数理解析研究所講究録
第2073巻 2018年 55-62

Bounds and operator monotonicity of a generalized Petz-Hasegawa function

伊藤 公智 (Masatoshi Ito) (前橋工科大学 Maebashi Institute of Technology)*1
古田 孝之 (Takayuki Furuta) (弘前大学 Hirosaki University)
柳田 昌宏 (Masahiro Yanagida) (東京理科大学 Tokyo University of Science)*2
山崎 丈明 (Takeaki Yamazaki) (東洋大学 Toyo University)*3

1. Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space $\mathcal{H}$. An operator A is positive semidefinite if and only if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, and we write $A \geq 0$. If an operator A is positive semidefinite and invertible, A is called positive definite. In this case, we write $A > 0$. For self-adjoint operators A and B, $B \leq A$ is defined by $0 \leq A - B$. A real-valued function f defined on an interval $I \subset \mathbb{R}$ is called an operator monotone function if

$$B \leq A \quad \text{implies} \quad f(B) \leq f(A)$$

for all self-adjoint operators A and B whose spectra are contained in I. Typical examples of operator monotone functions are $f(x) = x^{\lambda}$ and $f(x) = (1 - \lambda + \lambda x^{q})^{\frac{1}{q}}$ on $x > 0$ for $\lambda \in [0,1]$ and $q \in [-1,1] \setminus \{0\}$.

It was proven in Petz-Hasegawa [10] that the function $f_p(x)$ of $x > 0$ is operator monotone for $-1 \leq p \leq 2$, where

$$f_p(x) = p(1-p)\frac{(x-1)^2}{(x^p - 1)(x^{1-p} - 1)} \quad (p \neq 0, 1),$$

$f_0(x) = \lim_{p\to 0} f_p(x) = \frac{x-1}{\log x}$ and $f_1(x) = \lim_{p\to 1} f_p(x) = \frac{x-1}{\log x}$ (see also [1, 4]). In this report, we shall consider the function

$$f_{\boldsymbol{p}}(x) = x^{\gamma}\prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i} - 1}$$

for $\boldsymbol{p} = (p_1, \ldots, p_n)$ as a generalization of the Petz-Hasegawa function $f_p(x)$. In Section 2, we shall give upper and lower bounds of $f_{\boldsymbol{p}}(x)$. In Section 3, we shall introduce a new approach to showing operator monotonicity of functions such as $f_{\boldsymbol{p}}(x)$. This report is based on [5].

This work was supported by JSPS KAKENHI Grant Number JP16K05181 (M. Ito).
2010 Mathematics Subject Classification: 47A63, 47A64.
Keywords: Operator monotone function; Petz-Hasegawa theorem.
*1 e-mail: m-ito@maebashi-it.ac.jp
*2 e-mail: yanagida@rs.tus.ac.jp
*3 e-mail: t-yamazaki@toyo.jp

Remark. Professor Takayuki Furuta passed away on 26 June, 2016. He had obtained a small result (a part of Corollary 4), however it had not been submitted. The rest of the authors found his unpublished manuscript when we visited his home in order to arrange his notebooks. Then we added some results into Furuta's manuscript to make this report.

2. Upper and lower bounds of $f_{\boldsymbol{p}}(x)$

In what follows, we consider $p\dfrac{x-1}{x^p-1}$ for $p=0$ as $\dfrac{x-1}{\log x}$, the limit as $p \to 0$.

Theorem 1. *Let $n \geq 2$ be a natural number, and let $p_i \in [0,1]$ for $i = 0,1,2,\ldots,n$ such that $\sum_{i=0}^n p_i = n$. Then*

$$(1-p_0)x^\gamma \frac{x-1}{x^{1-p_0}-1} \leq f_{\boldsymbol{p}}(x) = x^\gamma \prod_{i=1}^n p_i \frac{x-1}{x^{p_i}-1}$$
$$\leq x^\gamma \left(\mu \frac{x-1}{x^\mu - 1}\right)^n \leq x^\gamma \left(\frac{x^\mu+1}{2}\right)^{\frac{p_0}{\mu}} \leq x^\gamma \left(\frac{x+1}{2}\right)^{p_0}$$

holds for $\gamma \in \mathbb{R}$ and $x > 0$, where $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$ and $\mu = \frac{1}{n}\sum_{i=1}^n p_i$.

