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ON REFLEXIVITY OF C–SYMMETRIC OR SKEW–C–SYMMETRIC OPERATORS

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ABSTRACT. We present results concerning reflexivity and hyperreflexivity of a subspace of all C–symmetric operators from [6] and a subspace of all skew–C–symmetric operators from [2] with a given conjugation C. We also give a description of theirs preanihilators.

1. INTRODUCTION

Let $\mathcal{H}$ denote a complex separable Hilbert space with an inner product $\langle .,. \rangle$ and let $B(\mathcal{H})$ be the Banach algebra of all bounded linear operators on $\mathcal{H}$.

Recall that C is a *conjugation* on $\mathcal{H}$ if $C : \mathcal{H} \longrightarrow \mathcal{H}$ is an antilinear, isometric involution, i.e., $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. An operator T in $B(\mathcal{H})$ is said to be *C-symmetric* if $CTC = T^*$. Denote by $\mathcal{C} = \{T \in B(\mathcal{H}) : CTC = T^*\}$ the subspace of all C–symmetric operators. Operators which are C–symmetric have been lately studied by many authors (see [3], [4], [5]). In this class there are for example Jordan blocs, truncated Toeplitz operators and Hankel operators. An operator $T \in B(\mathcal{H})$ is called to be *skew-C-symmetric* if $CTC = -T^*$. Denote by $\mathcal{C}^s = \{T \in B(\mathcal{H}) : CTC = -T^*\}$ the subspace of all skew–C–symmetric operators. It follows directly from the definition that $\mathcal{C}$ and $\mathcal{C}^s$ are weak* closed. It is worth to note that any operator $T \in B(\mathcal{H})$ can be written as a sum of a C–symmetric operator and a skew–C–symmetric operator. Indeed, $T = A + B$, where $A = \frac{1}{2}(T + CT^*C)$ and $B = \frac{1}{2}(T - CT^*C)$.

Recall that the predual to $B(\mathcal{H})$ is the space of trace class operators denoted by τc with the dual action $\langle T, f \rangle = tr(Tf)$, where $T \in B(\mathcal{H})$ and $f \in \tau c$. The norm in τc is denoted by $\|\cdot\|_1$ and called the trace norm. Denote by F_k the set of all operators which have rank at most k. Rank one operators are usually written as $x \otimes y$, where x, $y \in \mathcal{H}$,

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and $(x \otimes y)z = \langle z, y\rangle x$ for $z \in \mathcal{H}$. Moreover, $tr(T(x \otimes y)) = \langle Tx, y\rangle$. For a closed subspace $\mathcal{S} \subset B(\mathcal{H})$ denote by $\mathcal{S}_\perp$ the *preanihilator* of $\mathcal{S}$ defined by $\mathcal{S}_\perp = \{t \in \tau c : tr(St) = 0 \text{ for all } S \in \mathcal{S}\}$.

Recall that *the reflexive closure* of a subspace $\mathcal{S} \subset B(\mathcal{H})$ is given by

$$\operatorname{Ref} \mathcal{S} = \{T \in B(\mathcal{H}) : Tx \in [\mathcal{S}x] \text{ for all } x \in \mathcal{H}\},$$

here $[\cdot]$ is the norm-closure. A subspace $\mathcal{S}$ is called *reflexive*, if $\mathcal{S} = \operatorname{Ref} \mathcal{S}$ and $\mathcal{S}$ is called *transitive*, if $\operatorname{Ref} \mathcal{S} = B(\mathcal{H})$. A subspace $\mathcal{S} \subset B(\mathcal{H})$ is called *k-reflexive* if $\mathcal{S}^{(k)} = \{S^{(k)} : S \in \mathcal{S}\}$ is reflexive in $B(\mathcal{H}^{(k)})$, where $S^{(k)} = S \oplus \cdots \oplus S$ and $\mathcal{H}^{(k)} = \mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Recall after [10, 8] that a weak* closed subspace $\mathcal{S}$ is *k-reflexive* if and only if operators of rank at most k are linearly dense in $\mathcal{S}_\perp$, i.e., $\mathcal{S}_\perp = [\mathcal{S}_\perp \cap F_k]$. On the other hand, transitivity means that there are no rank-1 operators in the preanihilator of $\mathcal{S}$, i.e., $\mathcal{S}_\perp \cap F_1 = \{0\}$.

