t > 0$. Let $\mathbf{n}_t$ be the unit outer normal to Γ_t. Problem is to find domains $\Omega_{\pm t}$, velocities $\mathbf{v}_\pm = (v_{1\pm}, \ldots, v_{\pm N})$ and pressures $\mathfrak{p}_\pm$ satisfying the Navier-Stokes equations:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \mathrm{Div}\,(\mu \mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I}) = 0, \quad \mathrm{div}\, \mathbf{v} = 0 & \text{in } \Omega_t \text{ for } t \in (0,T), \\ [[\mu \mathbf{D}(\mathbf{v}) - \mathfrak{p}\mathbf{I}]]\mathbf{n}_t = \sigma H(\Gamma_t)\mathbf{n}_t, \quad [[\mathbf{v}]] = 0, \quad V_{\Gamma_t} = \mathbf{v} \cdot \mathbf{n}_t & \text{on } \Gamma_t \text{ for } t \in (0,T), \\ \mathbf{v}|_{t=0} = \mathbf{v}_0 & \text{in } \Omega_0 = \Omega. \end{cases} \tag{66}$$

Here, $\Omega_t = \Omega_{+t} \cup \Omega_{-t}$, $\Omega = \Omega_+ \cup \Omega_-$, $h = h_\pm$ for $x \in \Omega_{t\pm}$, $\mu = \mu_\pm$ for $x \in \Omega_{\pm t}$, $\mu_\pm$ being positive constants representing viscosity coefficients, σ positive constant (coefficient of surface tension), $H(\Gamma_t)$ $N-1\times$ the mean curvature of Γ_t, V_{Γ_t} the evolution speed of Γ_t in the $\mathbf{n}_t$ direction, $\mathbf{I}$ the $N \times N$ identity matrix, $\mathbf{D}(\mathbf{v}) = \nabla \mathbf{v} + (\nabla \mathbf{v})^\top$ the doubled deformation tensor whose $(i,j)^{\text{th}}$ component is $\partial_i v_j + \partial_j v_i$, and $[[f]](x_0) = \lim_{\substack{x \to x_0 \\ x \in \Omega_+}} f(x) - \lim_{\substack{x \to x_0 \\ x \in \Omega_-}} f(x)$, which is the jump quantity of f at $x_0 \in \Gamma_t$.

This problem has been studied by the following authors:

- V. Denisova, V. A. Solonnikov [1], [2] in the L_2 frame work and the Hölder space framework.
- J. Pruess et al [3], [4], [5], [6], L_p maximal regularity and Local well-posedness. Global well-posedness in the container by the combination of L_p-maximal regularity with Spectral analysis for the Laplace- Bertrami operator.

But, the global well-posedness in unbounded domains has not yet been studied well. In this note, the global well-posedness results are announced in the case that $\sigma = 0$ and $\sigma > 0$.

First of all, we mention that **Maximal L_p-L_q regularity** for the two phase problem for the Stokes equations does hold in a uniformly C^2 ($\sigma = 0$ case) or C^3 ($\sigma > 0$ case) domain under the assumption that weak Neumann problem is uniquely solvable (cf. Pruess et al [3, 4, 5, 6], Shibata and Shimizu [12], and Maryani and Saito [7]). Thus, **Local well-posedness** holds. Here, it is important that p and q can be chosen differently to prove Global well-posedness for free boundary problem in an unbounded domain. In fact as was seen in Sect. 1, in the unbounded domain case, we can get only polynomial decay properties for suitable L_q space norm of solutions, and so we have to choose p rather large to guarantee the L_p summability in time.

