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Renormalized Nelson model

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Abstract

In this paper renormalized Nelson Hamiltonian in quantum field theory is discussed. Gibbs measure associated with the ground state of the Nelson Hamiltonian is constructed, and the super exponential decay of the truncated number operator of the ground state is shown.

1 The Nelson model

1.1 Definition

This is a review article of the recent work [21, 13]. One of the simplest model in quantum field theory describing an interaction between non-relativistic quantum matters and a scalar Bose field is the so-called Nelson model which was introduced by Edward Nelson [22, 23] to describe a renormalization of ultraviolet cutoff functions. He fortunately proved the existence of the renormalized Hamiltonian by the operator theory. In this article we study the spectrum of the renormalized Nelson Hamiltonian by using functional integrations and Gibbs measures associated with the ground state. In particular we focus on investigating the properties of the ground state.

First we introduce the Nelson Hamiltonian with ultraviolet cutoff and secondly we define the Nelson Hamiltonian without the cutoff by removing the cutoff.

Let

$$H_{\mathrm{p}} = -\frac{1}{2}\Delta + V$$

be a Schrödinger operator in $L^2(\mathbb{R}^3)$. Let $D(T)$ be the domain of operator T. If V is relatively bounded with respect to $-\frac{1}{2}\Delta$ with a relative bound strictly smaller than one, i.e.,

$$D(V) \subset D(-\frac{1}{2}\Delta), \quad \|Vf\| \le a\| - \frac{1}{2}\Delta f\| + b\|f\|$$

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for $f \in D(V)$ with some $a < 1$ and $b \geq 0$. Then we say $V \in R_{\mathrm{Kato}}$. If $V \in R_{\mathrm{Kato}}$, then H_{p} is self-adjoint on $D(-\frac{1}{2}\Delta)$ and essentially self-adjoint on any core of $-\Delta$ so that $\overline{H_{\mathrm{p}}\lceil_D} = \overline{-\frac{1}{2}\Delta\lceil_D + V\lceil_D}$. See [14].

Let us introduce the scalar quantum field. The following are standing assumptions on dispersion relation ω, ultraviolet cutoff function $\hat{\varphi}$ and potential V throughout this section.

Assumption 1.1 (1) $\omega(k) = |k|$. (2) $\overline{\hat{\varphi}(k)} = \hat{\varphi}(-k)$, $\hat{\varphi}/\omega, \hat{\varphi}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$. (3) $V \in R_{\mathrm{Kato}}$.

We define $H_M = \{f | \hat{f}/\sqrt{\omega} \in L^2(\mathbb{R}^3)\}$ and $H_E = \{f | \hat{f}/\sqrt{\omega_E} \in L^2(\mathbb{R}^{3+1})\}$. Here $\omega_E = \omega_E(k, k_0) = \sqrt{|k|^2 + k_0^2}$. We also define the Fourier transform of H_M and H_E by $\hat{H}_M$ and $\hat{H}_E$, respectively. We also define real Hilbert spaces below:

$$\mathscr{M} = \{f \in H_M | f \text{ is real-valued}\}, \quad \mathscr{E} = \{f \in H_E | f \text{ is real-valued}\}.$$

Both $\mathscr{M}$ and $\mathscr{E}$ are Hilbert spaces over $\mathbb{R}$, and note that $\mathscr{M}_{\mathbb{C}} = H_M$ and $\mathscr{E}_{\mathbb{C}} = H_E$. Here $H_{\mathbb{C}}$ denotes the complexification of H. Let $(\phi(f), f \in \mathscr{M})$ be a family of Gaussian random variables on a probability space $(\mathrm{Q}, \Sigma, \mu)$ indexed by $f \in \mathscr{M}$. Thus it follows that

$$\mathbb{E}_\mu[\phi(f)] = 0, \quad \mathbb{E}_\mu[\phi(f)\phi(g)] = \frac{1}{2}(\hat{f}, \hat{g})_{\hat{H}_M}.$$

Here $\mathbb{E}_P[\cdots]$ describes the expectation with respect to probability measure P. The Hilbert space $L^2(Q)$ is called the boson Fock space in this paper. We define $\hat{\omega} = \omega(-i\nabla) = \sqrt{-\Delta}$. Let $H_{\mathrm{f}} = d\Gamma(\hat{\omega}) : L^2(Q) \to L^2(Q)$ be the free field Hamiltonian, where $d\Gamma(\hat{\omega})$ denotes the second quantization of $\hat{\omega}$. Thus H_{f} is the self-adjoint operator in $L^2(\mathrm{Q})$ and satisfies that $H_{\mathrm{f}}\mathbb{1} = 0$.

The Nelson Hamiltonian defined in the total Hilbert space

$$\mathcal{H}_{\mathrm{N}} = L^2(\mathbb{R}^3) \otimes L^2(\mathrm{Q})$$

is given by

$$H = H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}} + H_{\mathrm{I}}. \tag{1.1}$$

Here H_{I} describes the linear interaction and is given by

$$H_{\mathrm{I}} = \int_{\mathbb{R}^3}^{\oplus} \phi(\varphi(\cdot - x))dx$$

under the identification $\mathcal{H}_{\mathrm{N}} \cong \int_{\mathbb{R}^3}^{\oplus} L^2(\mathrm{Q})dx$. Here $\int_{\mathbb{R}^3}^{\oplus} \cdots dx$ denotes the constant fiber direct integral [24, XIII.16]. Notice that we can also define the Nelson Hamiltonian on $L^2(\mathbb{R}^3) \otimes \mathscr{F}$, where $\mathscr{F}$ is the boson Fock space over H_M. We refer to Appendix A and C. Suppose Assumption 1.1. Then H is self-adjoint on $D(-\frac{1}{2}\Delta \otimes \mathbb{1}) \cap D(\mathbb{1} \otimes H_{\mathrm{f}})$. This can be proven by using the inequality $\|H_{\mathrm{I}}\Phi\| \leq \|\hat{\varphi}/\omega\| \|(H_{\mathrm{f}} + \mathbb{1})^{1/2}\Phi\|$ and Kato-Rellich theorem.

1.2 Feynman-Kac-type formula

We define another Gaussian random variable to construct Feynman-Kac-type formula. Let $(\phi_E(f), f \in \mathscr{E})$ be the Gaussian random variable on a probability space $(\mathrm{Q_E}, \Sigma_E, \mu_E)$ indexed by $f \in \mathscr{E}$. $\phi_E(f)$ is called Euclidean field smeared by f. We will define a family of isometries J_t from $L^2(\mathrm{Q})$ to $L^2(\mathrm{Q_E})$ through the second quantization of a specific transformation j_t from $\mathscr{M}$ to $\mathscr{E}$. Define $j_t : \mathscr{M} \to \mathscr{E}$ by

$$j_t : f \mapsto \delta_t \otimes f.$$

Here $\delta_t(x) = \delta(x-t)$ is the delta function with mass at t. Thus $\overline{\delta_t \otimes f} = \delta_t \otimes f$, which implies that j_t preserves realness. It follows that

$$\tau_s^* \tau_t = e^{-|s-t|\hat{\omega}}, \quad s, t \in \mathbb{R}.$$

In particular, j_t is isometry between $\mathscr{M}$ and $\mathscr{E}$ for each $t \in \mathbb{R}$. Let

$$\mathrm{J}_t : L^2(\mathrm{Q}) \to L^2(\mathrm{Q_E}), \quad t \in \mathbb{R},$$

be the family of isometries connecting $L^2(\mathrm{Q})$ and $L^2(\mathrm{Q_E})$, i.e.,

$$\mathrm{J}_t \mathbb{1}_\mathrm{M} = \mathbb{1}_\mathrm{E}, \quad \mathrm{J}_t : \phi(f_1) \cdots \phi(f_n) :=: \phi_\mathrm{E}(j_t f_1) \cdots \phi_\mathrm{E}(j_t f_n) :$$

and it satisfies that $\mathrm{J}_t^* \mathrm{J}_s e^{-|s-t|H_\mathrm{f}}$ for $s, t \in \mathbb{R}$. Here $: \prod_{j=1}^n \phi_\mathrm{E}(f_j) :$ is the wick product of $\prod_{j=1}^n \phi_\mathrm{E}(f_j)$.

Let $(B_t)_{t \geq 0}$ be the Brownian motion on a Wiener space $(\Omega, \mathscr{F}, \mathscr{W}^x)$. Under Wiener measure $\mathscr{W}^x$, the Brownian motion starts from x almost surely at time $t = 0$. We denote $\mathbb{E}^x$ for $\mathbb{E}_{\mathscr{W}^x}$. Let $f, g \in L^2(\mathbb{R}^3)$. Then the Feynman-Kac formula of e^{-tH_p} is given by

$$(f, e^{-tH_\mathrm{p}} g)_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\int_0^t V(B_s) ds} \bar{f}(B_0) g(B_t) \right].$$

We can also construct Feynman-Kac-type formula for e^{-tH}.

Theorem 1.2 (Feynman-Kac-type formula) *Suppose Assumption 1.1. Then for $t \geq 0$ and $F, G \in \mathcal{H}_\mathrm{N}$,*

$$(F, e^{-tH} G)_{\mathcal{H}_\mathrm{N}} = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\int_0^t V(B_s) ds} (\mathrm{J}_0 F(B_0), e^{-\phi_E(\int_0^t j_s \varphi(\cdot - B_s) ds)} \mathrm{J}_t G(B_t))_{L^2(\mathrm{Q_E})} \right]. \tag{1.2}$$

Here $F, G \in \mathcal{H}_\mathrm{N}$ are regarded as $L^2(\mathrm{Q})$-valued L^2-functions on $\mathbb{R}^3$.