To give a proof of Theorem 1, we shall use the following theorem.

Theorem A ([11]). *Let $p, q \in [-1,1] \setminus \{0\}$. Then*

$$F_{p,q}(x) = \left[\int_0^1 (1-\lambda+\lambda x^p)^{\frac{q}{p}} d\lambda\right]^{\frac{1}{q}} = \left(\frac{p}{p+q}\frac{x^{p+q}-1}{x^p-1}\right)^{\frac{1}{q}}$$

is a positive operator monotone function on $x > 0$, and increasing on $p, q \in [-1,1] \setminus \{0\}$.

In [11], Theorem A is shown by using a technique of complex analysis. But it can be shown by the following facts easily: (i) $(1-\lambda+\lambda x^p)^{1/p}$ is operator monotone on $x > 0$ for $\lambda \in [0,1]$ and $p \in [-1,1] \setminus \{0\}$, and increasing on $p \in [-1,1] \setminus \{0\}$, and (ii) for operator monotone functions $f_i(x)$ $(i = 1,2,\ldots,n)$, $(\sum_{i=1}^n w_i f_i(x)^q)^{\frac{1}{q}}$ is operator monotone for $q \in [-1,1] \setminus \{0\}$ and $w_i > 0$ such that $\sum_{i=1}^n w_i = 1$, and increasing on $q \in [-1,1] \setminus \{0\}$.

Proof of Theorem 1. If $p_i = 0$ for some i, then $p_j = 1$ for all $j \neq i$ by the condition $\sum_{i=0}^n p_i = n$. $p_i \frac{x-1}{x^{p_i}-1} = 1$ when $p_i = 1$, so that we have only to consider the case $p_i \in (0,1)$. It is enough to show that

$$(1-p_0)\frac{x-1}{x^{1-p_0}-1} \leq \prod_{i=1}^n p_i \frac{x-1}{x^{p_i}-1}$$
$$\leq \left(\mu \frac{x-1}{x^\mu-1}\right)^n \leq \left(\frac{x^\mu+1}{2}\right)^{\frac{p_0}{\mu}} \leq \left(\frac{x+1}{2}\right)^{p_0} \tag{1}$$

holds for $x > 0$.

Firstly, we shall show the first inequality in (1).

$$\begin{aligned}
(1-p_0)\frac{x-1}{x^{1-p_0}-1} &= \prod_{i=0}^{n-1}\frac{\sum_{k=0}^{i}(1-p_k)}{\sum_{j=0}^{i+1}(1-p_j)}\frac{x^{\sum_{j=0}^{i+1}(1-p_j)}-1}{x^{\sum_{k=0}^{i}(1-p_k)}-1}\\
&= \prod_{i=0}^{n-1}\left(\frac{\sum_{k=0}^{i}(1-p_k)}{\sum_{j=0}^{i+1}(1-p_j)}\frac{x^{\sum_{j=0}^{i+1}(1-p_j)}-1}{x^{\sum_{k=0}^{i}(1-p_k)}-1}\right)^{\frac{1-p_{i+1}}{1-p_{i+1}}}\\
&= \prod_{i=0}^{n-1}F_{\sum_{k=0}^{i}(1-p_k),1-p_{i+1}}(x)^{1-p_{i+1}}\\
&\le \prod_{i=0}^{n-1}F_{p_{i+1},1-p_{i+1}}(x)^{1-p_{i+1}} = \prod_{i=1}^{n}p_i\frac{x-1}{x^{p_i}-1},
\end{aligned}$$

where the inequality follows from Theorem A and the following fact: $\sum_{k=0}^{n}(1-p_k)=1$ and $p_i\in(0,1)$ for $i=0,1,2,\dots,n$ imply

$$\sum_{k=0}^{i}(1-p_k) = 1-(1-p_{i+1})-\cdots-(1-p_n)\le p_{i+1}.$$

Secondly, we shall prove the second inequality in (1). To prove it, we show that for each $x>0$,

$$g(t)=\log\left(t\frac{x-1}{x^t-1}\right)\text{ is a concave function on }(0,1). \tag{2}$$