2.1 Global wellposedness, $\sigma > 0$ case

Let $B_R = \{x \in \mathbb{R}^N \mid |x| < R\}$ and $S_R = \{x \in \mathbb{R}^N \mid |x| = r\}$. We assume that
Assumption 1 $|\Omega_+| = |B_R| = R^N\omega_N/N$, where $|\cdot|$ denotes the Lebesgue measure and ω_N is the area of the unit sphere.
Assumption 2 $\displaystyle\int_\Omega x\,dx = 0$.
Assumption 3 $\Gamma = \{x = (R + \rho_0(R\omega))\omega \mid \omega \in S_1\}$ with given small function ρ_0 defined on S_R.
Let

$$\Gamma_t = \{x = (R + \rho(R\omega, t))\omega + \xi(t) \mid \omega \in S_1\}$$

where ρ is an unknown function and $\xi(t)$ is the barycenter point of the domain Ω_t defined by

$$\xi(t) = \frac{1}{|\Omega_+|}\int_{\Omega_{+t}} x\,dx.$$

Assume that $\Omega_+ \subset B_R$ with a large constant $R > 0$. Let $L \geq 3R$. Given $\rho \in W_q^{3-1/q}(S_R)$, let $H(\xi, t) \in H_q^3(\dot{B}_L)$ be a function such that $H|_{S_R} = R^{-1}\rho$, $\|H\|_{H_q^2(\dot{B}_L)} \leq C\|\rho\|_{W_q^{2-1/q}(S_R)}$, and $\|H\|_{H_q^3(\dot{B}_L)} \leq C\|\rho\|_{W_q^{3-1/q}(S_R)}$, where $\dot{B}_L = B_L \setminus S_R$.

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi(x) = 1$ for $|x| \leq L-2$ and $\varphi(x) = 0$ for $|x| \geq L-1$. We use the **Hanzawa transform** defined by

$$x = e_h(y, t) = y + \varphi(y)H(y, t)y + \xi(t) \quad \text{for } y \in B_R.$$

Let

$$\begin{aligned}
&\mathbf{u}(y, t) = \mathbf{v} \circ e_h, \quad \mathfrak{q}(y, t) = \mathfrak{p} \circ e_h - \frac{(N-1)\sigma}{R}.\\
&\Omega_t = \{x = y + \varphi(y)H(y, t)y + \xi(t) \mid y \in B_R\}, \quad \Gamma_t = \{x = (R + \rho(R\omega, t))\omega \mid \omega \in S_1\}.
\end{aligned}$$

And then, problem (66) is transformed to

$$\begin{cases}
\partial_t\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}) = F(\mathbf{u}, H) & \text{in } \Omega \times (0, T),\\
\mathrm{div}\,\mathbf{u} = F_d(\mathbf{u}, H) = \mathrm{div}\,\mathbf{F}_d(\mathbf{u}, H) & \text{in } \Omega \times (0, T),\\
[[\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}]]\omega - \sigma(\mathcal{B}_R\rho)\mathbf{n} = G(\mathbf{u}, \rho), \quad [[\mathbf{u}]] = 0 & \text{in } S_R \times (0, T),\\
\partial_t\rho - \mathbf{n} \cdot P\mathbf{u} = D(\mathbf{u}, \rho) & \text{on } S_R \times (0, T),\\
(\mathbf{u}, \rho)|_{t=0} = (\mathbf{u}_0, \rho_0) & \text{on } \Omega \times S_R.
\end{cases} \tag{67}$$

Here, $\Omega = B_R \cup B^R$ with $B^R = \{x \in \mathbb{R}^N \mid |x| > R\}$, $\mathbb{R}^N = \Omega \cup S_R$,

$$\mathcal{B}_R\rho = R^{-2}(\Delta_{S_1} + N - 1)\rho, \quad P\mathbf{u} = \mathbf{u} - \frac{1}{|B_R|}\int_{B_R} \mathbf{u}(y)\,dy.$$

Δ_{S_1} is the Laplace-Beltrami operator on S_1, and $F(\mathbf{u}, H)$, $F_d(\mathbf{u}, H)$, $\mathbf{F_d}(\mathbf{u}, H)$, $D(\mathbf{u}, \rho)$ are nonlinear functions. Then, we have the following theorem that is our global well-posedness theorem in the case that $\sigma > 0$.

