Riccati equation for positive semidefinite matrices

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1 Introduction
For given positive definite matrices $A$ and $B$, and an arbitrary matrix $T$, the matrix equation

$$X^*A^{-1}X - T^*X - X^*T = B$$

is said to be an algebraic Riccati equation. In particular, the case $T = 0$ in above, that is,

$$X^*A^{-1}X = B,$$

is called a Riccati equation.

In the preceding paper [3], we discussed them. In this paper, we extend them by the use of the Moore-Penrose generalized inverse. Precisely, we consider the following matrix equation;

$$X^*A^\dagger X - T^*X - X^*T = B,$$

where $A^\dagger$ is the Moore-Penrose generalized inverse of $A$. So the Riccati equation is of form

$$X^*A^\dagger X = B.$$ 

We call them a generalized algebraic Riccati equation and a generalized Riccati equation, respectively.

In this note, we first show that every generalized algebraic Riccati equation is reduced to a generalized Riccati equation, and that solutions of a generalized Riccati equation are analyzed. Next we show that under the kernel inclusion $\ker A \subset \ker B$, the geometric mean $A\#B$ is a solution of a generalized Riccati equation $XA^\dagger X = B$. As an application, we give another proof to equality conditions of matrix Cauchy-Schwarz inequality due to J. I. Fujii [2]: Let $X$ and $Y$ be $k \times n$ matrices and
$Y^*X = U\|Y^*X\|$ a polar decomposition of an $n \times n$ matrix $Y^*X$ with unitary $U$. Then

$$|Y^*X| \leq X^*X # U^*Y^*YU.$$ 

Finally we discuss an order relation between $A#B$ and $A^{1/2}((A^{1/2})^\dagger B(A^{1/2})^{\uparrow})^{1/2}A^{1/2}$ for positive semidefinite matrices $A$ and $B$.

2 Solutions of generalized algebraic Riccati equation

Following after [3], we discuss solutions of a generalized algebraic Riccati equation. Throughout this note, $P_X$ means the projection onto the range of a matrix $X$.

**Lemma 2.1.** Let $A$ and $B$ be positive semidefinite matrices and $T$ a arbitrary matrix. Then $W$ is a solution of a generalized Riccati equation

$$W^*A^\dagger W = B + T^*AT$$

if and only if $X = W + AT$ is a solution of a generalized algebraic Riccati equation

$$X^*A^\dagger X - T^*P_A X - X^*P_AT = B.$$ 

**Proof.** Put $X = W + AT$. Then it follows that

$$X^*A^\dagger X - T^*P_A X - X^*P_AT = W^*A^\dagger W - T^*AT,$$

so that we have the conclusion.

**Theorem 2.2.** Let $A$ and $B$ be positive semidefinite matrices. Then $W$ is a solution of a generalized Riccati equation

$$W^*A^\dagger W = B \quad \text{with } \operatorname{ran} W \subseteq \operatorname{ran} A$$

if and only if $W = A^{1/2}UB^{1/2}$ for some partial isometry $U$ such that $U^*U \geq P_B$ and $UU^* \leq P_A$.

**Proof.** Suppose that $W^*A^\dagger W = B$ and $\operatorname{ran} W \subseteq \operatorname{ran} A$. Since $\|(A^{1/2})^\dagger Wx\| = \|B^{1/2}x\|$ for all vectors $x$, there exists a partial isometry $U$ such that $UB^{1/2} = (A^{1/2})^\dagger W$ with $U^*U = P_B$ and $UU^* \leq P_A$. Hence we have

$$A^{1/2}UB^{1/2} = P_AW = W.$$
The converse is easily checked: If \( W = A^{\frac{1}{2}}U B^{\frac{1}{2}} \) for some partial isometry \( U \) such that \( U^*U \geq P_B \) and \( UU^* \leq P_A \), then \( \text{ran } W \subseteq \text{ran } A \) and
\[
W^*A^\dagger W = B^{\frac{1}{2}}U^*P_A UB^{\frac{1}{2}} = B^{\frac{1}{2}}U^*UB^{\frac{1}{2}} = B.
\]