For $0<t_1<t_2<1$,

$$\begin{aligned}
t_1\frac{x-1}{x^{t_1}-1}\cdot t_2\frac{x-1}{x^{t_2}-1} &= \frac{t_1}{\frac{t_1+t_2}{2}}\frac{x^{\frac{t_1+t_2}{2}}-1}{x^{t_1}-1}\cdot\frac{t_1+t_2}{2}\frac{x-1}{x^{\frac{t_1+t_2}{2}}-1}\cdot t_2\frac{x-1}{x^{t_2}-1}\\
&= F_{t_1,\frac{t_2-t_1}{2}}(x)^{\frac{t_2-t_1}{2}}F_{\frac{t_1+t_2}{2},1-\frac{t_1+t_2}{2}}(x)^{1-\frac{t_1+t_2}{2}}F_{t_2,1-t_2}(x)^{1-t_2}\\
&\le F_{\frac{t_1+t_2}{2},\frac{t_2-t_1}{2}}(x)^{\frac{t_2-t_1}{2}}F_{\frac{t_1+t_2}{2},1-\frac{t_1+t_2}{2}}(x)^{1-\frac{t_1+t_2}{2}}F_{t_2,1-t_2}(x)^{1-t_2}\\
&= \frac{\frac{t_1+t_2}{2}}{t_2}\frac{x^{t_2}-1}{x^{\frac{t_1+t_2}{2}}-1}\cdot\frac{t_1+t_2}{2}\frac{x-1}{x^{\frac{t_1+t_2}{2}}-1}\cdot t_2\frac{x-1}{x^{t_2}-1}\\
&= \left(\frac{t_1+t_2}{2}\frac{x-1}{x^{\frac{t_1+t_2}{2}}-1}\right)^2
\end{aligned}$$

holds by Theorem A. Then we have

$$\begin{aligned}
\frac{1}{2}\{g(t_1)+g(t_2)\} &= \frac{1}{2}\log\left(t_1\frac{x-1}{x^{t_1}-1}\cdot t_2\frac{x-1}{x^{t_2}-1}\right)\\
&\le \log\left(\frac{t_1+t_2}{2}\frac{x-1}{x^{\frac{t_1+t_2}{2}}-1}\right) = g\left(\frac{t_1+t_2}{2}\right),
\end{aligned}$$

that is, $g(t)$ is a concave function on $(0,1)$ since $g(t)$ is continuous. Therefore

$$\begin{aligned}\frac{1}{n}\log\left(\prod_{i=1}^{n} p_i\frac{x-1}{x^{p_i}-1}\right) &= \frac{1}{n}\left\{g(p_1)+g(p_2)+\cdots+g(p_n)\right\}\\ &\le g\left(\frac{p_1+p_2+\cdots+p_n}{n}\right) = g(\mu) = \log\left(\mu\frac{x-1}{x^{\mu}-1}\right),\end{aligned}$$

that is,

$$\prod_{i=1}^{n} p_i\frac{x-1}{x^{p_i}-1} \le \left(\mu\frac{x-1}{x^{\mu}-1}\right)^{n}.$$

Thirdly, we shall show the third inequality in (1). Since

$$\mu = \frac{1}{n}\sum_{i=1}^{n} p_i = 1-\frac{p_0}{n} \ge 1-\frac{p_0}{2} > \frac{1}{2},$$

that is, $1-\mu < \mu$, and

$$(1-\mu)n = n - \sum_{i=1}^{n} p_i = p_0,$$

Theorem A ensures that

$$\left(\mu\frac{x-1}{x^{\mu}-1}\right)^{n} = F_{\mu,1-\mu}(x)^{(1-\mu)n} \le F_{\mu,\mu}(x)^{(1-\mu)n} = \left(\frac{x^{\mu}+1}{2}\right)^{\frac{p_0}{\mu}}.$$

