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On localized trapped modes in a pipe-cavity system

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Abstract

The work is concerned with an analytical study of acoustic trapped modes in a cylindrical expansion chamber, placed in between two semi-infinite pipes (waveguides). Trapped mode solutions are expressed in terms of Fourier-Bessel series, with the expansion coefficients determined from a determinant condition. The roots of the determinant, expressed in terms of the wavenumber k, correspond to trapped modes. In the case of a shallow cavity, in the sense that the cavity radius is only slightly larger than the pipe radius, asymptotic approximations for the coefficients of the determinant can be applied. The determinant then reduces to a simple form with four-rowed minors placed on a diagonal, enabling analytical evaluation and a proof of existence of trapped modes. We consider here circumferential mode numbers $m \geq 1$. For a shallow cavity and for low values of the circumferential mode number there is just one trapped mode in the allowable wave number domain $k_{\min} < k < k_{\max}$, where $k_{\min}$ is the cutoff frequency for acoustic waves in the cavity and $k_{\max}$ is the corresponding cutoff frequency in the pipes. This mode is symmetric about a radial axis in the center of the cavity.

1 Introduction

The existence of *trapped modes* in a variety of physical systems has received much attention for more than sixty years and it is now known that trapped modes may be found across a wide range of length scales, from waveguides in the ocean, over acoustic and elastic waveguides, to quantum waveguides [5]. A trapped mode, also called a localized mode, refers to an infinite medium in a state of linear oscillations only in the vicinity of an object or a geometrical feature which can act as a resonator. There is no radiation of energy to infinity; the energy is also trapped, or localized. One reason for the attention these problems have received is mathematical: the trapped modes, characterized by a discrete wavenumber spectrum, co-exist with the continuous spectrum which is characteristic for guided waves in an infinite domain. The solutions to these problems are thus non-unique even though the problems are linear.

The present paper gives an analytical solution to the problem of acoustic trapped modes in a cylindrical expansion chamber, placed in between two semi-infinite pipes, i.e. waveguides, a problem which previously has been investigated just numerically [2]. The existence of trapped modes is proved in the case of a shallow cavity, shallow in the sense that the radius of the cavity is only slightly larger than the radius of the connecting pipes.

2 Problem formulation

Consider a cylindrical cavity – an expansion chamber – connected with two semi-infinite cylindrical pipes (wave guides), as shown in figure 1.

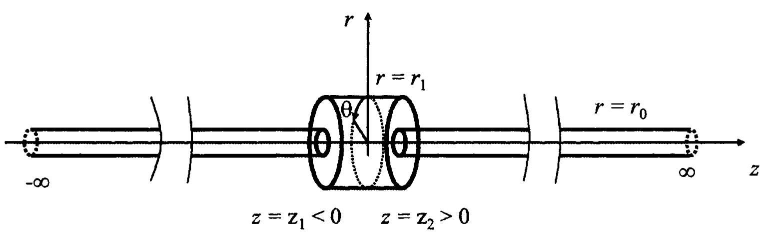

Figure 1: Sketch of the configuration.

Let the radius of the pipes be r_0 and let the radius of the cavity be r_1. The length of the cavity $L_1 = z_2 - z_1$. Cylindrical polar coordinates z, r, θ will be used. The propagation of sound through the system is governed by the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \nabla^2 \phi, \tag{1}$$

subject to the Neumann condition

$$\nabla \phi \cdot \boldsymbol{n} = 0 \tag{2}$$

on the solid boundaries. Here $\phi(t, z, r, \theta)$ is the velocity potential, with t being the time, c_0 is the speed of sound, and $\boldsymbol{n}$ is the inward pointing normal vector. The Laplacian in cylindrical polar coordinates is given by

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{3}$$

The sound pressure $p(t, z, r, \theta)$ is given by

$$p = -\rho \partial \phi / \partial t, \tag{4}$$

where ρ is the fluid density. The acoustic particle velocity $\boldsymbol{u} = (u_z, u_r, u_\theta)$ is given by

$$\boldsymbol{u} = \nabla \phi. \tag{5}$$

3 Construction of a trapped mode

We will here consider the construction of a trapped mode. By employing the method of separation of variables, the solution to (1) can be expressed in the form of a Fourier-Bessel series. In the cavity domain $z_1 < z < z_2$ the mode m solution can be written

$$\phi_{1m} = \sum_{n=0}^{\infty} \phi_{1mn} = \sum_{n=0}^{\infty} \left\{ C_{mn}\, \mathrm{e}^{\mathrm{i}k_{1mn}z} + D_{mn}\, \mathrm{e}^{-\mathrm{i}k_{1mn}z} \right\} J_m\left(j_{mn}\frac{r}{r_1} \right), \tag{6}$$

where m is the circumferential mode number, C_{mn} and D_{mn} are constants, $J_m(z)$ is the Bessel function of first kind and order m, j_{mn} are the zeros of $J'_m(z)$, where the dash denotes differentiation with respect to the argument z, and

$$k_{1mn}^2 = k^2 - \left(\frac{j_{mn}}{r_1}\right)^2, \quad k = \frac{\omega}{c_0}. \tag{7}$$

A factor $e^{-i\omega t}\, e^{im\theta}$ has been suppressed. The zeros j_{mn} are ordered such that

$$j_{m0} \le j_{m1} \le j_{m2} \le \cdots . \tag{8}$$

It is noted also that $m \le j_{m0}$ [1, p. 370]. The complete solution is given by $\phi_1 = \sum_m \phi_{1m}$. In the following a single, fixed value of m will be considered rather than the sum over m. This is equivalent to the approach in [8].