**Corollary 2.3.** Notation as in above. Then \( X \) is a solution of a generalized algebraic Riccati equation
\[
X^*A^\dagger X - T^*X - X^*T = B
\]
with \( \text{ran } X \subseteq \text{ran } A \) if and only if \( X = A^{\frac{1}{2}}U(B + T^*AT)^{\frac{1}{2}} + AT \) for some partial isometry \( U \) such that \( U^*U \geq P_{B+T^*AT} \) and \( UU^* \leq P_A \).

**Proof.** By Lemma 2.1, \( X \) is a solution of a generalized algebraic Riccati equation \( X^*A^\dagger X - T^*P_A X - X^*P_A T = B \) if and only if \( W = X - AT \) is a solution of \( W^*A^\dagger W = B + T^*AT \). Since \( \text{ran } X \subseteq \text{ran } A \) if and only if \( \text{ran } W \subseteq \text{ran } A \), we have the conclusion by Theorem 2.2.

### 3 Solutions of a generalized Riccati equation

Since \( A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \) for invertible \( A \), the geometric mean \( A\#B \) is the unique solution of the Riccati equation \( XA^{-1}X = B \) if \( A > 0 \), see [5] for an early work. So we consider it for positive semidefinite matrices by the use of the Moore-Penrose generalized inverse, that is,
\[
XA^\dagger X = B
\]
for positive semidefinite matrices \( A, B \).

**Theorem 3.1.** Let \( A \) and \( B \) be positive semidefinite matrices satisfying the kernel inclusion \( \ker A \subset \ker B \). Then \( A\#B \) is a solution of a generalized Riccati equation
\[
XA^\dagger X = B.
\]
Moreover, the uniqueness of its solution is ensured under the additional assumption \( \ker A \subset \ker X \).
proof. We first note that $(A^{1/2})^\dagger = (A^\dagger)^{1/2}$ and $P_A = P_{A^\dagger}$. Putting $X_0 = A\# B$, a recent result due to Fujimoto-Seo [4, Lemma 2.2] says that

$$X_0 = A^{1/2}[(A^{1/2})^\dagger B (A^{1/2})^\dagger]^{1/2}A^{1/2}.$$ 

Therefore we have

$$X_0 A^\dagger X_0 = A^{1/2}[(A^{1/2})^\dagger B (A^{1/2})^\dagger]^{1/2} P_A [(A^{1/2})^\dagger B (A^{1/2})^\dagger]^{1/2} A^{1/2}
= A^{1/2}[(A^{1/2})^\dagger B (A^{1/2})^\dagger]
= P_A BP_A = B$$

Since $\text{ran } X_0 \subset \text{ran } A^{1/2}$, $X_0$ is a solution of the equation.

The second part is proved as follows: If $X$ is a solution of $XA^\dagger X = B$, then

$$(A^{1/2})^\dagger X A^\dagger X (A^{1/2})^\dagger = (A^{1/2})^\dagger B (A^{1/2})^\dagger,$$

so that

$$(A^{1/2})^\dagger X (A^{1/2})^\dagger = [(A^{1/2})^\dagger B (A^{1/2})^\dagger]^{1/2}.$$ 

Hence we have

$$P_A XP_A = A^{1/2}[(A^{1/2})^\dagger B (A^{1/2})^\dagger]^{1/2} A^{1/2} = X_0.$$ 

Since $P_A XP_A = X$ by the assumption, $X = X_0$ is obtained.

As an application, we give a simple proof of the case where the equality holds in matrix Cauchy-Schwarz inequality, see [4, Lemma 2.5].