The last inequality in (1) follows from the fact that $F_{q,q}(x) = (\frac{x^q+1}{2})^{\frac{1}{q}}$ is monotone increasing on $q \in [-1,1]\setminus\{0\}$ by Theorem A. $\square$

We remark that we prove (2) by using Theorem A here, while it can be shown by differential calculations.

Corollary 2. *Let $n \ge 2$ be a natural number, and let $p_i \in [0,1]$ for $i = 1,2,\ldots,n$ such that $\sum_{i=1}^{n} p_i = n-1$. Then*

$$\begin{aligned}x^{\gamma}\frac{x-1}{\log x} &\le f_{\boldsymbol{p}}(x) = x^{\gamma}\prod_{i=1}^{n} p_i\frac{x-1}{x^{p_i}-1}\\ &\le x^{\gamma}\left(\mu\frac{x-1}{x^{\mu}-1}\right)^{n} \le x^{\gamma}\left(\frac{x^{\mu}+1}{2}\right)^{\frac{1}{\mu}} \le x^{\gamma}\frac{x+1}{2}\end{aligned}$$

holds for $\gamma \in \mathbb{R}$ and $x > 0$, where $\boldsymbol{p} = (p_1,p_2,\ldots,p_n)$ and $\mu = \frac{1}{n}\sum_{i=1}^{n} p_i = 1-\frac{1}{n}$.

Proof. By taking the limit $p_0 \to 1$ in Theorem 1, we have the desired inequality since $\lim_{\alpha\to 0}\frac{x^{\alpha}-1}{\alpha} = \log x$ holds for all $x > 0$. $\square$

We also obtain another upper bound of $f_{\boldsymbol{p}}(x)$.

Theorem 3. *Let $n \geq 2$ be a natural number, and let $p_i \in [0,1]$ for $i = 0,1,2,\ldots,n$ such that $\sum_{i=0}^{n} p_i = n$. Then*

$$\begin{aligned} f_{\boldsymbol{p}}(x) = x^\gamma \prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i}-1} &\leq x^\gamma \left(\prod_{i=1}^{n-1} p_i \frac{x-1}{x^{p_i}-1} \right) \frac{1}{2-p_n} \frac{x^{2-p_n}-1}{x-1} \\ &\leq x^\gamma \prod_{i=1}^{n} \frac{1}{2-p_i} \frac{x^{2-p_i}-1}{x-1} \leq x^\gamma \left(\frac{x+1}{2} \right)^{p_0} \end{aligned}$$

holds for $\gamma \in \mathbb{R}$ and $x > 0$, where $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$.

Proof. It is enough to show that

$$\begin{aligned} \prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i}-1} &\leq \left(\prod_{i=1}^{n-1} p_i \frac{x-1}{x^{p_i}-1} \right) \frac{1}{2-p_n} \frac{x^{2-p_n}-1}{x-1} \\ &\leq \prod_{i=1}^{n} \frac{1}{2-p_i} \frac{x^{2-p_i}-1}{x-1} \leq \left(\frac{x+1}{2} \right)^{p_0} \end{aligned} \tag{3}$$

holds for $x > 0$. The first inequality can be shown as follows:

$$\begin{aligned} \prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i}-1} &= \prod_{i=1}^{n} F_{p_i,1-p_i}(x)^{1-p_i} \\ &\leq \left(\prod_{i=1}^{n-1} F_{p_i,1-p_i}(x)^{1-p_i} \right) F_{1,1-p_n}(x)^{1-p_n} \quad \text{by Theorem A} \\ &= \left(\prod_{i=1}^{n-1} p_i \frac{x-1}{x^{p_i}-1} \right) \frac{1}{2-p_n} \frac{x^{2-p_n}-1}{x-1}. \end{aligned}$$

