数理解析研究所講究録
第2074巻 2018年 63-67

A DUAL FORM OF THE SHARP NASH INEQUALITY AND ITS WEIGHTED GENERALIZATION

Eric A. Carlen[1] and Elliott H. Lieb[2]

[1] Department of Mathematics, Hill Center,
Rutgers University, 110 Frelinghuysen Road Piscataway NJ 08854-8019 USA
[2] Departments of Mathematics and Physics, Jadwin Hall, Princeton University
Washington Rd., Princeton, NJ 08544.

September 15, 2017

Abstract

The well known duality between the Sobolev inequality and the Hardy-Littlewood-Sobolev inequality suggests that the Nash inequality should also have an interesting dual form. We provide one here. This dual inequality relates the L^2 norm to the infimal convolution of the L^∞ and H^{-1} norms. The computation of this infimal convolution is a minimization problem, which we solve explicitly, thus providing a new proof of the sharp Nash inequality itself. This proof, via duality, also yields the sharp form of some weighted generalizations of the Nash inequality and the dual of these weighted variants.

1 Introduction

The subject of this talk is an example of how Kato motivated others by asking good questions. The story starts with a letter from Kato to Eric Carlen and Michael Loss, in which he asks whether it is possible to compute the sharp constant in Nash's inequality [5].

Eric and Michael solved that problem in 1993 [2] and showed, surprisingly, that every optimal function has compact support. The unanswered question hanging in the air was *What is the dual of Nash's inequality?* We have a solution of this problem and the result is even more surprising – as one might expect.

Let us review the situation by starting with Sobolev's inequality.

---

[0]This is a summary of a talk given by Elliott Lieb on September 4, 2017 at the conference at the University of Tokyo in honor of the centenary of Tosio Kato. The full paper is at arXiv:1704.08720.

[1]Work partially supported by U.S. National Science Foundation grant DMS 1501007.

[2]Work partially supported by U.S. National Science Foundation grant PHY 1265118

2 From Sobolev to Nash

The Sobolev inequality in $\mathbb{R}^n$, ($n \geq 3$ *only*), (see [1, 7], DD) is $\|\nabla f\|_2 \geq S_n \|f\|_{2n/(n-2)}$.
This is an inequality between two (convex) integrals and has an unambiguous dual, which is the Hardy-Littlewood inequality (HLS) (see [4]) and which is valid for *all* n and $0 < \lambda < n$,

$$\int\int g(x)|x-y|^{-\lambda} g(y)dxdy \leq C_n(\lambda)\|g\|^2_{2n/(2n-\lambda)}.$$

The special case $\lambda = n-2$ is the dual of Sobolev, but we see that HLS covers many more cases. We learn here that it is sometimes useful to study duals because they can lead us to new mathematics. When $n=3$, Sobolev tells us about *kinetic energy*, while its dual, HLS, is the story of the *Coulomb potential* and 'potential theory', which has quite a different flavor.

Nash's inequality involves *three* integrals and is valid for all n.

$$\boxed{C_n \|\nabla f\|_2^{n/(n+2)} \ \|f\|_1^{2/(n+2)} \geq \|f\|_2.}$$

Carlen and Loss [2] found the sharp C_n and the optimizers, which always have compact support.

For $n \geq 3$, Nash's inequality can be derived from Sobolev's inequality (but with a bad constant) by using Hölder's inequality. Thus, *Nash is weaker than Sobolev* – but it is extremely useful for problems in which the L^1-norm is either conserved or monotone decreasing.

Kato was interested in the two-dimensional Navier Stokes equation in the vorticity formulation, which is just such a problem. Nash had applications to fluid dynamics in mind when he wrote his famous 1958 parabolic regularity paper in which his inequality first appeared. Many applications have been found in probability theory.

3 Our dual Nash inequality

$$\boxed{L_n \|g\|_2^{\frac{2n+4}{n+4}} \geq \inf_h \left\{ \tfrac{1}{2}\|(-\Delta)^{-1/2}(g-h)\|_2^2 + \|h\|_\infty \right\} .}$$

What this says is, given a function $g \in L^2(\mathbb{R}^n)$, try to minimize its Coulomb energy by subtracting another function h. The price to be paid, however, is the L^∞-norm of h.