**Corollary 3.2.** Let $X$ and $Y$ be $k \times n$ matrices and $Y^* X = U |Y^* X|$ a polar decomposition of an $n \times n$ matrix $Y^* X$ with unitary $U$. If $\ker X \subset \ker YU$, then

$$|Y^* X| = X^* X \# U^* Y^* Y U$$

if and only if $Y = XW$ for some $n \times n$ matrix $W$.

proof. Since $\ker X^* X \subset \ker U^* Y^* Y U$, the preceding theorem implies that $|Y^* X|$ is a solution of a generalized Riccati equation, i.e.,

$$U^* Y^* Y U = |Y^* X|(X^* X)^\dagger |Y^* X| = U^* Y^* X (X^* X)^\dagger X^* Y U,$$
or consequently
\[ Y^*Y = Y^*X(X^*X)^\dagger X^*Y. \]
Noting that \( X(X^*X)^\dagger X^* \) is the projection \( P_X \), we have \( Y^*Y = Y^*P_XY \) and hence \( Y = P_XY = X(X^*X)^\dagger X^*Y \) by \((Y - P_XY)^*(Y - P_XY) = 0\), so that \( Y = XW \) for \( W = (X^*X)^\dagger X^*Y \).

4 Geometric mean in operator Cauchy-Schwarz inequality
The origin of Corollary 3.2 is the operator Cauchy-Schwarz inequality due to J.I.Fujii [2], which says as follows:

**OCS inequality** If \( X, Y \in B(H) \) and \( Y^*X = U |Y^*X| \) is a polar decomposition of \( Y^*X \), then

\[ |Y^*X| \leq X^*X \# U^*Y^*YU. \]

In his proof of it, the following well-known fact due to Ando [1] is used: For \( A, B \geq 0 \), the geometric mean \( A \# B \) is given by

\[
A \# B = \max \left\{ X \geq 0; \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0 \right\}
\]

First of all, we discuss the case \( Y^*X \geq 0 \) in (OCS). That is,

\[ Y^*X \leq X^*X \# Y^*Y \]

is shown: Noting that \( Y^*X = X^*Y \geq 0 \), we have

\[
\begin{pmatrix} X^*X & X^*Y \\ Y^*X & Y^*Y \end{pmatrix} = \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} \geq 0,
\]
which means \( Y^*X \leq X^*X \# Y^*Y \).

The proof for a general case is presented by applying the above: Noting that \((YU)^*X = |Y^*X| \geq 0\), it follows that

\[ |Y^*X| = (YU)^*X \leq X^*X \# (YU)^*YU. \]

**Remark 1.** We can give a direct proof to the general case:

\[
\begin{pmatrix} X^*X & |Y^*X| \\ |Y^*X| & U^*Y^*YU \end{pmatrix} = \begin{pmatrix} X & YU \\ 0 & 0 \end{pmatrix} \begin{pmatrix} X & YU \\ 0 & 0 \end{pmatrix} \geq 0.
\]

**Remark 2.** An equivalent condition which the equality holds in the matrix C-S inequality is known by Fujimoto-Seo [4]: Under the assumption \( \ker X \subset \ker YU \),
the equality holds if and only if $YU = XW$ for some $W$. In the proof, they use
(1) If $\ker A \subset \ker B$, then $A \# B A^\dagger B = B$.
(2) If $A \# B = A \# C$ and $\ker A \subset \ker B \cap \ker C$, then $B = C$.

Related to matrix Cauchy-Schwarz inequality, the following result is obtained by Fujimoto-Seo [4]:

Let $A = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$ be positive definite matrix. Then $B \geq C^* A^{-1} C$ holds. Furthermore it is known by them:

**Theorem 4.1.** Let $A$ be as in above and $C = U|C|$ a polar decomposition of $C$ with unitary $U$. Then

$$|C| \leq U^* A U \# C^* A^{-1} C.$$  

**Proof.** It can be also proved as similar as in above: Since $|C| = U^* C = C^* U$, we have

$$\begin{pmatrix} U^* A U & |C| \\ |C| & C^* A^{-1} C \end{pmatrix} = \begin{pmatrix} A^{1/2} U & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A^{1/2} U & 0 \\ 0 & 0 \end{pmatrix} \geq 0.$$