The second and third inequalities in (3) are obtained by Theorem A as follows:

$$\begin{aligned} \left(\prod_{i=1}^{n-1} p_i \frac{x-1}{x^{p_i}-1} \right) \frac{1}{2-p_n} \frac{x^{2-p_n}-1}{x-1} &= \left(\prod_{i=1}^{n-1} F_{p_i,1-p_i}(x)^{1-p_i} \right) F_{1,1-p_n}(x)^{1-p_n} \\ &\leq \prod_{i=1}^{n} F_{1,1-p_i}(x)^{1-p_i} = \prod_{i=1}^{n} \frac{1}{2-p_i} \frac{x^{2-p_i}-1}{x-1} \\ &\leq \prod_{i=1}^{n} F_{1,1}(x)^{1-p_i} \\ &= \prod_{i=1}^{n} \left(\frac{x+1}{2} \right)^{1-p_i} = \left(\frac{x+1}{2} \right)^{p_0}, \end{aligned}$$

where the last equality holds by $\sum_{i=1}^{n}(1-p_i) = p_0$. □

Especially, we have upper and lower bounds of the Petz-Hasegawa function by Corollary 2 and Theorem 3 as follows. We note that the function $\frac{p}{p+1} \frac{x^{p+1}-1}{x^p-1}$ has been considered in [2, 3].

Corollary 4. *Let $p \in [0,1]$.*

(i) *The inequality*

$$f_0(x) = f_1(x) = \frac{x-1}{\log x} \le f_p(x) = p(1-p)\frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)} \le \left(\frac{\sqrt{x}+1}{2}\right)^2 \le \frac{x+1}{2}$$

holds for $x > 0$.

(ii) *The inequality*

$$\begin{aligned} f_0(x) = f_1(x) = \frac{x-1}{\log x} \le f_p(x) &= p(1-p)\frac{(x-1)^2}{(x^p-1)(x^{1-p}-1)} \\ &\le \frac{p}{p+1}\frac{x^{p+1}-1}{x^p-1} \\ &\le \frac{1}{(p+1)(2-p)}\frac{(x^{p+1}-1)(x^{2-p}-1)}{(x-1)^2} \le \frac{x+1}{2} \end{aligned}$$

holds for $x > 0$.

Proof. (i) and the first inequality in (ii) are obtained by putting $n = 2$, $\gamma = 0$, $p_1 = p$ and $p_2 = 1-p$ in Corollary 2. The other inequalities in (ii) are obtained by putting $n = 2$, $\gamma = 0$, $p_0 = 1$, $p_1 = p$ and $p_2 = 1-p$ in Theorem 3. □

3. Operator monotonicity of $f_p(x)$

First of all, we shall give an elementary proof of the following known result.

Theorem B ([6]). *For $-2 \le p \le 2$,*

$$s_p(x) = \left(p\frac{x-1}{x^p-1}\right)^{\frac{1}{1-p}}$$

is an operator monotone function on $x > 0$, where $s_0(x)$ and $s_1(x)$ are defined by the limit as $s_0(x) = \lim_{p\to 0} s_p(x) = \dfrac{x-1}{\log x}$ and $s_1(x) = \lim_{p\to 1} s_p(x) = \dfrac{1}{e}x^{\frac{x}{x-1}}$.

In [6], Theorem B has been proven by using a technique of complex analysis. Here we give an alternative proof by using only Theorem A and the following well-known fact:

Lemma C (e.g. [7]). *Let $f(x)$ and $g(x)$ be operator monotone functions. Then the following functions are also operator monotone:*

(i) $f(x)^\alpha g(x)^\beta$ *for $\alpha, \beta \ge 0$ such that $\alpha + \beta \le 1$,*

(ii) $f(x^{-1})^{-1}$.