The definition of k-hyperreflexivity was introduced in [1, 7] and is a stronger property than k-reflexivity, which means that each k-hyperreflexive subspace is k-reflexive. A subspace $\mathcal{S}$ is called *k-hyperreflexive* if there is a constant $c > 0$ such that

$$\operatorname{dist}(T, \mathcal{S}) \le c \cdot \sup\{|tr(Tt)| : t \in F_k \cap \mathcal{S}_\perp, \|t\|_1 \le 1\}, \tag{1}$$

for all $T \in B(\mathcal{H})$. Note that $\operatorname{dist}(T, \mathcal{S}) = \inf\{\|T - S\| : S \in \mathcal{S}\}$ and the supremum on the right hand side of (1) we denote by $\alpha_k(T, \mathcal{S})$. The smallest constant for which the inequality (1) is satisfied is called the *k-hyperreflexivity constant* and is denoted $\kappa_k(\mathcal{S})$. If $k = 1$, the letter k will be omitted.

In this paper we present results concerning reflexivity and hyperreflexivity of subspaces $\mathcal{C}$ and $\mathcal{C}^s$ proved in [6] and [2]. It is shown that the subspace of all C-symmetric operators is transitive (hence far from being reflexive) and 2-reflexive or even 2-hyperreflexive. It means that the preanihilator of $\mathcal{C}$ does not contain any rank-one operators and rank-two operators are linearly dense in the preanihilator. Moreover, we describe all rank-two operators in this preanihilator. However, the subspace of all skew–C–symmetric operators have much better properties: it is reflexive (so very far from being transitive) and hyperreflexive.

2. Preanihilator

Let $\mathcal{H}$ be a complex separable Hilbert space with an antilinear involution C. Now we will present results describing the structure of preanihilator of the subspace $\mathcal{C}$. First theorem says that there are no rank-1 operators in the preanihilator.

Theorem 2.1 (Theorem 2.1 [6]). *Let $\mathcal{C}$ be the set of C–symmetric operators. The subspace $\mathcal{C}$ is transitive.*

The next theorem gives a full description of rank-2 operators in $\mathcal{C}_\perp$.

Theorem 2.2 (Theorem 3.1 [6]). *Let $\mathcal{C}$ be the set of all C-symmetric operators. Then*

$$F_2 \cap \mathcal{C}_\perp = \{h \otimes g - Cg \otimes Ch : h, g \in \mathcal{H}\}.$$

Let now consider some examples of conjugations given in [3] in the context of Theorem 2.2.

Example 2.3. A natural example of a conjugation in $l^2(\mathbb{N})$ is given by

$$C(z_0, z_1, z_2, \dots) = (\overline{z}_0, \overline{z}_1, \overline{z}_2, \dots).$$

In this case

$$\mathcal{C}_\perp \cap F_2 = \{h \otimes g - \overline{g} \otimes \overline{h} : h, g \in l^2(\mathbb{N})\}.$$

Example 2.4. Consider the classical Hardy space H^2 and take a non-constant inner function α. Denote by $K_\alpha^2 = H^2 \ominus \alpha H^2$. For $f \in K_\alpha^2$ and $h \in H^2$ the formula

$$C_\alpha f = \alpha \overline{zf}$$

defines a conjugation $C = C_\alpha$ on K_α^2. Then

$$\mathcal{C}_\perp \cap F_2 = \{h \otimes g - \alpha \overline{zg} \otimes \alpha \overline{zh} : h, g \in K_\alpha^2\}.$$

Example 2.5. Let ϱ be a bounded, positive continuous weight on the interval $[-1, 1]$, symmetric with respect to the midpoint of the interval: $\varrho(t) = \varrho(-t)$ for $t \in [0, 1]$. Then the formula

$$Cf(t) = \overline{f(-t)}$$

defines a conjugation on $L^2([-1, 1], \varrho\, dt)$. In this case

$$\mathcal{C}_\perp \cap F_2 = \{h(\cdot) \otimes g(\cdot) - \overline{g(-(\cdot))} \otimes \overline{h(-(\cdot))} : h, g \in L^2([-1, 1], \varrho\, dt)\}.$$

Example 2.6. Consider the isometric antilinear operator

$$C(z_1, z_2) = (\overline{z}_2, \overline{z}_1)$$

on $\mathbb{C}^2$. Then

$$\mathcal{C}_\perp \cap F_2 = \{(h_1, h_2) \otimes (g_1, g_2) - (\overline{g}_2, \overline{g}_1) \otimes (\overline{h}_2, \overline{h}_1) : (h_1, h_2), (g_1, g_2) \in \mathbb{C}^2\}.$$

Now we will consider the preanihilator of the subspace of all skew–C–symmetric operators.

Lemma 2.7. *Let C be a conjugation in a complex Hilbert space $\mathcal{H}$ and $h, g \in \mathcal{H}$. Then*

(1) $C(h \otimes g)C = Ch \otimes Cg$,
(2) $h \otimes g - Cg \otimes Ch \in \mathcal{C}^s$.