PROOF. For $\varepsilon \geq 0$ let $H_\mathrm{I}^\varepsilon = y_\varepsilon(H_\mathrm{I})$, where $y_\varepsilon(X) = X + \varepsilon X^2$. Then H_I^ε is bounded below for $\varepsilon > 0$. We can also see that $H^\varepsilon = H_\mathrm{p} \otimes \mathbb{1} + 1 \otimes H_\mathrm{f} + H_\mathrm{I}^\varepsilon$ is self-adjoint on $D(H_\mathrm{p} \otimes \mathbb{1} + 1 \otimes H_\mathrm{f})$ and essentially self-adjoint on any core of $H_\mathrm{p} \otimes \mathbb{1} + 1 \otimes H_\mathrm{f}$ for $0 \leq \varepsilon < c$ with some c. In particular $e^{-tH^\varepsilon} \to e^{-tH}$ strongly as $\varepsilon \downarrow 0$. For simplicity, first we assume that $V \in C_0^\infty(\mathbb{R}^3)$. Let $h = -\frac{1}{2}\Delta$. By the Trotter-Kato product formula [17, 18, 19] and the factorization formula $e^{-|s-t|H_{\mathrm{f}}} = \mathrm{J}_s^* \mathrm{J}_t$, we have

$$e^{-tH^\varepsilon} = \mathrm{s-}\lim_{n\to\infty} \mathrm{J}_0^* \left(\prod_{j=0}^{n-1} \mathrm{J}_{\frac{jt}{n}} e^{-\frac{t}{n}H_{\mathrm{I}}^\varepsilon} e^{-\frac{t}{n}h} e^{-\frac{t}{n}V} \mathrm{J}_{\frac{jt}{n}}^* \right) \mathrm{J}_t, \tag{1.3}$$

and we insert (1.3) into $(F, e^{-tH^\varepsilon}G)$. Hence we have

$$(F, e^{-tH^\varepsilon}G) = \lim_{n\to\infty} \left(\mathrm{J}_0 F, \left(\prod_{j=0}^{n-1} \mathrm{J}_{\frac{jt}{n}} e^{-\frac{t}{n}H_{\mathrm{I}}^\varepsilon} e^{-\frac{t}{n}h} e^{-\frac{t}{n}V} \mathrm{J}_{\frac{jt}{n}}^* \right) \mathrm{J}_t G \right).$$

Here $\prod_{j=1}^n t_j = t_1 \cdots t_n$. Using the identity $\mathrm{J}_s e^{-H_{\mathrm{I}}^\varepsilon} \mathrm{J}_s = E_s e^{-H_{\mathrm{I}}^\varepsilon(s)} E_s$ for $s \in \mathbb{R}$, where $H_{\mathrm{I}}^\varepsilon(s) = \int_{\mathbb{R}^3}^{\oplus} y_\varepsilon\left(\phi_E(j_s \varphi(\cdot - x))\right) dx$ and $E_s = \mathrm{J}_s \mathrm{J}_s^*$ is a projection, we can see that

$$(F, e^{-tH^\varepsilon}G) = \lim_{n\to\infty} \left(\mathrm{J}_0 F, \left(\prod_{j=0}^{n-1} E_{\frac{jt}{n}} e^{-\frac{t}{n}H_{\mathrm{I}}^\varepsilon(\frac{jt}{n})} e^{-\frac{t}{n}h} e^{-\frac{t}{n}V} E_{\frac{jt}{n}} \right) \mathrm{J}_t G \right).$$

By the Markov property [26] of E_s's we can neglect E_s's on the right-hand side above. Then

$$(F, e^{-tH^\varepsilon}G) = \lim_{n\to\infty} \left(\mathrm{J}_0 F, \left(\prod_{j=0}^{n-1} e^{-\frac{t}{n}H_{\mathrm{I}}^\varepsilon(\frac{jt}{n})} e^{-\frac{t}{n}h} e^{-\frac{t}{n}V} \right) \mathrm{J}_t G \right).$$

The right-hand side above can be represented in terms of the Wiener measure by

$$(F, e^{-tH^\varepsilon}G) = \lim_{n\to\infty} \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\sum_{j=0}^{n-1} V(B_{\frac{jt}{n}})} \left(\mathrm{J}_0 F(B_0) e^{-\sum_{j=0}^{n-1} \frac{t}{n} H_{\mathrm{I}}^\varepsilon(\frac{jt}{n})} \mathrm{J}_t G(B_t) \right) \right].$$

Note that $s \mapsto j_s\varphi(\cdot - B_s)$ is strongly continuous as a map $\mathbb{R} \to \mathscr{E}$, almost surely. Hence $s \mapsto \phi_E(j_s\varphi(\cdot - B_s))$ is also strongly continuous as a map $\mathbb{R} \to L^2(\mathscr{Q}_E)$. Then we can compute the limit as

$$(F, e^{-tH^\varepsilon}G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\int_0^t V(B_s)ds} \left(\mathrm{J}_0 F(B_0), e^{-\varepsilon Q - \phi_E(\int_0^t j_s \varphi(\cdot - B_s) ds)} \mathrm{J}_t G(B_t) \right) \right].$$

Here $Q_t = \int_0^t \phi_E(j_s \varphi(\cdot - B_s))^2 ds$. Take $\varepsilon \downarrow 0$ on both sides above we have (1.2). Then the theorem follows for $V \in C_0^\infty(\mathbb{R}^3)$. By a simple limiting argument we can prove (1.2) for $V \in R_{\mathrm{Kato}}$. □

2 Ultraviolet-renormalization

2.1 Pair interactions

We introduce a cutoff function by

$$\hat{\varphi}_\varepsilon(k) = e^{-\varepsilon|k|^2/2} \mathbb{1}_{|k| \geq \kappa}$$

and the Nelson Hamiltonian H with cutoff function above is denoted by H_ε. Here $\kappa > 0$ is the sharp infrared cutoff parameter which is fixed throughout this paper, and ε ultraviolet cutoff parameter.We note that it is not necessarily fix the cutoff function as above, and the discussion below can be verified for more general cutoff functions. We also suppose that

$$V \in L^\infty(\mathbb{R}^3).$$

In this section we consider the limit of $\varepsilon \downarrow 0$. Nelson shows in [22] that there exists a self-adjoint operator $H_{\rm ren}$ such that $H_\varepsilon - g^2 E_\varepsilon \to H_{\rm ren}$ as $\varepsilon \downarrow 0$ in the strong resolvent sense by the operator theory. Here

$$E_\varepsilon = -\int_{\mathbb{R}^3} \frac{e^{-\varepsilon|k|^2}}{2\omega(k)} \beta(k) \mathbb{1}_{|k|\geq\kappa} dk, \quad \varepsilon > 0$$

is a renormalization term which goes to $-\infty$ as $\varepsilon \downarrow 0$, where $\beta(k)$ is given by

$$\beta(k) = \frac{1}{\omega(k) + |k|^2/2}. \tag{2.1}$$

By Theorem 1.2, for $F = f \otimes \mathbb{1}$ and $G = h \otimes \mathbb{1}$ we have

$$(f \otimes \mathbb{1}, e^{-2TH_\varepsilon} h \otimes \mathbb{1}) = \int_{\mathbb{R}^3} dx\, \mathbb{E}^x \left[\overline{f(B_{-T})} h(B_T) e^{-\int_{-T}^T V(B_s)ds} e^{\frac{g^2}{2} S_\varepsilon^T} \right],$$

where $(B_t)_{t\in\mathbb{R}}$ is two-sided 3-dimensional Brownian motion,

$$S_\varepsilon^T = \int_{-T}^T ds \int_{-T}^T dt\, W_\varepsilon(B_s - B_t, s-t)$$

is called the pair interaction, and

$$W_\varepsilon(x,t) = \int_{\mathbb{R}^3} \frac{e^{-\varepsilon|k|^2} e^{-ikx} e^{-|t|\omega(k)}}{2\omega(k)} \mathbb{1}_{|k|\geq\kappa} dk.$$

2.2 Functional integral representations

Consider the function on $\mathbb{R}^3 \times \mathbb{R}$:

$$\varphi_\varepsilon(x,t) = \int_{\mathbb{R}^3} \frac{e^{-\varepsilon|k|^2} e^{-ikx-|t|\omega(k)}}{2\omega(k)} \beta(k) \mathbb{1}_{|k|\geq\kappa} dk, \quad \varepsilon \geq 0.$$

Note that $E_\varepsilon = -\varphi_\varepsilon(0,0)$. Next proposition is a key ingredient.

Proposition 2.1 ([7] and [12, Chapter 8]) *It follows that*

$$\lim_{\varepsilon\downarrow 0} \mathbb{E}^x \left[e^{-\int_{-T}^T V(B_s)ds} e^{\frac{g^2}{2}(S_\varepsilon^T - 4T\varphi_\varepsilon(0,0))} \right] = \mathbb{E}^x \left[e^{-\int_{-T}^T V(B_s)ds} e^{\frac{g^2}{2} S_0^{\rm ren}} \right].$$

where $S_0^{\rm ren}$ is the random process defined by

$$S_0^{\rm ren} = -2\int_{-T}^T \varphi_0(B_s - B_T, s-T)ds + 2\int_{-T}^T \left(\int_{-T}^t \nabla\varphi_0(B_s - B_t, s-t)ds \right) dB_t.$$