Theorem 14. *Let $N \geq 3$. Let q_1 and q_2 be exponents such that $N < q_2 < \infty$ and $1/q_1 = 1/q_2 + 1/N$. Let b be a number such that $N/q_1 > b \geq N/(2q_2)$. Then, there exists an $\epsilon > 0$ such that if initial data $\mathbf{u}_0 \in B_{q_2,p}^{2(1-1/p)} \cap B_{q_1/2,p}^{2(1-1/p)} = D_{p,q_1,q_2}$ and $\rho_0 \in B_{q_2,p}^{3-1/p-1/q_2}(S_R)$ satisfy the smallness condition:*

$$\|\mathbf{u}_0\|_{D_{p,q_1,q_2}} + \|\rho_0\|_{B_{q_2,p}^{3-1/p-1/q_2}(S_R)} \leq \epsilon$$

and the compatibility condition:

$$\mathrm{div}\,\mathbf{u}_0 = 0 \ \textit{in } \Omega, \quad [[\mu\mathbf{D}(\mathbf{u}_0)]]\omega - <[[\mu\mathbf{D}(\mathbf{u}_0)]]\omega, \omega>= 0 \ \textit{on } S_R,$$

then problem (67) *admits unique solutions* $\mathbf{u}$ *and* ρ *with*

$$\mathbf{u} \in L_p((0,\infty), H^2_{q_2}(\Omega)^N \cap H^2_{q_1/2}(\Omega)^N) \cap H^1_p((0,\infty), L_{q_2}(\Omega)^N \cap L_{q_1/2}(\Omega)^N),$$
$$\rho \in L_p((0,\infty), W^{3-1/q_2}_{q_2}(S_R)) \cap H^1_p((0,\infty), W^{2-1/q_2}_{q_2}(S_R))$$

possessing the estimate: $E(\mathbf{u},\rho)(0,\infty) \leq C\epsilon$. *Here,*

$$\begin{aligned} E(\mathbf{u},\rho)(0,T) &= \| < t >^b (\mathbf{u}, H)\|_{L_\infty((0,T),H^1_\infty(\Omega)\times H^2_\infty(\dot B_L))} + \| < t >^{b-\frac{N}{2q_1}} \mathbf{u}\|_{L_p((0,T),H^1_{q_1}(\Omega))} \\ &+ \| < t >^{\frac{N}{2q_1}} \mathbf{u}\|_{L_\infty((0,T),L_{q_1}(\Omega))} + \| < t >^{b-\frac{N}{2q_2}} \partial_t(\mathbf{u},H)\|_{L_p((0,T),L_{q_2}(\Omega)\times H^2_{q_2}(\dot B_L))} \\ &+ \| < t >^{b-\frac{N}{2q_2}} (\mathbf{u},H)\|_{L_p((0,T),H^2_{q_2}(\Omega)\times H^3_{q_2}(\dot B_L))}. \end{aligned}$$

Here, $< t >= (1+t^2)^{1/2}$.

2.2 Global wellposedness, $\sigma = 0$ case

In this case, we can not use the Hanzawa transform, because of the lack of regularity for the height function ρ. Thus, we use the partial Lagrange transform like Sect. 1. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi(x) = 1$ for $|x| \leq L-2$ and $\varphi(x) = 0$ for $|x| \geq L-1$ for $L \geq 3R$. Let $\mathbf{u}(\xi, s) = \mathbf{u}_\pm(\xi, s)$ for $\xi \in \Omega_\pm$ be the Lagrange velocity fields, and the partial Lagrange transform is defined by

$$x = \xi + \varphi(\xi)\int_0^t \mathbf{u}(\xi, s)\,ds = X_\mathbf{u}(\xi, t) \quad \text{for } \xi \in \Omega_\pm.$$

There exists a small constant $\sigma > 0$ such that if

$$\int_0^T \|\nabla(\varphi(\cdot)\mathbf{u}(\cdot, s))\|_{L_\infty(\Omega)}\,ds \leq \sigma$$

then, the partial Lagrange transform is a diffeomorphism from $\Omega = \Omega_+ \cup \Omega_+ = \mathbb{R}^N \setminus \Gamma$ onto $\Omega_t = \{x = X_\mathbf{u}(\xi, t) \mid \xi \in \Omega\}$.