The preceding result is generalized a bit by the use of the Moore-Penrose generalized inverse, for which we note that $(A^{1/2})^\dagger = (A^\dagger)^{1/2}$ for $A \geq 0$:

**Theorem 4.2.** Let $A$ be of form as in above and positive semidefinite, and $C = U|C|$ a polar decomposition of $C$ with unitary $U$. If $\operatorname{ran} C \subseteq \operatorname{ran} A$, then

$$|C| \leq U^* A U \# C^* A^\dagger C.$$  

**Proof.** Let $P_A$ be the projection onto the range of $A$. Since $P_A C = C$ and $C^* P_A = C^*$, we have $|C| = U^* P_A U = C^* P_A C$. Hence it follows that

$$\begin{pmatrix} U^* A U & |C| \\ |C| & C^* A^\dagger C \end{pmatrix} = \begin{pmatrix} A^{1/2} U & (A^\dagger)^{1/2} C \\ (A^\dagger)^{1/2} C & 0 \end{pmatrix} \begin{pmatrix} A^{1/2} U & (A^\dagger)^{1/2} C \\ (A^\dagger)^{1/2} C & 0 \end{pmatrix} \geq 0.$$

5 A generalization of formulae for geometric mean

Since $A \# B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$ for invertible $A$, the geometric mean $A \# B$ for positive semidefinite matrices $A$ and $B$ might be expected the same formulae as for positive definite matrices, i.e.,

$$A \# B = A^{1/2} ((A^{1/2})^\dagger B (A^{1/2})^\dagger)^{1/2} A^{1/2}.$$
As a matter of fact, the following result is known by Fujimoto and Seo:

**Theorem 5.1.** Let $A$ and $B$ be positive semidefinite matrices. Then

$$A \# B \leq A^{1/2}((A^{1/2})^\dagger B(A^{1/2})^\dagger)^{1/2}A^{1/2},$$

If the kernel inclusion $\ker A \subset \ker B$ is assumed, then the equality holds in above.

**Proof.** For the first half, it suffices to show that if

$$
\begin{pmatrix}
A & X \\
X & B \\
\end{pmatrix} 
\geq 0,
$$

then

$$X \leq A^{1/2}((A^{1/2})^\dagger B(A^{1/2})^\dagger)^{1/2}A^{1/2},$$

because of Ando’s definition of the geometric mean. We here use the facts that

$$(A^{1/2})^\dagger = (A^\dagger)^{1/2},$$

and that if

$$
\begin{pmatrix}
A & X \\
X & B \\
\end{pmatrix} 
\geq 0
$$

for positive semdefinite $X$, then $X = AA^\dagger X = P_A X$ and $B \geq XA^\dagger X$. Now, since $B \geq XA^\dagger X$, we have

$$
(A^{1/2})^\dagger B(A^{1/2})^\dagger \geq [(A^{1/2})^\dagger X(A^{1/2})^\dagger]^2,
$$

so that Löwner–Heinz inequality implies

$$[(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2} \geq (A^{1/2})^\dagger X(A^{1/2})^\dagger.$$

Hence it follows from $X = P_A X$ that

$$A^{1/2}[(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2}A^{1/2} \geq X.$$

Next suppose that $\ker A \subset \ker B$. Then we have ran $B \subset \text{ran } A$ and so

$$A^{1/2}(A^{1/2})^\dagger B(A^{1/2})^\dagger A^{1/2} = B.$$

Therefore, putting $C = (A^{1/2})^\dagger B(A^{1/2})^\dagger$ and

$$Y = A^{1/2}((A^{1/2})^\dagger B(A^{1/2})^\dagger)^{1/2}A^{1/2} = A^{1/2}C^{1/2}A^{1/2},$$

we have

$$
\begin{pmatrix}
A & Y \\
Y & B \\
\end{pmatrix} = 
\begin{pmatrix}
A^{1/2} & 0 \\
0 & A^{1/2} \\
\end{pmatrix} 
\begin{pmatrix}
I & C^{1/2} \\
C^{1/2} & C \\
\end{pmatrix} 
\begin{pmatrix}
A^{1/2} & 0 \\
0 & A^{1/2} \\
\end{pmatrix} \geq 0,
$$

which implies that $Y \leq A \# B$ and thus $Y = A \# B$ by combining the result in the first half.