We recall that $(\phi(f), f \in \mathscr{M})$ and $(\phi_E(F), F \in \mathscr{E})$ are families of Gaussian random variables indexed by $\mathscr{M}$ and $\mathscr{E}$, respectively. Then it follows that

$$\phi_E(F) \cong \frac{1}{\sqrt{2}}(a_E^*(\hat{F}) + a_E(\tilde{\hat{F}})), \quad \hat{F} \in \hat{H}_E, \tag{2.2}$$

$$\phi(f) \cong \frac{1}{\sqrt{2}}(a_M^*(\hat{f}) + a_M(\tilde{\hat{f}})), \quad \hat{f} \in \hat{H}_M, \tag{2.3}$$

where $a_E^*(\hat{F})$ and $a_M^*(\hat{f})$ (resp. $a_E(\hat{F})$ and $a_M(\hat{f})$) are creation operators (resp. annihilation operators) on boson Fock space $\mathscr{F}(\hat{H}_E)$ and $\mathscr{F}(\hat{H}_M)$, respectively, i.e.,

$$[a_M^*(\hat{f}), a_M(\hat{g})] = (\hat{f}, \hat{g})_{\hat{H}_M} = (\hat{f}/\sqrt{\omega}, \hat{g}/\sqrt{\omega}),$$
$$[a_E^*(\hat{F}), a_E(\hat{G})] = (\hat{F}, \hat{G})_{\hat{H}_E} = (\hat{F}/\omega_E, \hat{G}/\omega_E),$$

where $\omega_E = \sqrt{\omega(k)^2 + |k_0|^2}$. Note that $a_M^*(f)$, $a_M(f)$, $a_E^*(f)$ and $a_E(f)$ are linear in f. We given a functional integral representation of $(F, e^{-TH_{\rm ren}}G)$ in [7] but only for $F, G \in \mathcal{D}$, where $\mathcal{D}$ is some dense subset. Let $F, G \in \mathcal{H}_{\rm N}$ and we define

$$U_T^\varepsilon(k) = -\frac{g}{\sqrt{2}} \int_{-T}^{T} e^{-|s+T|\omega(k)} e^{-ikB_s} e^{-\varepsilon|k|^2/2} 1\!\!1_{|k| \geq \kappa} ds,$$
$$\tilde{U}_T^\varepsilon(k) = -\frac{g}{\sqrt{2}} \int_{-T}^{T} e^{-|s-T|\omega(k)} e^{ikB_s} e^{-\varepsilon|k|^2/2} 1\!\!1_{|k| \geq \kappa} ds.$$

Set $U_T^0 = U_T$ and $\tilde{U}_T^0 = \tilde{U}_T$. Exponential of annihilation operators and creation operators $e^{a_M(f)}$ and $e^{a_M^*(f)}$ are discussed in Appendix. These are closed operators and $e^{a_M^*(f)} e^{-tH_{\rm f}}$ and $e^{-tH_{\rm f}} e^{a_M(f)}$ are bounded operators for $t > 0$ if $f/\sqrt{\omega} \in \hat{H}_M$, i.e., $f/\omega, f/\sqrt{\omega} \in L^2(\mathbb{R}^3)$. Set

$$S_\varepsilon^{\rm ren} = S_\varepsilon^T - 4T\varphi_\varepsilon(0,0).$$

Theorem 2.2 ([21]) *It follows that $U_T, \tilde{U}_T \in \hat{H}_M$ a.s. i.e., $U_T/\sqrt{\omega}, \tilde{U}_T/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, and*

$$(F, e^{-2TH_{\rm ren}}G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\int_{-T}^T V(B_s)ds} e^{\frac{g^2}{2} S_0^{\rm ren}} \left(F(B_{-T}), e^{a_M^*(U_T)} e^{-2TH_{\rm f}} e^{a_M(\tilde{U}_T)} G(B_T) \right) \right]. \tag{2.4}$$

Proof. We show only outline of the proof. Refer to see [12, Section 8.10] and [21]. Let $\varrho_\varepsilon = (e^{-\varepsilon|k|^2/2} 1\!\!1_{|k| \geq \kappa})\check{}$. Let $\varepsilon > 0$. We then have

$$(F, e^{-2T(H_\varepsilon - E_\varepsilon)}G)$$
$$= \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\int_{-T}^T V(B_s)ds} e^{\frac{g^2}{2} S_\varepsilon^{\rm ren}} (F(B_{-T}), \mathrm{J}_{-T}^* e^{-g\phi_E(\int_{-T}^T j_s \varrho_\varepsilon(\cdot - B_s)ds)} \mathrm{J}_T G(B_T)) \right].$$

By the identification

$$\phi_E\left(\int_{-T}^{T} j_s\varrho_\varepsilon(\cdot - B_s)ds\right) \cong \frac{1}{\sqrt{2}}\left\{a_E^*\left(\int_{-T}^{T}\widehat{j_s\varrho_\varepsilon}e^{-ikB_s}ds\right) + a_E\left(\int_{-T}^{T}\widetilde{\widehat{j_s\varrho_\varepsilon}}e^{ikB_s}ds\right)\right\}$$

and $\frac{1}{2}\left\|\int_{-T}^{T}\widehat{j_s\varrho_\varepsilon}e^{-ikB_s}ds\right\|_{\hat{H}_E}^2 = S_\varepsilon^T$, the Baker-Campbell-Hausdorff formula yields that

$$e^{-g\phi(\int_{-T}^{T} j_s\varrho_\varepsilon(\cdot-B_s)ds)} = e^{\frac{g^2}{2}S_\varepsilon^T}e^{-a_M^*(\frac{g}{\sqrt{2}}\int_{-T}^{T}\widehat{j_s\varrho_\varepsilon}e^{-ikB_s}ds)}e^{-a_M(\frac{g}{\sqrt{2}}\int_{-T}^{T}\widetilde{\widehat{j_s\varrho_\varepsilon}}e^{ikB_s}ds)}.$$

Then we can compute as

$$\mathrm{J}_{-T}^*e^{-g\phi(\int_{-T}^{T} j_s\varrho_\varepsilon(\cdot-B_s)ds)}\mathrm{J}_T = e^{\frac{g^2}{2}S_\varepsilon^T}e^{a_M^*(U_T^\varepsilon)}e^{-2TH_{\mathrm{f}}}e^{a_M(\tilde{U}_T^\varepsilon)}.$$

Thus we have a functional integral representation of semigroup $e^{-2T(H_\varepsilon - E_\varepsilon)}$ in terms of $e^{a_M^*(U_T^\varepsilon)}e^{-2TH_{\mathrm{f}}}e^{a_M(\tilde{U}_T^\varepsilon)}$ by $(F, e^{-2T(H_\varepsilon-E_\varepsilon)}G) = \int_{\mathbb{R}^3} P_\varepsilon(x)dx$, where

$$P_\varepsilon(x) = \mathbb{E}^x\left[e^{-\int_{-T}^{T}V(B_s)ds}e^{\frac{g^2}{2}S_\varepsilon^{\mathrm{ren}}}(F(B_{-T}), e^{a_M^*(U_T^\varepsilon)}e^{-2TH_{\mathrm{f}}}e^{a_M(\tilde{U}_T^\varepsilon)}G(B_T))\right].$$

We can check that $\mathbb{E}^x[\int_{\mathbb{R}^3}|U_T/\sqrt{\omega}|^2dk] < \infty$. Then we can conclude that $U_T/\sqrt{\omega} \in L^2(\mathbb{R}^3)$ and $\tilde{U}_T/\sqrt{\omega} \in L^2(\mathbb{R}^3)$ a.s. By using the uniform continuity of the map $f \mapsto e^{a_M^*(f)}e^{-TH_{\mathrm{f}}}$ discussed in Proposition B.8 we can show that $P_\varepsilon \in L^1(\mathbb{R}^3)$ and $P_\varepsilon \to P$ in L^1 as $\varepsilon \downarrow 0$. Then the proof is complete. □

One crucial corollary of Theorem 2.2 is the positivity improving property [5] of the semigroup $e^{-tH_{\mathrm{ren}}}$.

Corollary 2.3 ([21]) *Let $t > 0$. Then $e^{-tH_{\mathrm{ren}}}$ is positivity improving. In particular if the ground state of H_{ren} exists, then it is unique.*

PROOF. Let $\Phi \in L^2(\mathrm{Q})$ be non-negative. Then Φ can be approximated by functions $\{\Phi_n\}_{n=1}^\infty$ such that $\Phi_n = F_n(\phi(f_1^n), \cdots, \phi(f_{m_n}^n))$, where $F_n \in \mathscr{S}(\mathbb{R}^{m_n})$ is a non-negative function, and $f_i^j \in \mathscr{M}$. Suppose that $\Psi = F_n(\phi(f_1), \cdots, \phi(f_m))$. For $g \in \mathscr{M}$ we have

$$e^{a_M(g)}\Psi = F_n(\phi(f_1) + (g, f_1)_{H_M}, \cdots, \phi(f_m) + (g, f_m)_{H_M}) \geq 0.$$

The linear hull of functions like Ψ is dense, and $e^{-tH_{\mathrm{f}}}$ is positivity improving [24, XIII.12]. Then $\overline{e^{-tH_{\mathrm{f}}}e^{a_M(g)}}$ is positivity improving for any $t > 0$. In particular the bounded operator $e^{a_M^*(U_T)}e^{-2TH_{\mathrm{f}}}e^{a_M(\tilde{U}_T)}$ is also positivity improving for any $T > 0$. Let $F, G \in L^2(\mathbb{R}^3 \times \mathrm{Q})$ be non-negative functions. By formula (2.4) we have

$$(F, e^{-2TH_{\mathrm{ren}}}G) = \int_{\mathbb{R}^3} dx\mathbb{E}^x\left[e^{-\int_{-T}^{T}V(B_s)ds}e^{\frac{g^2}{2}S_0^{\mathrm{ren}}}\left(F(B_{-T}), e^{a_M^*(U_T)}e^{-2TH_{\mathrm{f}}}e^{a_M(\tilde{U}_T)}G(B_T)\right)\right] > 0.$$

Then the corollary follows. □

2.3 Kato-class potentials

A potential $V : \mathbb{R}^d \to \mathbb{R}$ is said to belong to Kato-class relative to the Laplacian [16, 4] whenever

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \mathbb{E}^x_{\mathscr{W}}\left[\int_0^t |V(B_s)|ds\right] = 0$$

We denote by $\mathscr{K}_d$ the set of all such potentials. Let $0 \leq V \in \mathscr{K}_d$. Then there exist $\beta, \gamma > 0$ such that

$$\sup_{x\in R^d} \mathbb{E}[\mathrm{e}^{\int_0^t V(W_s^x)ds}] \leq \gamma \mathrm{e}^{t\beta}.$$

See [27] and [20, Lemma 3.38] for this.