In [3, Lemma 2] it was shown that

$$\mathcal{C} \cap F_1 = \{\alpha \cdot h \otimes Ch : h \in \mathcal{H}, \alpha \in \mathbb{C}\}.$$

The next proposition shows that it is also a description of the rank-one operators in the preanihilator of $\mathcal{C}^s$.

Proposition 2.8 (Proposition 2.2 [2]). *Let C be a conjugation in a complex Hilbert space $\mathcal{H}$. Then*

$$\mathcal{C}^s_\perp \cap F_1 = \mathcal{C} \cap F_1 = \{\alpha \cdot h \otimes Ch : h \in \mathcal{H}, \alpha \in \mathbb{C}\}.$$

Lemma 2.9. *Let C be a conjugation in a complex Hilbert space $\mathcal{H}$. Then*

$$\mathcal{C}^s_\perp \cap F_2 \supset \{h \otimes g + Cg \otimes Ch : h, g \in \mathcal{H}\}.$$

The following examples illustrate the result presented in Proposition 2.8.

Example 2.10. Note that for different conjugations we obtain different subspaces. Let $C_1(x_1, x_2, x_3) = (\bar{x}_3, \bar{x}_2, \bar{x}_1)$ be a conjugation on $\mathbb{C}^3$. Then

$$\mathcal{C}^s_1 = \left\{ \begin{pmatrix} a & b & 0 \\ c & 0 & -b \\ 0 & -c & -a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$$

and

$$\mathcal{C}_1 = \left\{ \begin{pmatrix} a & b & * \\ c & * & b \\ * & c & a \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Rank-one operators in $\mathcal{C}_1$ and in $(\mathcal{C}^s_1)_\perp$ are of the form $\alpha(x_1, x_2, x_3) \otimes (\bar{x}_3, \bar{x}_2, \bar{x}_1)$ for $\alpha \in \mathbb{C}$.

If we now consider another conjugation $C_2(x_1, x_2, x_3) = (\bar{x}_2, \bar{x}_1, \bar{x}_3)$ on $\mathbb{C}^3$, then

$$\mathcal{C}^s_2 = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & -a & c \\ -c & -b & 0 \end{pmatrix} : a, b, c \in \mathbb{C} \right\},$$

and

$$\mathcal{C}_2 = \left\{ \begin{pmatrix} a & * & b \\ * & a & c \\ c & b & * \end{pmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Similarly, rank-one operators in $\mathcal{C}_2$ and also in $(\mathcal{C}^s_2)_\perp$ are of the form $\alpha(x_1, x_2, x_3) \otimes (\bar{x}_2, \bar{x}_1, \bar{x}_3)$.

Example 2.11. Let C be a conjugation in $\mathcal{H}$. Consider the conjugation $\tilde{C} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$ in $\mathcal{H} \oplus \mathcal{H}$ (see [9]). An operator $T \in B(\mathcal{H} \oplus \mathcal{H})$ is skew–$\tilde{C}$–symmetric, if and only if $T = \begin{pmatrix} A & B \\ D & -CA^*C \end{pmatrix}$, where $A, B, D \in B(\mathcal{H})$ and B, D are skew–C–symmetric. Moreover, rank-one operators in $\tilde{\mathcal{C}}^s_\perp$ are of the form $\alpha(f \oplus g) \otimes (Cg \oplus Cf)$ for $f, g \in \mathcal{H}$ and $\alpha \in \mathbb{C}$.

The following example gives a description of skew–C–symmetric operators in a case of model space K^2_α equipped with the conjugation C_α defined in Example 2.4.

Example 2.12. Let H^2 be the Hardy space, and let α be a non-constant inner function. As in Example 2.4 consider the conjugation $C_\alpha h = \alpha \overline{zh}$ on the space $K^2_\alpha = H^2 \ominus \alpha H^2$. By S_α and S^*_α denote the compressions of the unilateral shift S and the backward shift S^* to K^2_α, respectively. Recall after [11] that the kernel functions in K^2_α for $\lambda \in \mathbb{C}$ are projections of appropriate kernel functions k_λ onto K^2_α, namely $k^\alpha_\lambda = k_\lambda - \overline{\alpha(\lambda)} \alpha k_\lambda$. Denote by $\tilde{k}^\alpha_\lambda = C_\alpha k^\alpha_\lambda$. Since S_α and S^*_α are C_α–symmetric (see [3]), for a skew–C_α–symmetric operator $A \in B(K^2_\alpha)$ we have