There are three topics to be discussed:

(1.) Where does this funny inequality come from and what is its connection to Nash?
(2.) Does there exist a minimizing h for this new problem and what does it look like?
(3.) Does there exist an optimizing g (and h) that gives the smallest value of L_n? How is this g related to the optimizer for Nash?

4 Generalities about dual inequalities

Suppose we have two convex functionals, $A(f)$, $B(f)$ and $A(f) - B(f) \geq 0$, $\forall f$, as in the Sobolev inequality. We can then take the Legendre transforms:

$$A^*(g) := \sup_f \left\{\int fg - A(f)\right\}, \qquad B^*(g) := \sup_f \left\{\int fg - B(f)\right\}.$$

Let F be an (approximate) maximizer for $B^*(g)$, whence we have the dual inequality:

$$B^*(g) - A^*(g) \geq \int Fg - B(F) - \int Fg + A(F) \geq 0.$$

Thus, the dual of $A \geq B$ is $B^* \geq A^*$. Since A, B are convex, the 'dual of the dual' is the original inequality $A \geq B$.)

In the case of Nash, there are 3 functionals and the right side is *not* convex. Help! We must combine 2 of them into one convex functional, and this will lead us to the strange construction called *infimal convolution.* (see [6].)

5 Second law of thermodynamics and infimal convolution

Let systems A and B have energy dependent entropy functions $S_A(E)$ and $S_B(E)$. These functions are concave, of course. The systems are brought into equilibrium with total energy U. According to the *second law* they distribute the energy so that the *total entropy is maximized*. Thus

$$S_{AB}(U) = \sup_E \{S_A(U - E) + S_B(E)\}.$$

The amazing thing is this: Despite the *supremum*$_E$, the resulting S_{AB} is a concave function — as required by the second law. (For **convex functions** everything is reversed.)

The general theorem, (1 line proof!) of which this 'convolution' is a special case, is this:

If $F(X,Y)$ is a jointly concave function of X, Y then $\sup_Y F(X,Y)$ is concave!

Let us apply this to the product $\|\nabla f\|_2^{n/(n+2)} \|f\|_1^{2/(n+2)}$ of functions of f, that appear on the 'large side' of Nash. This product is **NOT** a convex functional. To deal with this problem we shall first reformulate Nash.

To convert the product into one convex function using infimal convolution, we must first convert them into a sum of functions.

By using the f–scaling properties of the various norms, we can rewrite this inequality as

$$\boxed{C_n^{(2n+4)/n} \|\nabla f\|_2^2 + \Phi(f) \geq \|f\|_2^{(2n+4)/n},}$$

where $\Phi(f) = \begin{cases} 0 & \|f\|_1 \leq 1 \\ \infty & \|f\|_1 > 1 \end{cases}$, and whose Legendre transform is $\|g\|_\infty$.

The Legendre transform of of $\|\nabla f\|_2$ is our beloved Coulomb potential $\|(-\Delta)^{-1/2} g\|_2^2$.

The fundamental theorem of convex analysis is: *the Legendre transform of the sum of two convex functions is the infimal convolution of the two Legendre transforms.*

Conclusion: By taking the infimal convolution of these two convex functions, and scaling g, we get a dual of the Nash inequality (in which both sides are convex in g):

$$\boxed{L_n \|g\|_2^{\frac{2n+4}{n+4}} \geq \inf_h \left\{ \tfrac{1}{2}\|(-\Delta)^{-1/2}(g-h)\|_2^2 + \|h\|_\infty \right\} .}$$

Unfortunately, because of the '$\inf_h$', this is useless unless we can find h

6 Facts about h

This is the fun part! We cannot compute h (except in one case), but we can say, more or less, what h looks like.

As a preliminary step we can try to minimize $\|(-\Delta)^{-1/2}(g-h)\|_2^2$ under the condition that $\|h\|_\infty \leq c$. Call this $K(c)$ and, as a second, easy step, minimize $K(c)+c$. So let us discuss only the first step, with c fixed and $|h(x)| \leq c, \ \forall x$.