Theorem 2.4 *Let $V \in \mathscr{K}_3$. Let us define the quadratic form on $\mathcal{H}_{\mathrm{N}} \times \mathcal{H}_{\mathrm{N}}$ by*

$$Q(F,G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\int_{-T}^T V(B_s)ds} e^{\frac{g^2}{2} S_0^{\mathrm{ren}}} \Big(F(B_{-T}), e^{a_M^*(U_T)} e^{-2TH_{\mathrm{f}}} e^{a_M(\tilde{U}_T)} G(B_T) \Big) \right].$$

Then there exists a self-adjoint operator K such that $Q(F,G) = (F, e^{-2TK}G)$.

Proof. Let

$$P_\varepsilon(x) = \mathbb{E}^x \left[e^{-\int_{-T}^T V(B_s)ds} e^{\frac{g^2}{2} S_0^{\mathrm{ren}}} \Big(F(B_{-T}), e^{a_M^*(U_T^\varepsilon)} e^{-2TH_{\mathrm{f}}} e^{a_M(\tilde{U}_T^\varepsilon)} G(B_T) \Big) \right].$$

We then see that $|P_\varepsilon(x)| \leq C\|F(x)\|\|G(B_{2T})\|$ with some constant C independent of $x \in \mathbb{R}^3$, furthermore it can be seen that $|P_\varepsilon(x) - P(x)| \leq C_\varepsilon \|F(x)\|\|G(B_{2T})\|$ with C_ε such that $\lim_{\varepsilon\downarrow 0} C_\varepsilon = 0$. Thus $Q(F,G) = (F, S_T G)$ with a bounded operator S_T by the Riesz representation theorem [15, p.322]. We can see that

$$S_0 = \mathbb{1}.$$

Let

$$Q_\varepsilon(F,G) = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\int_{-T}^T V(B_s)ds} e^{\frac{g^2}{2} S_\varepsilon^{\mathrm{ren}}} \Big(F(B_{-T}), e^{a_M^*(U_T^\varepsilon)} e^{-2TH_{\mathrm{f}}} e^{a_M(\tilde{U}_T^\varepsilon)} G(B_T) \Big) \right].$$

Then for each $\varepsilon > 0$, we see that $Q_\varepsilon(F,G) = (F, e^{2T(H_\varepsilon - E_\varepsilon)}G)$ and $(F, e^{2T(H_\varepsilon - E_\varepsilon)}G) \to (F, S_T G)$ as $\varepsilon \downarrow 0$. In particular

$$(F, S_S S_T G) = \lim_{\varepsilon\downarrow 0}(F, e^{2S(H_\varepsilon - E_\varepsilon)} e^{2T(H_\varepsilon - E_\varepsilon)}G) = \lim_{\varepsilon\downarrow 0}(F, e^{2(S+T)(H_\varepsilon - E_\varepsilon)}G) = (F, S_{S+T}G).$$

Hence

$$S_S S_T = S_{S+T}$$

for any $S, T \geq 0$. Then S_T satisfies the semi-group property. It is also easily seen that $T \mapsto (F, S_T G)$ is continuous. This implies that $T \mapsto S_T$ is strongly continuous. Thus by the Stone theorem for semigroup [20], the theorem follows. □

3 Gibbs measures

3.1 Local convergence

Under some assumptions in e.g., [1, 2, 28, 6] it is shown that H has the unique ground state. In [10, 13] the existence of the ground state is shown for $H_{\rm ren}$. In this section we assume the existence of ground state of $H_{\rm ren}$. On the other hand properties of ground state of H is shown in [3] by using a path measure. In this paper we can see the properties of ground state of $H_{\rm ren}$ by using the so-called Gibbs measure.

The ground state of $H_{\rm ren}$ is denoted by $\Psi_{\rm g}$ and $H_{\rm ren}\Psi_{\rm g} = {\rm E}\Psi_{\rm g}$. Then $\Psi_{\rm g} > 0$ is proven. In particular

$$\Psi_{\rm g} = \lim_{t\to 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\|e^{-tH_\varepsilon} f\otimes \mathbb{1}\|} e^{-tH_\varepsilon} f\otimes \mathbb{1},$$

and it follows that

$$(\Psi_{\rm g}, O\Psi_{\rm g}) = \lim_{t\to 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\|e^{-tH_\varepsilon} f\otimes \mathbb{1}\|^2}(e^{-tH_\varepsilon} f\otimes \mathbb{1}, Oe^{-tH_\varepsilon} f\otimes \mathbb{1}).$$

For some operator O we can construct a functional integral representation of the right-hand side above, and which has the form of $\mathbb{E}_{\mu_T^{\rm ren}}[f_{T,O}]$ with some probability measure $\mu_T^{\rm ren}$ and some integrand $f_{T,O}$. Formally we have

$$(\Psi_{\rm g}, O\Psi_{\rm g}) = \mathbb{E}_{\mu_\infty^{\rm ren}}[f_{\infty,O}]. \tag{3.1}$$

The purpose of this section is to construct $\mu_T^{\rm ren}$ and to show the convergence $\mu_T^{\rm ren} \to \mu_\infty^{\rm ren}$ as $t\to\infty$ in the local sense. Using the formula (3.1) we can study the properties of ground state $\Psi_{\rm g}$. This type of formulas are actually established for the Nelson model in [3] and [12, Section 8.8.], the so-called spin-boson model in [9] and semi-relativistic Pauli-Fierz model in [11]. We summarize them in [12]. The procedure is similar to those in [9, 11]. We shall show only the result concerning the existence of limit measure $\mu_\infty^{\rm ren}$.

Let $\mathscr{X}$ be the set of $\mathbb{R}^3$-valued continuous paths on $\mathbb{R}$:

$$\mathscr{X} = C(\mathbb{R};\mathbb{R}^3).$$

Let $\mathscr{F}_T = \sigma(B_r, -t\le r\le t)$ be the natural filtration of Brownian motion $(B_t)_{t\in\mathbb{R}}$. Then we set $\mathscr{G}_T = \bigcup_{0\le s\le T}\mathscr{F}_s$ and $\mathscr{G} = \bigcup_{0\le s}\mathscr{F}_s$ are finitely additive families of sets. We define

$$\mathscr{L}_T^{\rm ren} = f(B_{-T})f(B_T)e^{S_0^{\rm ren}}e^{-\int_{-T}^T V(B_s)ds}$$

and define the family of path measures $\mu_T^{\rm ren}$, $T>0$, on $(\mathscr{X}, \mathcal{B}(\mathscr{X}))$ by

$$\mu_T^{\rm ren}(A) = \frac{1}{Z_T}\int_{\mathbb{R}^3} dx\mathbb{E}^x[\mathbb{1}_A\mathscr{L}_T^{\rm ren}],$$

where Z_T is the normalizing constant. We can show that $\mu_T^{\rm ren}$ converges to a probability measure $\mu_\infty^{\rm ren}$ in the local sense.

Theorem 3.1 ([13]) *The family of probability measures* $\{\mu_T^{\rm ren}\}_{T\geq 0}$ *on* $(\mathscr{X},\mathcal{B}(\mathscr{X}))$ *converges to a probability measure* $\mu_\infty^{\rm ren}$ *in the local sense, i.e.,* $\mu_T^{\rm ren}(A) \to \mu_\infty^{\rm ren}(A)$ *as* $T\to\infty$ *for each* $A\in\mathscr{G}$, *and* $\mu_\infty^{\rm ren}$ *is independent of* f.

In Theorem 3.1 we do not know the explicit form of $\mu_\infty^{\rm ren}$ but we see that

$$\mu_\infty^{\rm ren}(A) = e^{2Es}\int_{\mathbb{R}^3} dx \mathbb{E}^x\left[\mathbb{1}_A\left(\Psi_{\rm g}(B_{-s}), {\rm J}_s^{\rm ren}\Psi_{\rm g}(B_s)\right)\right] = \mu_\infty^{\rm ren}(A)$$

for $A\in\mathscr{F}_s$. Here ${\rm J}_s^{\rm ren} = e^{-\int_{-s}^s V(B_r)dr}e^{\frac{g^2}{2}S_0^{\rm ren}}e^{a_M^*(U_s)}e^{-2TH_{\rm f}}e^{a_M(\tilde U_s)}$.

3.2 Applications

By using the measure $\mu_\infty^{\rm ren}$ constructed in the previous section we can also express $(\Psi_{\rm g}, O\Psi_{\rm g})$. In order to factorize $e^{-tH_{\rm f}}$ we need an extra Hilbert space H_E in addition to H_M, and define $j_s: H_M\to H_E$ such that $j_s^*j_t = e^{-|s-t|\hat\omega}$, by which we can construct ${\rm J}_t = \Gamma(j_t)$ and it satisfies ${\rm J}_s^*{\rm J}_t = e^{-|s-t|H_{\rm f}}$. In a similar way we can construct a functional integral representation of $(e^{-TH}F, e^{-\beta d\Gamma(\rho)}e^{-TH}F)$, where ρ is a non-negative measurable function. In order to have a functional integral representation of $(e^{-TH}F, e^{-\beta d\Gamma(\rho)}e^{-TH}F)$, we prepare an extra Hilbert space H_ρ to factorize $e^{-\beta d\Gamma(\rho)}$. We set $H_\rho = L^2(\mathbb{R}^{3+2})$ and the Fourier transform of H_ρ is denoted by $\hat H_\rho = FL^2(\mathbb{R}^{3+2})$, where F denotes the Fourier transform on H_ρ. The scalar product on $\hat H_\rho$ (resp. H_ρ) is denoted by $(\cdot,\cdot)_{\hat\rho}$ (resp. $(\cdot,\cdot)_\rho$). Define a family of Gaussian random variables $(\phi_\rho(f), f\in L^2_{\rm real}(\mathbb{R}^{3+2}))$ on a probability space $({\rm Q}_\rho,\Sigma_\rho,\mu_\rho)$ indexed by $L^2_{\rm real}(\mathbb{R}^{3+2})$. For $\hat f\in\hat H_\rho$ the variables of $\hat f$ is denoted by $(k,k_0,k_1)\in\mathbb{R}^3\times\mathbb{R}\times\mathbb{R}$. Define a family of isometries $\xi_s: H_E\to H_\rho$ by

$$\widehat{\xi_s f}(k,k_0,k_1) = \frac{e^{-isk_1}}{\sqrt{\pi}}\frac{1}{\sqrt{\omega(k)^2+|k_0|^2}}\sqrt{\frac{\rho(k)}{\rho(k)^2+|k_1|^2}}\hat f(k,k_0).$$