$$(2) \quad \langle AS^n_\alpha k^\alpha_\lambda, (S^*_\alpha)^m \tilde{k}^\alpha_\lambda \rangle = \langle C_\alpha (S^*_\alpha)^m \tilde{k}^\alpha_\lambda, C_\alpha A S^n_\alpha k^\alpha_\lambda \rangle = \\ - \langle S^m_\alpha C_\alpha \tilde{k}^\alpha_\lambda, A^* C_\alpha S^n_\alpha k^\alpha_\lambda \rangle = -\langle AS^m_\alpha k^\alpha_\lambda, (S^*_\alpha)^n \tilde{k}^\alpha_\lambda \rangle,$$

for all $n, m \in \mathbb{N}$. Note that if $n = m$, then

$$(3) \qquad \langle AS^n_\alpha k^\alpha_\lambda, (S^*_\alpha)^n \tilde{k}^\alpha_\lambda \rangle = 0.$$

In particular, we may consider the special case $\alpha = z^k$, $k > 1$. Then the equality (3) implies that a skew–C_α–symmetric operator $A \in B(K^2_{z^k})$ has the matrix representation in the canonical basis with 0 on the diagonal orthogonal to the main diagonal. Indeed, let $A \in B(K^2_{z^k})$ have the matrix $(a_{ij})_{i,j=0,\dots,k-1}$ with respect to the canonical basis. Note that $C_{z^k} f = z^{k-1} \bar{f}$, $k^{z^k}_0 = 1$, $\tilde{k}^{z^k}_0 = z^{k-1}$. Hence for $0 \le n \le k-1$ we have

$$0 = \langle AS^n_\alpha 1, (S^*_\alpha)^n z^{k-1} \rangle = \langle Az^n, z^{k-n-1} \rangle = a_{n,k-n-1}.$$

Moreover, from the equality (2) we can obtain that

$$\langle Az^n, z^{k-m-1} \rangle = -\langle Az^m, z^{k-n-1} \rangle,$$

which implies that $a_{n,k-m-1} = -a_{m,k-n-1}$ for $0 \le m, n \le k-1$.

3. Reflexivity

In this section we present results concerning reflexivity of the space of all C–symmetric operators and the subspace of all skew–C–symmetric operators.

Theorem 3.1 (Theorem 4.1 [6]). *Let $\mathcal{H}$ be a complex separable Hilbert space with an antilinear involution C. The subspace $\mathcal{C} \subset B(\mathcal{H})$ of all C-symmetric operators is 2-reflexive.*

In the case of the space of all skew–C–symmetric operators we can obtain a stronger result.

Theorem 3.2 (Theorem 3.1 [2]). *Let C be a conjugation in a complex Hilbert space $\mathcal{H}$. The subspace $\mathcal{C}^s$ of all skew-C-symmetric operators on $\mathcal{H}$ is reflexive.*

Recall that a single operator $T \in B(\mathcal{H})$ is called reflexive if the weakly closed algebra generated by T and the identity is reflexive. In [9] authors characterized normal skew symmetric operators and by [12] we know that every normal operator is reflexive. Hence one may wonder, if all skew–C–symmetric operators are reflexive. The following simple example shows that it is not true.

Example 3.3. Consider the space $\mathbb{C}^2$ and a conjugation $C(x,y) = (\bar{x},\bar{y})$. Note that operator $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is skew–C–symmetric. The weakly closed algebra $\mathcal{A}(T)$ generated by T consists of operators of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Hence $\mathcal{A}(T)_\perp = \left\{ \begin{pmatrix} t & s \\ s & -t \end{pmatrix} : t, s \in \mathbb{C} \right\}$. It is easy to see, that $\mathcal{A}(T)_\perp \cap F_1 = \{0\}$, which implies that T is not reflexive.

4. Hyperreflexivity

Hyperreflexivity is a stronger property than reflexivity. Here we present results concerning hyperreflexivity of the subspaces $\mathcal{C}$ and $\mathcal{C}^s$. Since $\mathcal{C}$ is transitive, it cannot be hyperreflexive. However, we can prove the following:

Theorem 4.1 (Theorem 4.2 [6]). *Let $\mathcal{H}$ be a complex separable Hilbert space and let C be a conjugation on $\mathcal{H}$. The subspace $\mathcal{C} \subset B(\mathcal{H})$ of all C-symmetric operators is 2-hyperreflexive with constant 1.*

The subspace $\mathcal{C}^s$ is reflexive. It can be proved that it also has the stronger property – hyperreflexivity.

Theorem 4.2 (Theorem 4.1 [2]). *Let C be a conjugation in a complex Hilbert space $\mathcal{H}$. Then the subspace $C^s \subset B(\mathcal{H})$ of all skew-C-symmetric operators is hyperreflexive with the constant $\kappa(C^s) \leqslant 3$ and 2-hyperreflexive with $\kappa_2(C^s) = 1$.*

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