It is not hard to prove (everyone here can surely do it) that a *unique* minimizing h exists for $K(c)$. Let us then move on to the *Euler-Lagrange equation* for h, which is

$$\psi(x) \begin{cases} \geq 0, & \text{if } h(x) = c \\ = 0, & \text{if } -c < h(x) < c \\ \leq 0, & \text{if } h(x) = -c\ , \end{cases} \qquad \text{with } \ \psi = (-\Delta)^{-1}(g-h).$$

An important fact about Laplacians (in the sense of distributions) is that $\Delta f = 0$ almost everywhere on the set $\{x: \ f(x) = 0\}$. Since $\Delta\psi = h - g$, we conclude that almost everywhere

$$\boxed{\underline{\text{either}} \ \ h(x) = \pm c \ \ \underline{\text{or else}} \ \ h(x) = g(x) \ \text{ and } \ |g(x)| < c}$$

This kind of argument goes back to the 2016 *'no-flat-spots for strictly subharmonic functions'* theorem of Frank & Lieb [3].

In case $g \geq 0$ one can also show that $h \geq 0$.

Another thing that one can easily prove is that $\int h = \int g$ for any $c > 0$. (Otherwise the Coulomb energy would be infinite.)

Unfortunately, we cannot find a formula for h except in one special, but important case: The case in which g is a symmetric decreasing, non-negative radial function. Trivial proof!

$$h(x) = \begin{cases} c & \text{if} |x| \leq R, \\ g(x) & \text{if} |x| > R. \end{cases} \qquad \text{and volume of } \{x : |x| < R\} = \tfrac{1}{c}\int g.$$

7 The sharp constant

] The *sharp* constant C_n in Nash and L_n in dual Nash (dN) are trivially related, just as are the sharp constants for Sobolev and HLS. Assume you have not read the Carlen-Loss paper for C_n, and let us compute L_n directly. This will gives us an alternative proof of C_n.

Let G be the maximizing g in dN. By Faber-Krahn (i.e., rearrangement inequality for the Laplacian) the optimizers for N are symmetric decreasing. By the 1:1 correspondence between optimizers for N and dN, we see that G also wants to be symmetric decreasing. In this case, we know the optimum H, as we just saw at the end of the previous slide.

Let us compute the Euler-Lagrange equation for G. $L_n G = (-\Delta)^{-1}(G - H)$

(Note: The variation w.r.t. $H = H_G$ vanishes since H is a minimizer for G).

With $\phi = G - H$ we have $\boxed{L_n\,(-\Delta)\phi = \phi}$ in a ball of radius R, and ϕ satisfies Dirichlet, and also Neumann boundary conditions on the ball. This eigenvalue problem is exactly what Carlen-Loss found for Nash, and which they solved explicitly.

8 The weighted version

] To conclude this story, let me briefly explain the word 'weighted' in the title.

The **sharp weighted Nash inequality** for $p > 0$ generalizes Carlen/Loss/Nash:

$$\boxed{\|f\|_2^{2+n/(4+2p)} \leq C_{n,p}\,\|\nabla f\|_2^2 \cdot \|\,|x|^p f\|_1^{n/(4+2p)}\ .}$$

Legendre transforming, as before, the equivalent **dual weighted Nash inequality** is:

$$\boxed{L_{n,p}\,\|g\|_2^{\frac{2n+4}{n+4}} \geq \inf_h \left\{ \tfrac{1}{2}\|(-\Delta)^{-1/2}(g-h)\|_2^2 + \|\,|x|^{-p}h\|_\infty \right\}\ .}$$

In contrast to the unweighted case, neither sharp constant was known. When p is an even integer, however, the method we just described can be applied and yields a *new result: The sharp values of $C_{n,p}$ and $L_{n,p}$*.

Weights different from $|x|^p$ are possible, but only in this case can we easily find a formula for the sharp constants.

References

[1] T. Aubin, *Problemes isopérimétriques et espaces de Sobolev.* Journal of differential geometry **11**, No. 4 (1976), pp. 573-598.

[2] E. A. Carlen and M. Loss *Sharp constant in Nash's inequality*, Internat. Math. Res. Notices, **1993**, (1993) , pp. 213-215

[3] R. L. Frank and E. H. Lieb, *A 'liquid-solid' phase transition in a simple model for swarming, based on the 'no flat-spots' theorem for subharmonic functions*, arXiv preprint arXiv:1607.07971.

[4] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. Math. (2), 118 (1983), pp. 349–374.

[5] J. F. Nash *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math., **80**, (1958), PP. 931-954.

[6] Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, 1970

[7] G. Talenti, *Best constants in Sobolev Inequality.* Ann. Mat. Pura Appl. **110** (1976), pp. 353-372.