It follows that $\xi_s^*\xi_t = e^{-|s-t|\rho(-i\nabla)\otimes\mathbb{1}}$ for $s,t\in\mathbb{R}$. Here we used identification $L^2(\mathbb{R}^{d+1})\cong L^2(\mathbb{R}^3)\otimes L^2(\mathbb{R})$. We define a family of second quantizations Ξ_s by $\Xi_s = \Gamma(\xi_s): L^2({\rm Q})\to L^2(Q_\rho)$. Let $\hat\rho = \rho(-i\nabla)$. We can see that Ξ_s is isometry for each s and furthermore it factorize $e^{-td\Gamma(\hat\rho\otimes\mathbb{1})}$. It follows that $\Xi_s^*\Xi_t = e^{-|s-t|d\Gamma(\hat\rho\otimes\mathbb{1})}$ for $s,t\in\mathbb{R}$, and the intertwining property ${\rm J}_te^{-sd\Gamma(\hat\rho)} = e^{-sd\Gamma(\hat\rho\otimes\mathbb{1})}{\rm J}_t$ follows. Using these facts we can have the theorem below. Suppose Assumption 1.1. Let ρ be a positive function on $\mathbb{R}^3$. Let $F,G\in L^2({\rm Q})$ and $\beta>0$. Then it follows that

$$\begin{aligned}&(e^{-TH}F, e^{-\beta d\Gamma(\hat\rho)}e^{-TH}G)\\ &= \int_{\mathbb{R}^3} dx\mathbb{E}^x\left[e^{-\int_{-T}^T V(B_s)ds}(\Xi_0{\rm J}_{-T}F(B_{-T}), e^{-\phi_\rho(K_T^\rho)}\Xi_\beta{\rm J}_TG(B_T))_{L^2({\rm Q}_\rho)}\right],\end{aligned}\tag{3.2}$$

where

$$K_T^\rho = \int_{-T}^0 \xi_0 j_s\varphi(\cdot - B_s)ds + \int_0^T \xi_\beta j_s\varphi(\cdot - B_s)ds.$$

Lemma 3.2 *Suppose Assumption 1.1. Let ρ be a positive function on $\mathbb{R}^3$. Let $f_T = e^{-TH} f \otimes 1\!\!1$. Then it follows that*

$$\frac{(f_T, e^{-\beta d\Gamma(\hat{\rho})} f_T)}{\|f_T\|^2} = \mathbb{E}_{\mu_T^{\rm ren}} \left[e^{-\int_{-T}^0 ds \int_0^T dt W_\beta(B_s - B_t, s-t)} \right],$$

where $W_\beta(x,t) = \int_{\mathbb{R}^3} \frac{|\hat{\varphi}(k)|^2}{2\omega(k)} e^{-|t|\omega(k)} e^{-ikx} (1 - e^{-\beta\rho(k)}) dk$.

PROOF. We have

$$(f_T, e^{-\beta d\Gamma(\hat{\rho})} f_T) = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\int_{-T}^T V(B_s) ds} (1\!\!1, e^{-\phi_\rho(K_T^\rho)} 1\!\!1)_{L^2(Q_\rho)} \right]. \tag{3.3}$$

We can compute as $(1\!\!1, e^{-\phi_\rho(K_T^\rho)} 1\!\!1) = e^{\frac{1}{2}\|K_T^\rho\|_\rho^2}$ and

$$\begin{aligned} \|K_T^\rho\|_\rho^2 = & \left\| \int_{-T}^0 \xi_0 j_s \varphi(\cdot - B_s) ds \right\|^2 + \left\| \int_0^T \xi_\beta j_s \varphi(\cdot - B_s) ds \right\|^2 \\ & + 2\Re \left(\int_{-T}^0 \xi_0 j_s \varphi(\cdot - B_s) ds, \int_0^T \xi_\beta j_s \varphi(\cdot - B_s) ds \right). \end{aligned}$$

Since $\widehat{\xi_0^* \xi_\beta f}(k, k_0) = e^{-\beta\rho(k)} \hat{f}(k, k_0)$, we have

$$\|K_T^\rho\|_\rho^2 = \int_{-T}^T ds \int_{-T}^T dt W(B_s - B_t, s-t) - 2 \int_{-T}^0 ds \int_0^T dt W_\beta(B_s - B_t, s-t).$$

Then

$$\begin{aligned} \frac{(f_T, e^{-\beta d\Gamma(\rho)} f_T)}{\|f_T\|^2} &= \frac{\int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\int_{-T}^T V(B_s) ds} e^{\frac{1}{2}\int_{-T}^T ds \int_{-T}^T dt W} e^{-\int_{-T}^0 ds \int_0^T dt W_\beta} \right]}{\int_{\mathbb{R}^3} dx \mathbb{E}^x \left[e^{-\int_{-T}^T V(B_s) ds} e^{\frac{1}{2}\int_{-T}^T ds \int_{-T}^T dt W} \right]} \\ &= \mathbb{E}_{\mu_T^{\rm ren}} \left[e^{-\int_{-T}^0 ds \int_0^T dt W_\beta(B_s - B_t, s-t)} \right] \end{aligned}$$

and the lemma follows. $\square$

Let $f \in L^2(\mathbb{R}^3)$ and $T > 0$. We define $f_T = e^{-TH_{\rm ren}} f \otimes 1\!\!1$ and $f_T^\varepsilon = e^{-TH_\varepsilon} f \otimes 1\!\!1$. We define Λ-truncated number operator N_Λ by $d\Gamma(1\!\!1_{|k|<\Lambda})$ which is formally written as

$$N_\Lambda = \int_{|k|<\Lambda} a_M^*(k) a_M(k) dk$$

and N_Λ counts the number of bosons with momentum smaller than Λ. We have

$$(f_T, e^{-\beta N_\Lambda} f_T) = \int_{\mathbb{R}^3} dx \mathbb{E}^x \left[f(B_{-T}) f(B_T) e^{-\int_{-T}^T V(B_s) ds} e^{-(1-e^{-\beta}) \int_{-T}^0 ds \int_0^T dt W_\Lambda(B_s - B_t, s-t)} e^{S_0^{\rm ren}} \right]. \tag{3.4}$$

Here $W_\Lambda(X,t) = \int_{|k|<\Lambda} \frac{e^{-|s-t|\omega(k)}}{\omega(k)} e^{-ik(B_s - B_t)} dk$.

Corollary 3.3 (Super-exponential decay [13]) *Let $\beta \in \mathbb{C}$ and suppose Assumption 1.1. Then $\Psi_g \in D(e^{-\beta N})$ and it follows that*

$$(\Psi_g, e^{-\beta N_\Lambda}\Psi_g) = \mathbb{E}_{\mu_\infty^{\rm ren}}\left[e^{-(1-e^{-\beta})\int_{-T}^0 ds \int_0^T dt W_\Lambda(B_s-B_t, s-t)}\right]. \tag{3.5}$$

In a similar manner to the proof of the super-exponential decay of Ψ_g we can also show a Gaussian domination of the ground state Ψ_g by the path measure $\mu_\infty^{\rm ren}$. We only mention the statement.

Corollary 3.4 (Gaussian dominations [13]) *Let $\hat{g}/\sqrt{\omega} \in L^2(\mathbb{R}^3)$, $\hat{g}/\omega^2 \in L^1(\mathbb{R}^3)$ and $\beta < 1/\|\hat{g}/\sqrt{\omega}\|^2$. Suppose Assumption 1.1. Then $\Psi_g \in D(e^{(\beta/2)\phi(g)^2})$ and*

$$\|e^{(\beta/2)\phi(g)^2}\Psi_g\|^2 = \frac{1}{\sqrt{1-\beta\|\hat{g}/\sqrt{\omega}\|^2}}\mathbb{E}_{\mu_\infty^{\rm ren}}\left[e^{\frac{\beta K(g)^2}{1-\beta\|\hat{g}/\sqrt{\omega}\|^2}}\right], \tag{3.6}$$

where $K(g)$ denotes the random variable defined by

$$K(g) = \frac{1}{2}\int_{-\infty}^{\infty} dr \int_{\kappa \geq |k|} dk \frac{e^{-|r|\omega(k)}\hat{g}(k)e^{-ikB_r}}{\omega(k)}.$$

In particular $\lim_{\beta \to \|\hat{g}/\sqrt{\omega}\|^{-2}} \|e^{(\beta/2)\phi(g)^2}\Psi_g\| = \infty$.