Another proof of Theorem B. (i) The case $0 \leq p \leq 1$. $s_p(x)$ is operator monotone for $0 < p < 1$ by Theorem A since $s_p(x) = F_{p,1-p}(x)$. As for the case $p = 0, 1$, $s_p(x)$ is still operator monotone by taking its limit $p \to +0$ and $p \to 1-0$ (see [11]).

(ii) In the case $1 < p \leq 2$, $s_p(x)$ is operator monotone by Theorem A since

$$s_p(x) = \left(\frac{1}{p}\frac{x^p - 1}{x - 1}\right)^{\frac{1}{p-1}} = F_{1,p-1}(x).$$

(iii) In the case $-1 \leq p < 0$,

$$s_p(x) = \left(-|p|\frac{x-1}{x^{-|p|}-1}\right)^{\frac{1}{1+|p|}} = \left(x^{|p|}|p|\frac{x-1}{x^{|p|}-1}\right)^{\frac{1}{1+|p|}} = x^{\frac{|p|}{1+|p|}} s_{|p|}(x)^{\frac{1-|p|}{1+|p|}}$$

is operator monotone by (i) and Lemma C since $\frac{|p|}{1+|p|}, \frac{1-|p|}{1+|p|} \in [0,1]$.

(iv) In the case $-2 \leq p \leq -1$,

$$s_p(x) = \left(-|p|\frac{x-1}{x^{-|p|}-1}\right)^{\frac{1}{1+|p|}} = \left(x|p|\frac{x^{-1}-1}{x^{-|p|}-1}\right)^{\frac{1}{1+|p|}} = x^{\frac{1}{1+|p|}} \left\{s_{|p|}(x^{-1})^{-1}\right\}^{\frac{|p|-1}{1+|p|}}$$

is operator monotone by (ii) and Lemma C since $\frac{1}{1+|p|}, \frac{|p|-1}{1+|p|} \in [0,1]$. □

By the same way, we can obtain operator monotonicity of $f_{\boldsymbol{p}}(x)$.

Theorem 5. *Let* $\boldsymbol{p} = (p_1, \ldots, p_n) = (a_1, \ldots, a_l, b_1, \ldots, b_m, c_1, \ldots, c_u, d_1, \ldots, d_v)$ $(n = l + m + u + v)$ *and* $\gamma \in \mathbb{R}$ *such that*

$$-2 \leq d_1 \leq \cdots \leq d_v < -1 \leq c_1 \leq \cdots \leq c_u < 0 \leq b_1 \leq \cdots \leq b_m < 1 \leq a_1 \leq \cdots \leq a_l \leq 2,$$

$$0 \leq \gamma + (l+v) - \sum_{i=1}^{l} a_i - \sum_{i=1}^{u} c_i \leq 1 \quad \text{and} \quad 0 \leq \gamma + (m+u) - \sum_{i=1}^{m} b_i - \sum_{i=1}^{v} d_i \leq 1.$$

Then $f_{\boldsymbol{p}}(x) = x^\gamma \prod_{i=1}^{n} p_i \frac{x-1}{x^{p_i}-1}$ *is operator monotone on* $x > 0$.

We notice that the fact shown in Theorem 5 is included in the results in Kawasaki-Nagisa [8] and Nagisa-Wada [9].

Proof. Since

$$\begin{aligned}
a_i\frac{x-1}{x^{a_i}-1} &= x^{1-a_i}\left\{a_i\frac{x^{-1}-1}{(x^{-1})^{a_i}-1}\right\}^{\frac{-1}{1-a_i}\cdot\{-(1-a_i)\}} &&= x^{1-a_i}\left\{s_{a_i}(x^{-1})^{-1}\right\}^{a_i-1}, \\
b_i\frac{x-1}{x^{b_i}-1} & &&= s_{b_i}(x)^{1-b_i}, \\
c_i\frac{x-1}{x^{c_i}-1} &= x^{-c_i}\left\{(-c_i)\frac{x-1}{x^{-c_i}-1}\right\}^{\frac{1}{1+c_i}\cdot(1+c_i)} &&= x^{-c_i}s_{-c_i}(x)^{1+c_i} \text{ and} \\
d_i\frac{x-1}{x^{d_i}-1} &= x\left\{(-d_i)\frac{x^{-1}-1}{(x^{-1})^{-d_i}-1}\right\}^{\frac{-1}{1+d_i}\cdot\{-(1+d_i)\}} &&= x\left\{s_{-d_i}(x^{-1})^{-1}\right\}^{-(1+d_i)}
\end{aligned}$$