By checking the proof carefully, we have an improvement:
Theorem 5.2. Let \( A \) and \( B \) be positive semidefinite matrices. Then

\[
A \# B \leq A^{1/2}(A^{1/2})^\dagger B(A^{1/2})^\dagger A^{1/2},
\]

In particular, the equality holds in above if and only if \( P_A = AA^\dagger \) commutes with \( B \).

Proof. Notation as in above. If \( P_A = AA^\dagger (= A^{1/2}(A^{1/2})^\dagger) \) commutes with \( B \), we have \( P_A B P_A \leq B \). Therefore we have

\[
\begin{pmatrix}
A & Y \\
Y & B
\end{pmatrix} \geq \begin{pmatrix}
A & Y \\
Y & P_A B P_A
\end{pmatrix} = \begin{pmatrix}
A^{1/2} & 0 \\
0 & A^{1/2}
\end{pmatrix} \begin{pmatrix}
I & C^{1/2} \\
C^{1/2} & C
\end{pmatrix} \begin{pmatrix}
A^{1/2} & 0 \\
0 & A^{1/2}
\end{pmatrix} \geq 0,
\]

Conversely assume that the equality holds. Then \( \begin{pmatrix}
A & Y \\
Y & B
\end{pmatrix} \geq 0 \). Hence we have

\[
B \geq YA^\dagger = A^{1/2}CA^{1/2} = P_A B P_A,
\]

which means \( P_A \) commutes with \( B \).

Finally we cite the following lemma which we used in the proof of Theorem 5.1.

Lemma 5.3. If \( \begin{pmatrix}
A & X \\
X^* & B
\end{pmatrix} \geq 0 \), then \( X = AA^\dagger X = P_A X \) and \( B \geq XA^\dagger X \).

Proof. The assumption implies that

\[
\begin{pmatrix}
(A^{1/2})^\dagger & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
A & X \\
X^* & B
\end{pmatrix} \begin{pmatrix}
(A^{1/2})^\dagger & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
P_A & (A^{1/2})^\dagger X \\
X^*(A^{1/2})^\dagger & B
\end{pmatrix} \geq 0.
\]

Moreover, since

\[
0 \leq \begin{pmatrix}
1 & -(A^{1/2})^\dagger X \\
0 & 1
\end{pmatrix} \begin{pmatrix}
P_A & (A^{1/2})^\dagger X \\
X^*(A^{1/2})^\dagger & B
\end{pmatrix} \begin{pmatrix}
1 & -(A^{1/2})^\dagger X \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
P_A & 0 \\
0 & B - X^*A^\dagger X
\end{pmatrix},
\]

we have \( B \geq X^*A^\dagger X \).

Next we show that \( X = P_A X \). It is equivalent to \( \ker A \subseteq \ker X^* \). Suppose that \( Ax = 0 \). Putting \( y = -\frac{1}{\|B\|}X^*x \), we have

\[
0 \leq \begin{pmatrix}
A & X \\
X^* & B
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix}
= (Xy, x) + (X^*x, y) + (By, y)
= -\frac{2}{\|B\|}\|X^*x\|^2 + \frac{1}{\|B\|^2}(BX^*x, X^*x)
\leq -\frac{\|X^*x\|^2}{\|B\|} \leq 0.
\]

Hence we have \( X^*x = 0 \), that is, \( \ker A \subseteq \ker X^* \) is shown.
REFERENCES


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