A Boson Fock space

In this appendix we quickly review boson Fock space for reader's convenient. Let $\mathscr{W}$ be a separable Hilbert space over $\mathbb{C}$. Consider the operation $\otimes_s^n$ of n-fold symmetric tensor product defined through the symmetrization operator

$$S_n(f_1 \otimes \cdots \otimes f_n) = \frac{1}{n!}\sum_{\pi \in \wp_n} f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}, \quad n \geq 1,$$

where $f_1, \ldots, f_n \in \mathscr{W}$ and $\wp_n$ denotes the permutation group of order n. Define $\mathscr{F}^{(n)} = S_n(\otimes^n \mathscr{W})$, where $\otimes_s^0 \mathscr{W} = \mathbb{C}$. The space $\mathscr{F} = \oplus_{n=0}^\infty \mathscr{F}^{(n)}$, where $\oplus_{n=0}^\infty$ is understood to be completed direct sum, is called boson Fock space over $\mathscr{W}$. $\mathscr{F}$ is a Hilbert space endowed with the scalar product $(\Psi, \Phi)_{\mathscr{F}} = \sum_{n=0}^\infty (\Psi^{(n)}, \Phi^{(n)})_{\mathscr{F}^{(n)}}$. The vector $\Omega = (1, 0, 0, \ldots)$ is called Fock vacuum. There are two fundamental boson particle operators, the creation operator denoted by $a^*(f)$ and the annihilation operator by $a(f)$ defined by

$$(a^*(f)\Psi)^{(0)} = 0, \quad (a^*(f)\Psi)^{(n)} = \sqrt{n}S_n(f \otimes \Psi^{(n-1)}), n \geq 1$$

with domain $D(a^*(f)) = \left\{(\Psi^{(n)})_{n\geq 0} \in \mathscr{F} \;\middle|\; \sum_{n=1}^\infty n\|S_n(f \otimes \Psi^{(n-1)})\|^2_{\mathscr{F}^{(n)}} < \infty\right\}$ and $a(f) = (a^*(\bar{f}))^*$. It is known that

$$\mathscr{F}_{\rm fin} = \{(\Psi^{(n)})_{n\geq 0} \in \mathscr{F} | \ \Psi^{(m)} = 0 \text{ for all } m \geq M \text{ with some } M\}$$

is dense. The field operators a, a^* leave $\mathscr{F}_{\text{fin}}$ invariant and satisfy the canonical commutation relations

$$[a(f), a^*(g)] = (\bar{f}, g)1, \quad [a(f), a(g)] = 0, \quad [a^*(f), a^*(g)] = 0$$

on $\mathscr{F}_{\text{fin}}$. Given a bounded operator T on $\mathscr{W}$, the second quantization of T is the operator $\Gamma(T)$ on $\mathscr{F}$ defined by $\Gamma(T) = \oplus_{n=0}^{\infty}(\otimes^n T)$. Here it is understood that $\otimes^0 T = \mathbb{1}$. For a contraction operator T, the second quantization $\Gamma(T)$ is also a contraction on $\mathscr{F}$. For a self-adjoint operator h on $\mathscr{W}$, $\{\Gamma(e^{ith}) : t \in \mathbb{R}\}$ is a strongly continuous one-parameter unitary group on $\mathscr{F}$. Then by the Stone theorem there exists a unique self-adjoint operator $d\Gamma(h)$ on $\mathscr{F}$ such that $\Gamma(e^{ith}) = e^{itd\Gamma(h)}$. The operator $d\Gamma(h)$ is called the second quantization of h. Thus the action of $d\Gamma(h)$ is given by $d\Gamma(h)\Omega = 0$ and

$$d\Gamma(h)a^*(f_1)\cdots a^*(f_n)\Omega = \sum_{j=1}^{n} a^*(f_1)\cdots a^*(hf_j)\cdots a^*(f_n)\Omega.$$

We use the following facts below. The superscript in $a^\sharp$ indicates that either of the creation or annihilation operators is meant.

Proposition A.1 (Relative bounds) *Let h be a positive self-adjoint operator, and $f \in D(h^{-1/2})$, $\Psi \in D(d\Gamma(h)^{1/2})$. Then $\Psi \in D(a^\sharp(f))$ and*

$$\|a(f)\Psi\| \leq \|h^{-1/2}f\|\|d\Gamma(h)^{1/2}\Psi\|, \tag{A.1}$$

$$\|a^*(f)\Psi\| \leq \|h^{-1/2}f\|\|d\Gamma(h)^{1/2}\Psi\| + \|f\|\|\Psi\|. \tag{A.2}$$

In particular, $D(d\Gamma(h)^{1/2}) \subset D(a^\sharp(f))$, whenever $f \in D(h^{-1/2})$.

To obtain the commutation relations between $a^\sharp(f)$ and $d\Gamma(h)$, suppose that $f \in D(h^{-1/2}) \cap D(h)$. Then

$$[d\Gamma(h), a^*(f)]\Psi = a^*(hf)\Psi, \quad [d\Gamma(h), a(f)]\Psi = -a(hf)\Psi, \tag{A.3}$$

for $\Psi \in D(d\Gamma(h)^{3/2}) \cap \mathscr{F}_{\text{fin}}$.

The Segal field $\Phi(f)$ on the boson Fock space $\mathscr{F}$ is defined by

$$\Phi(f) = \frac{1}{\sqrt{2}}(a^*(\bar{f}) + a(f)), \quad f \in \mathscr{W},$$

and its conjugate momentum by

$$\Pi(f) = \frac{i}{\sqrt{2}}(a^*(\bar{f}) - a(f)), \quad f \in \mathscr{W}.$$

Here $\bar{f}$ denotes the complex conjugate of f. It is straightforward to check that $[\Phi(f), \Pi(g)] = i\mathrm{Re}(f, g)$, $[\Phi(f), \Phi(g)] = i\mathrm{Im}(f, g)$ and $[\Pi(f), \Pi(g)] = i\mathrm{Im}(f, g)$. In particular, for real-valued f and g the canonical commutation relationsbecome

$$[\Phi(f), \Pi(g)] = i(f, g), \quad [\Phi(f), \Phi(g)] = [\Pi(f), \Pi(g)] = 0.$$

B Exponential of annihilation operators and creation operators

In this appendix we discuss exponent of annihilation operators and creation operators. We learned this in [8, 21]. Let $f \in \mathscr{W}$ and we define the exponential of creation operators F_f by

$$F_f = \sum_{n=0}^{\infty} \frac{1}{n!} a^*(f)^n$$

and $D(F_f) = \left\{ \Phi \in \cap_{n=1}^{\infty} D(a^*(f)^n) \middle| \sum_{n=0}^{\infty} \frac{1}{n!} \|a^*(f)^n \Phi\| < \infty \right\}$. Let $\Phi \in \mathscr{F}^{(m)}$. Thus we have

$$\|F_f \Phi\| \leq \|\Phi\| + \sum_{n=1}^{\infty} \frac{\sqrt{m+n-1} \cdots \sqrt{m}}{n!} \|f\|^n \|\Phi\| < \infty.$$

Then $\mathscr{F}_{\text{fin}} \subset D(F_f)$ follows. We also define the exponential of annihilation operators by

$$G_f = \sum_{n=0}^{\infty} \frac{1}{n!} a(f)^n$$

with the domain $D(G_f) = \left\{ \Phi \in \cap_{n=1}^{\infty} D(a(f)) \middle| \sum_{n=0}^{\infty} \frac{1}{n!} \|a(f)^n \Phi\| < \infty \right\}$. We simply write $F_f = e^{a^*(f)}$ and $G_f = e^{a(\bar{f})}$ whenever confusion may arise. Then we can see that $(e^{a^*(f)})^* \supset e^{a(\bar{f})}$ and this implies that $e^{a^*(f)}$ is closable. The closure of $e^{a^*(f)}$ is denoted by the same symbol. Similarly the closure of $e^{a(f)}$ is denoted by the same symbol. The vector defined by $C(f) = e^{a^*(f)}\Omega$ is called the coherent vector.

Proposition B.1 (Algebraic properties) *Let $f, g \in \mathscr{W}$ and P be a polynomial. Then*

$$(1)\ e^{a^*(g)} e^{a^*(f)} \Omega = e^{a^*(f+g)} \Omega,$$
$$(2)\ P(a(g)) e^{a^*(f)} \Omega = P((\bar{g}, f)) e^{a^*(f)} \Omega,$$
$$(3)\ e^{a(g)} e^{a^*(f)} \Omega = e^{(\bar{g}, f)} e^{a^*(f)} \Omega.$$

PROOF. It can be seen that $C(g) \in \cap_{n=0}^{\infty} D(a^*(f)^n)$ and that

$$\left\| C(f+g) - \sum_{n=0}^{M} \frac{a^*(f)^n}{n!} C(g) \right\| \to 0$$

as $M \to \infty$. Then $C(f) \in D(e^{a^*(g)})$ and $e^{a^*(f)} C(g) = C(f+g)$ follow by the closedness of $e^{a^*(f)}$. We can also see that $a(g) e^{a^*(f)} = e^{a^*(f)} a(g) + (\bar{g}, f) e^{a^*(f)}$ on $\mathscr{F}_{\text{fin}}$. In particular we have $a(g) C(f) = (\bar{g}, f) C(f)$ and recursively we can get (2) for any polynomial P. Then (1) and (2) are proven. Since

$$\sum_{n=0}^{M} \frac{a(g)^n}{n!} e^{a^*(f)} \Omega = \sum_{n=0}^{M} \frac{(\bar{g}, f)^n}{n!} e^{a^*(f)} \Omega$$

and the right-hand side converges to $e^{(\bar{g},f)}e^{a^*(f)}\Omega$ as $M \to \infty$. Then (3) follows from the closedness of $e^{a(g)}$. □

Next we see the continuity of map $\mathscr{W} \ni f \mapsto e^{a^\sharp(f)}\Phi \in \mathscr{F}$.