By the partial Lagrange transform, problem (66) is transformed to

$$\begin{cases} \partial_t\mathbf{u} - \mathrm{Div}\,(\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}) = F(\mathbf{u}), \quad \mathrm{div}\,\mathbf{v} = f(\mathbf{u}) = \mathrm{div}\,\mathbf{f}(\mathbf{u}) & \text{in } \Omega \times (0,T), \\ [[\mu\mathbf{D}(\mathbf{u}) - \mathfrak{q}\mathbf{I}]]\mathbf{n} = \mathbf{g}(\mathbf{u}), \quad [[\mathbf{u}]] = 0 & \text{on } \Gamma \times (0,T), \\ \mathbf{u}|_{t=0} = \mathbf{v}_0, \quad \Omega_t|_{t=0} = \Omega, \end{cases} \tag{68}$$

with suitable nonlinear functions $F(\mathbf{u})$, $f(\mathbf{u})$, $\mathbf{f}(\mathbf{u})$ and $\mathbf{g}(\mathbf{u})$. Then, we have the following theorem.

Theorem 15. *Let* $N \geq 3$ *and let* q_1 *and* q_2 *be exponents such that* $N < q_2 < \infty$ *and* $1/q_1 = 1/q_2 + 1/N$ *and* $q_1 > 2$. *Let* b, p *and* $p' = p/(p-1)$ *be numbers satisfying the conditions:*

$$\begin{aligned} &\frac{N}{q_1} > b > \frac{1}{p'}, \ \Big(\frac{N}{q_1} - b\Big)p > 1, \ \Big(b - \frac{N}{2q_2}\Big)p > 1, \ b > \frac{N}{2q_1}, \\ &\Big(\frac{N}{2q_2} + \frac{1}{2}\Big)p' < 1, \ bp' > 1, \ \Big(b - \frac{N}{2q_2}\Big)p' > 1, \ \frac{N}{q_2} + \frac{2}{p} < 1. \end{aligned} \tag{69}$$

Then, there exists an $\epsilon > 0$ *such that if initial data* $\mathbf{v}_0$ *satisfies the compatibility condition and the smallness condition:* $\|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q_2,p}(\Omega)} + \|\mathbf{v}_0\|_{B^{2(1-1/p)}_{q_1/2,p}(\Omega)} \leq \epsilon$, *then problem* (68) *admits a unique solution* $\mathbf{u} \in L_p((0,\infty), H^2_{q_2}(\Omega)^N) \cap H^1_p((0,\infty), L_{q_2}(\Omega)^N)$, *possessing the estimate:* $[\mathbf{u}]_\infty < C\epsilon$ *with some constant* $C > 0$ *independent of* ϵ. *Here*

$$\begin{aligned} [\mathbf{u}]_T = \Big\{ &\int_0^T ((1+t)^b \|\mathbf{u}(\cdot, s)\|_{H^1_\infty(\Omega)})^p\,ds \\ &+ \int_0^T ((1+s)^{(b-\frac{N}{2q_1})} \|\mathbf{u}(\cdot, s)\|_{H^1_{q_1}(\Omega)})^p\,ds + (\sup_{0<s<T} (1+s)^{\frac{N}{2q_1}} \|\mathbf{u}(\cdot, s)\|_{L_{q_1}(\Omega)})^p \\ &+ \int_0^T ((1+s)^{(b-\frac{N}{2q_2})} (\|\mathbf{u}(\cdot, s)\|_{H^2_{q_2}(\Omega)} + \|\partial_t\mathbf{u}(\cdot, s)\|_{L_{q_2}(\Omega)}))^p\,ds \Big\}^{1/p}. \end{aligned}$$

Remark 16. We can prove Theorem 15 by using the similar argument to that in Sect. 1.

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