hold for each i, we have

$$
\begin{aligned}
f_p(x) &= x^\gamma \prod_{i=1}^{l} a_i \frac{x-1}{x^{a_i}-1} \prod_{i=1}^{m} b_i \frac{x-1}{x^{b_i}-1} \prod_{i=1}^{u} c_i \frac{x-1}{x^{c_i}-1} \prod_{i=1}^{v} d_i \frac{x-1}{x^{d_i}-1} \\
&= x^w \prod_{i=1}^{l} \left\{ s_{a_i}(x^{-1})^{-1} \right\}^{a_i-1} \prod_{i=1}^{m} s_{b_i}(x)^{1-b_i} \prod_{i=1}^{u} s_{-c_i}(x)^{1+c_i} \prod_{i=1}^{v} \left\{ s_{-d_i}(x^{-1})^{-1} \right\}^{-(1+d_i)},
\end{aligned}
$$

where $w = \gamma + \sum_{i=1}^{l}(1-a_i) + \sum_{i=1}^{u}(-c_i) + \sum_{i=1}^{v} 1 = \gamma + (l+v) - \sum_{i=1}^{l} a_i - \sum_{i=1}^{u} c_i$.

By the assumption, $w, a_i - 1, 1 - b_i, 1 + c_i, -(1+d_i) \in [0,1]$ for every i and

$$
w + \sum_{i=1}^{l}(a_i - 1) + \sum_{i=1}^{m}(1-b_i) + \sum_{i=1}^{u}(1+c_i) - \sum_{i=1}^{v}(1+d_i) = \gamma + (m+u) - \sum_{i=1}^{m} b_i - \sum_{i=1}^{v} d_i \in [0,1].
$$

Hence $f_p(x)$ is operator monotone by Theorem B and Lemma C. □

References

[1] L. Cai and F. Hansen, *Metric-adjusted skew information: convexity and restricted forms of superadditivity*, Lett. Math. Phys. **93** (2010), 1–13.

[2] J. I. Fujii, *Interpolatinality for symmetric operator means*, Sci. Math. Jpn. **75** (2012), 267–274.

[3] J. I. Fujii and M. Fujii, *Upper estimations on integral operator means*, Sci. Math. Jpn. **75** (2012), 217–222.

[4] T. Furuta, *Elementary proof of Petz-Hasegawa theorem*, Lett. Math. Phys. **101** (2012), 355–359.

[5] T. Furuta, M. Ito, T. Yamazaki and M.Yanagida, *Upper and lower bounds, and operator monotonicity of an extension of the Petz-Hasegawa function*, Math. Inequal. Appl. **21** (2018), 154–164.

[6] F. Hiai and D. Petz, *Introduction to matrix analysis and applications*, Universitext. Springer, Cham; Hindustan Book Agency, New Delhi, 2014.

[7] S. Izumino and N. Nakamura, *Elementary proofs of operator monotonicity of some functions*, Sci. Math. Jpn. **77** (2015), 363–370.

[8] M. Kawasaki and M. Nagisa, *Transforms on operator monotone functions*, `arXiv:0706.1234 [math.FA]`.

[9] M. Nagisa and S. Wada, *Operator monotonicity of some functions*, Linear Algebra Appl. **486** (2015), 389–408.

[10] D. Petz and H. Hasegawa, *On the Riemannian metric of α-entropies of density matrices*, Lett. Math. Phys. **38** (1996), 221–225.

[11] Y. Udagawa, S. Wada, T. Yamazaki and M. Yanagida, *On a family of operator means involving the power difference means*, Linear Algebra Appl. **485** (2015), 124–131.