Proposition B.2 (Continuity) *Let $\Phi \in \mathscr{F}_{\mathrm{fin}}$. Then the map $\mathscr{W} \ni f \mapsto e^{a^\sharp(f)}\Phi \in \mathscr{F}$ is continuous.*

PROOF. Suppose that $f_m \to f$ strongly in $\mathscr{W}$ as $m \to \infty$. Let $\Phi \in \mathscr{F}^{(N)}$. Then $e^{a(f_m)}\Phi = \sum_{n=0}^N \frac{a(f_m)^n}{n!}\Phi$. Since $\sum_{n=0}^N \frac{a(f_m)^n}{n!}\Phi \to \sum_{n=0}^N \frac{a(f)^n}{n!}\Phi$ as $m \to \infty$,

$$\lim_{m\to\infty} e^{a(f_m)}\Phi = e^{a(f)}\Phi$$

follows. Next we consider the continuity of $f \mapsto e^{a^*(f)}\Phi$. Let $\varepsilon > 0$ and $\|f_m - f\| < \varepsilon$ for sufficiently large m. We fix $c > 0$ such that $\|f_m\| < c$ for all m. Then we can see that

$$\begin{aligned}
&\left\|e^{a^*(f_m)}\Phi - e^{a^*(f)}\Phi\right\| \\
&\leq \sum_{n=1}^\infty \frac{\sqrt{(N+n-1)}\cdots\sqrt{N}}{n!}\sum_{k=0}^{n-1}\|f_m\|^k\|f_m - f\|\|f\|^{n-k-1}\|\Phi\| \\
&\leq \varepsilon\sum_{n=1}^\infty \frac{c^{n-1}\sqrt{(N+n-1)}\cdots\sqrt{N}}{(n-1)!}\|\Phi\|.
\end{aligned}$$

Then $\left\|e^{a^*(f_n)}\Phi - e^{a^*(f)}\Phi\right\| \to 0$ as $n \to \infty$ follows, and the proof is complete. □

Proposition B.3 (Differentiability) *Let h be a self-adjoint operator in $\mathscr{W}$, $f \in D(h)$ and $\Phi \in \mathscr{F}_{\mathrm{fin}}$. Then the map $\mathbb{R} \ni t \mapsto e^{a^\sharp(e^{ith}f)}\Phi \in \mathscr{F}$ is strongly differentiable with*

$$\frac{d}{dt}e^{a^\sharp(e^{ith}f)}\Phi = a^\sharp(ihe^{ith}f)e^{a^\sharp(e^{ith}f)}\Phi.$$

PROOF. Let $\varepsilon \in \mathbb{R}$. Suppose that $\Phi \in \mathscr{F}^{(N)}$. We show only the case of $a^*(f)$. The proof for $a(f)$ is similar. We set $a^*(e^{i(t+\varepsilon)h}f) = a^*(\varepsilon)$ for notational simplicity. We have

$$\begin{aligned}
&\frac{1}{\varepsilon}\left(e^{a^*(\varepsilon)} - e^{a^*(0)}\right)\Phi - a^*(ihe^{ith}f)e^{a^*(0)}\Phi \\
&= a^*\left(\left(\frac{e^{i\varepsilon h}-1}{\varepsilon} - ih\right)e^{ith}f\right)\sum_{n=1}^\infty\frac{1}{n!}\sum_{k=0}^{n-1}a^*(\varepsilon)^{n-k-1}a^*(0)^k\Phi \\
&+ a^*(ihe^{ith}f)\left(\sum_{n=1}^\infty\frac{1}{n!}\sum_{k=0}^{n-1}a^*(\varepsilon)^{n-k-1}a^*(0)^k - \sum_{n=0}^\infty\frac{1}{n!}a^*(0)^n\right)\Phi = A + B.
\end{aligned}$$

We see that

$$\|A\| \leq \left\|\frac{e^{i\varepsilon h}-1}{\varepsilon}f - ihf\right\|\sum_{n=1}^\infty\frac{\sqrt{N+n-1}\cdots\sqrt{N}}{n!}\|f\|^{n-1}\|\Phi\|$$

and

$$\|B\| \leq \|hf\|\|(e^{i\varepsilon h}-1)f\| \sum_{n=1}^{\infty} \frac{\sqrt{N+n-1}\cdots\sqrt{N}}{(n-1)!} \frac{n(n-1)}{2} \|f\|^{n-2}\|\Phi\|.$$

Hence $\lim_{\varepsilon\to 0}\|A\| = 0$ and $\lim_{\varepsilon\to 0}\|B\| = 0$ follow. Then the proposition follows. □
We discuss relationships between $e^{a^*(f)}$ and the second quantization $\Gamma(T)$. Let h be a self-adjoint operator in $\mathscr{W}$ and we define

$$\mathscr{D} = \text{L.H.}\{a^*(f_1)\cdots a^*(f_n)\Omega, \Omega |\, f_j \in \mathscr{W}, j=1,\ldots,n, n\geq 1\},$$
$$\mathscr{D}_h = \text{L.H.}\{a^*(f_1)\cdots a^*(f_n)\Omega, \Omega |\, f_j \in D(h), j=1,\ldots,n, n\geq 1\}.$$

Proposition B.4 (Intertwining properties) (1) *Let T be a contraction operator on $\mathscr{W}$. Then it follows that on $\mathscr{F}_{\text{fin}}$*

$$\Gamma(T)e^{a^*(f)} = e^{a^*(Tf)}\Gamma(T),$$
$$\Gamma(T)e^{a(\overline{T^*f})} = e^{a(f)}\Gamma(T).$$

(2) *Let h be self-adjoint in $\mathscr{W}$ and $f \in D(h)$. Then it follows that on $\mathscr{D}_h$*

$$d\Gamma(h)e^{a^*(f)} = a^*(hf)e^{a^*(f)} + e^{a^*(f)}d\Gamma(h),$$
$$d\Gamma(h)e^{a(f)} = -a(\overline{hf})e^{a^*(f)} + e^{a(f)}d\Gamma(h).$$

PROOF. Let $\Phi = \prod_{j=1}^m a^*(g_j)\Omega \in \mathscr{D}$. Then $\Phi \in D(e^{a^*(f)})$ and $\Gamma(T)\mathscr{D} \subset \mathscr{D}$. We have

$$\Gamma(T)e^{a^*(f)}\Phi = \sum_{n=0}^{\infty} \frac{a^*(Tf)^n}{n!} \prod_{j=1}^m a^*(Tg_j)\Omega = e^{a^*(Tf)}\Gamma(T)\Phi.$$

Then the first statement of (1) is proven on $\mathscr{D}$. Let $\Phi \in \mathscr{F}^{(N)}$. Then there exists $\Phi_n \in \mathscr{D}$ such that $\Phi_n \in \mathscr{F}^{(N)}$ and $\|\Phi_n - \Phi\| \to 0$ as $n \to \infty$. We can also see that $e^{a^*(f)}\Phi_n \to e^{a^*(f)}\Phi$ as $n\to\infty$. Then the limit of $\Gamma(T)e^{a^*(f)}\Phi_n = e^{a^*(Tf)}\Gamma(T)\Phi_n$ implies that $\Gamma(T)e^{a^*(f)}\Phi = e^{a^*(Tf)}\Gamma(T)\Phi$. Then the first statement of (1) is proven on $\mathscr{F}_{\text{fin}}$. The second statement of (1) can be show by taking the adjoint of both sides of the first statement. Next let us prove (2). Let $\Phi \in \mathscr{D}_h$ and $T = e^{ith}$. Then

$$\Gamma(e^{ith})e^{a^*(f)}\Phi = e^{a^*(e^{ith}f)}\Gamma(e^{ith})\Phi. \tag{B.1}$$

In a similar way to Proposition B.3 it can be seen that the right-hand side above is differentiable with respect to t at $t=0$, the result is

$$\frac{d}{dt}e^{a^*(e^{ith}f)}\Gamma(e^{ith})\Phi = ia^*(hf)e^{a^*(f)}\Phi + ie^{a^*(f)}d\Gamma(h)\Phi. \tag{B.2}$$

This implies that the left-hand side of (B.1) is also differentiable with respect to t, and thus $e^{a^*(f)}\Phi \in D(d\Gamma(h))$ (see [25, Theorem VIII.7 (d)]) and the derivative of the left-hand side at $t = 0$ is

$$\frac{d}{dt}\Gamma(e^{ith})e^{a^*(f)}\Phi = id\Gamma(h)e^{a^*(f)}\Phi. \tag{B.3}$$

Comparing (B.2) and (B.3), we can conclude the first statement of (2). The second statement can be show by taking the adjoint of both sides of the first statement. □

Finally we discuss the representation of $e^{\Phi(f)}$ in terms of both $e^{a^*(f)}$ and $e^{a(f)}$. Let $\mathscr{D}_{\mathrm{b}} = \mathrm{L.H.}\{C(g), \Phi |\, g, \Phi \in \mathscr{F}_{\mathrm{fin}}\}$.

Proposition B.5 (Baker-Campbell-Hausdorff formula) *Let $f \in \mathscr{W}$. Then it follows that on on $\mathscr{D}_{\mathrm{b}}$*

$$e^{a^*(f)+a(\bar{f})} = e^{a^*(f)}e^{a(\bar{f})}e^{\frac{1}{2}\|f\|^2}. \tag{B.4}$$

PROOF. We shall show (B.4) on $C(g)$. The proof of (B.4) on $\mathscr{F}_{\mathrm{fin}}$ is similar. We have

$$e^{a^*(f)}e^{a(\bar{f})}C(g) = e^{(f,g)}C(f+g). \tag{B.5}$$

Let $\psi(f) = a^*(f) + a(\bar{f})$. Then $\psi(f)$ is self-adjoint and it holds that

$$e^{\psi(f)} = \sum_{n=0}^{\infty}\frac{\psi(f)^n}{n!} \tag{B.6}$$

on the finite particle subspace. Let $C_m(g) = \sum_{n=0}^{m}\frac{a^*(g)^n}{n!}\Omega$. By using the expansion (B.6) we can compute as

$$e^{\psi(f)}C_m(g) = e^{\psi(f)}\sum_{n=0}^{m}\frac{a^*(g)^n}{n!}\Omega = \sum_{n=0}^{m}\frac{(a^*(g)+(f,g))^n}{n!}e^{\psi(f)}\Omega.$$

Together with $e^{\psi(f)}\Omega = e^{\frac{1}{2}\|f\|^2}e^{a^*(f)}\Omega$ we see that

$$e^{\psi(f)}C_m(g) = \sum_{n=0}^{m}\frac{(a^*(g)+(f,g))^n}{n!}e^{\frac{1}{2}\|f\|^2}e^{a^*(f)}\Omega.$$

Then we have

$$e^{\psi(f)}C(g) = e^{(f,g)}e^{\frac{1}{2}\|f\|^2}C(f+g). \tag{B.7}$$

By (B.5) and (B.7) the proposition follows. □

Now we shall show that $e^{a^*(g)}e^{-tH_\mathrm{f}}$ for $t > 0$ is bounded for $g \in L^2(\mathbb{R}^3)$. In order to see this we evaluate $\| \prod_{j=1}^m a^*(f_j)\Phi\|$. We have a general formula. Let $\Phi \in \mathscr{F}_{\mathrm{fin}}$ and $f_i, g_j \in \mathscr{W}$ for $i, j = 1, ..., m$. Then

$$\prod_{j=1}^n a(\bar{g}_j)\prod_{j=1}^n a^*(f_j)\Phi$$
$$= \sum_{m=0}^n \sum_{C_m \ni A} \sum_{C_{n-m}\ni B} \sum_{\sigma: B\to B, bijection} \left(\prod_{l\in A^c}(g_l, f_{\sigma(\tau(l))})\right)\left(\prod_{p\in B^c} a^*(f_p)\right)\left(\prod_{q\in A} a(\bar{g}_q)\right)\Phi. \tag{B.8}$$

Here $C_k = \{A \subset \{1, ..., n\} | \# A = k\}$ and $C_0 = \emptyset$, τ is an identification map between A^c and B, finally $\sum_{\sigma:B\to B, bijection}$ is understood to take all bijections from B to itself. From now on we consider the case where $\mathscr{W} = L^2(\mathbb{R}^3)$. Then ω is the multiplication operator by ω in $L^2(\mathbb{R}^3)$.

Proposition B.6 (Boundedness) *Let $t > 0$ and $f \in D(1/\sqrt{\omega})$. Then $e^{a^*(f)}e^{-tH_\mathrm{f}}$ and $\overline{e^{-tH_\mathrm{f}}e^{a(f)}}$ are bounded.*

PROOF. Let $\Psi \in \cap_{n=1}^\infty D(H_\mathrm{f}^n)$. Suppose that $t < 1$. Let $f_i, g_j \in D(1/\sqrt{\omega})$ for $i, j = 1, ..., n$ and $\Phi \in D(H_\mathrm{f}^{n/2})$. Then by (B.8) we have

$$\left|\left(\prod_{j=1}^n a^*(g_j)\Phi, \prod_{j=1}^n a^*(f_j)\Phi\right)\right| \le n!2^n \left(\prod_{l=1}^n \|f_l\|_\omega \|g_l\|_\omega\right) \sum_{m=0}^n \frac{1}{m!}\|H_\mathrm{f}^{m/2}\Phi\|^2,$$

where $\|f\|_\omega = \|f\| + \|f/\sqrt{\omega}\|$. In particular we have the bound

$$\left\|\prod_{j=1}^n a^*(f_j)\Phi\right\| \le \sqrt{n!}2^{n/2}\left(\prod_{l=1}^n \|f_l\|_\omega\right)\left(\sum_{m=0}^n \frac{1}{m!}\|H_\mathrm{f}^{m/2}\Phi\|^2\right)^{1/2}.$$

Then for any $s < 1$ we have

$$\left\|\prod_{j=1}^n a^*(f_j)\Psi\right\| \le \sqrt{n!}2^{n/2}s^{-n/2}\left(\prod_{l=1}^n \|f_l\|_\omega\right)\left(\sum_{m=0}^n \frac{1}{m!}\|(sH_\mathrm{f})^{m/2}\Psi\|^2\right)^{1/2}.$$

Hence we observe that for $\Phi \in \mathscr{F}$,

$$\left\|\sum_{n=0}^m \frac{1}{n!}a^*(f)^n e^{-tH_\mathrm{f}}\Phi\right\| \le \sum_{n=0}^m \frac{1}{\sqrt{n!}}2^{n/2}s^{-n/2}\|f\|_\omega^n\left(\sum_{k=0}^n \frac{1}{k!}\|(sH_\mathrm{f})^{k/2}e^{-tH_\mathrm{f}}\Phi\|^2\right)^{1/2}.$$

We can see that $\{\sum_{n=0}^m \frac{1}{n!}a^*(f)^n e^{-tH_\mathrm{f}}\Phi\}_{m=0}^\infty$ is a Cauchy sequence. Hence $e^{-tH_\mathrm{f}}\Phi \in D(e^{a^*(f)})$ and as $m \to \infty$ on both sides above we have

$$\|e^{a^*(f)}e^{-tH_\mathrm{f}}\Phi\| \le A(f,s)\|e^{-\frac{1}{2}(t-s)H_\mathrm{f}}\Phi\|,$$

where $A(f, s) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} 2^{n/2} s^{-n/2} \|f\|_\omega^n$. Choosing s such that $s \le t$, we can see that $\|e^{-\frac{1}{2}(t-s)H_{\mathrm{f}}}\Phi\| \le \|\Phi\|$ and $e^{a^*(f)}e^{-tH_{\mathrm{f}}}$ for $t < 1$ is bounded. Suppose $1 \le t$. Choosing $s = 1$ in the above discussion, we have

$$\|e^{a^*(f)}e^{-tH_{\mathrm{f}}}\Phi\| \le A(f,1)\|e^{-\frac{1}{2}(t-1)H_{\mathrm{f}}}\Phi\| \le A(f,1)\|\Phi\|.$$

Thus $e^{a^*(f)}e^{-tH_{\mathrm{f}}}$ for $t \ge 1$ is bounded. Finally since $\left(e^{-tH_{\mathrm{f}}}e^{a(f)}\right)^* \supset e^{a^*(\bar{f})}e^{-tH_{\mathrm{f}}}$, the second statement follows. Then the proposition follows. □

We can also estimate the bound of $\|e^{a^*(f)}e^{-tH_{\mathrm{f}}}\|$ and $\|e^{a^*(f)}e^{-tN}\|$, which can be derived from the estimates in the proof of Proposition B.6.

Proposition B.7 (Bound) *Let $f \in D(1/\sqrt{\omega})$. Then*

$$\begin{aligned} \|e^{a^*(f)}e^{-tH_{\mathrm{f}}}\| &\le \sqrt{2}e^{4/s\|f\|_\omega^2}\|e^{-\frac{1}{2}(t-s)H_{\mathrm{f}}}\|, \quad 0 < s < t \le 1, \\ \|e^{a^*(f)}e^{-tH_{\mathrm{f}}}\| &\le \sqrt{2}e^{4\|f\|_\omega^2}\|e^{-\frac{1}{2}(t-1)H_{\mathrm{f}}}\|, \quad 1 < t. \end{aligned}$$

In particular we have

$$\begin{aligned} \|e^{a^*(f)}e^{-2tH_{\mathrm{f}}}e^{a(f)}\| &\le 2e^{8/s\|f\|_\omega^2}, \quad 0 < s < t \le 1, \\ \|e^{a^*(f)}e^{-2tH_{\mathrm{f}}}e^{a(f)}\| &\le 2e^{8\|f\|_\omega^2}, \quad 1 < t. \end{aligned}$$

PROOF. We can estimate $A(f,s)$ as $A(f,s) \le \sqrt{2}e^{4/s\|f\|_\omega^2}$. Then the corollary follows from Proposition B.6. □

We have already seen the strong continuity of map $L^2(\mathbb{R}^3) \ni f \mapsto e^{a^\sharp(f)}\Phi$. We can furthermore prove the uniform continuity of map $f \mapsto e^{a^*(f)}e^{-tH_{\mathrm{f}}}$ for $t > 0$.

Proposition B.8 (Uniform continuity) *Let $f, g \in D(1/\sqrt{\omega})$. Then*

$$\begin{aligned} \|e^{a^*(f)}e^{-tH_{\mathrm{f}}} - e^{a^*(g)}e^{-tH_{\mathrm{f}}}\| &\le \sqrt{2}\|f-g\|_\omega e^{4/s(\|f\|_\omega+\|g\|_\omega+1)^2}, \quad 0 < s < t \le 1, \\ \|e^{a^*(f)}e^{-tH_{\mathrm{f}}} - e^{a^*(g)}e^{-tH_{\mathrm{f}}}\| &\le \sqrt{2}\|f-g\|_\omega e^{4(\|f\|_\omega+\|g\|_\omega+1)^2}, \quad 1 < t. \end{aligned} \tag{B.9}$$

In particular let $f, f_n \in D(1/\sqrt{\omega})$ for $n \ge 1$ such that $\|f - f_n\|_\omega \to 0$ as $n \to \infty$. Then $e^{a^*(f_n)}e^{-tH_{\mathrm{f}}}$ uniformly converges to $e^{a^*(f)}e^{-tH_{\mathrm{f}}}$ as $n \to \infty$.

PROOF. We can straightforwardly see that

$$\begin{aligned} &\|(a^*(f)^n - a^*(g)^n)\Psi\| \\ &\le \sqrt{n!}2^{n/2}s^{-n/2}(\|f\|_\omega + \|g\|_\omega + 1)^n \left(\sum_{m=0}^{n} \frac{1}{m!}\|(sH_{\mathrm{f}})^{m/2}\Psi\|^2\right)^{1/2} \|f-g\|_\omega. \end{aligned}$$

Hence (B.9) follows. □

C Fock space and Gaussian random variables

In this appendix we state the equivalence between $L^2(\mathrm{Q})$ and a boson Fock space $\mathscr{F}$. Let $\mathscr{F}$ be the boson Fock space over $\hat{H}_M$. Let $(\phi(f), f \in \mathscr{M})$ be the Gaussian random variable on a probability space (Q, Σ, μ) indexed by $f \in \mathscr{M}$. Note that $H_M = \mathscr{M}_{\mathbb{C}}$. Then there exists a unitary operator $U : L^2(\mathrm{Q}) \to \mathscr{F}$ such that

(1) $U\mathbb{1} = \Omega$,

(2) $U^{-1}\Phi(\hat{f})U = \phi(f)$, where $\Phi(\hat{f}) = \frac{1}{\sqrt{2}}(a_M^*(\hat{f}) + a_M(\tilde{\hat{f}}))$,

(3) $U^{-1}d\Gamma(\omega)U = d\Gamma(\hat{\omega})$.

Using this equivalence we can define the Nelson Hamiltonian both on $L^2(\mathbb{R}^3) \otimes \mathscr{F}$ and $L^2(\mathbb{R}^3) \otimes L^2(\mathrm{Q})$. In this paper for constructing a functional integral representation we adopt the Nelson Hamiltonian defined on $L^2(\mathbb{R}^3) \otimes L^2(\mathrm{Q})$.

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