The function $J_m(j_{mn}r/r_1)$, $0 \le r/r_1 \le 1$, is shown in figure 2 for $n = 0$, 1, 2, and 3, with $m = 1$ in part (a) and with $m = 3$ in part (b). For any value of m the value of n corresponds to the number of nodal points in the domain $0 \le r/r_1 \le 1$; thus there are no nodal point for $n = 0$, one nodal point for $n = 1$, etc. As part (b) indicates, the larger the value of m the 'flatter' are the graphs by small values of r/r_1.

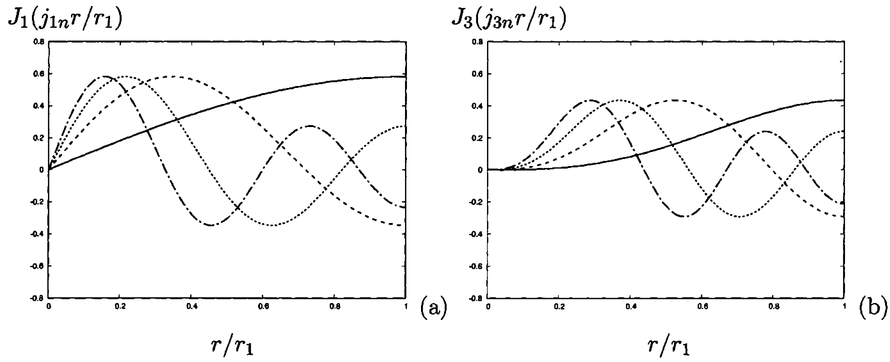

Figure 2: The function $J_m(j_{mn}r/r_1)$ plotted in the range $0 \le r/r_1 \le 1$ with $m = 1$ in (a) and $m = 3$ in (b). Plain lines: $n = 0$; broken lines: $n = 1$; dotted lines: $n = 2$; dash-dot lines: $n = 3$.

In order to construct a trapped mode we consider the following mode m solutions in the pipe domains. In the left hand domain $-\infty < z < z_1$, the left-going (outgoing) wave

$$\phi_{0m}^- = \sum_{n=0}^{\infty} A_{mn}\, J_m\left(j_{mn}\frac{r}{r_0}\right) e^{-ik_{0mn}z}, \quad -\infty < z < z_1 < 0, \tag{9}$$

where

$$k_{0mn}^2 = k^2 - \left(\frac{j_{mn}}{r_0}\right)^2, \tag{10}$$

and in the right-hand domain $z_2 < z < \infty$, the right-going (and again outgoing) wave

$$\phi_{0m}^+ = \sum_{n=0}^{\infty} B_{mn}\, J_m\left(j_{mn}\frac{r}{r_0}\right) e^{ik_{0mn}z}, \quad 0 < z_2 < z < \infty. \tag{11}$$

The functions k_{0mn} and k_{1mn} in (10) and (7), respectively, are specified as

$$k_{\alpha mn} = \begin{cases} \sqrt{k^2 - \left(\frac{j_{mn}}{r_\alpha}\right)^2}, & k > \dfrac{j_{mn}}{r_\alpha} \\ \mathrm{i}\sqrt{\left(\frac{j_{mn}}{r_\alpha}\right)^2 - k^2}, & k < \dfrac{j_{mn}}{r_\alpha} \end{cases} \qquad \alpha = 0, 1. \tag{12}$$

It is thus seen that a trapped mode may be present by mode (m, n) if the wavenumber k is in the range

$$\frac{j_{mn}}{r_1} < k < \frac{j_{mn}}{r_0}, \tag{13}$$

where j_{mn}/r_0 is the cutoff frequency for mode (m, n) for a pipe of radius r_0 and j_{mn}/r_1 is the cutoff frequency for mode (m, n) for a pipe of radius r_1. This is because, with this specification of k, the solution has the form $e^{\pm \mathrm{i}|k_{1mn}z|}$ in the cavity domain and the form $e^{-|k_{0mn}z|}$ in the pipe domains.

It is noted that the expansions (6), (9), (11) constitute complete sets of functions [7]. These functions must satisfy matching and boundary conditions at the steps. At $z = z_1$ these conditions are given by

$$\begin{aligned} \phi_{m0}^-(z_{1-}) &= \phi_{1m}(z_{1+}), \quad 0 < r < r_0, & (14)\\ \frac{\partial \phi_{0m}^-}{\partial z}(z_{1-}) &= \frac{\partial \phi_{1m}}{\partial z}(z_{1+}), \quad 0 < r < r_0, & (15)\\ \frac{\partial \phi_{1m}}{\partial z}(z_{1+}) &= 0, \quad r_0 < r < r_1. & (16) \end{aligned}$$

Similar conditions apply at $z = z_2$. It is noted that the boundary conditions

$$\left.\frac{\partial \phi_{0m}}{\partial r}\right|_{r=r_0} = 0 \quad \text{for} \quad z_1 < z < z_2, \qquad \left.\frac{\partial \phi_{1m}}{\partial r}\right|_{r=r_1} = 0 \quad \text{for} \quad z < z_1, z > z_2, \tag{17}$$

which apply as well, are exactly satisfied by the solutions (6), (9), and (11). The conditions (14) (at $z = z_1$, z_2) are Dirichlet conditions, while (15) and (16) (also at $z = z_1$, z_2) are Neumann conditions.

The mixed boundary value problem (6), (9), (11) and (14)-(16) (again with similar conditions applying at $z = z_2$) can be discretized by converting the boundary conditions (14)-(16) to a Galerkin weak formulation, using the weight functions $J_m\left(j_{mq}\frac{r}{r_0}\right) r$ and $J_m\left(j_{mq}\frac{r}{r_1}\right) r$. The residual equations take the forms

$$\int_0^{r_0} \{\phi_{0m}(z_{1-}) - \phi_{1m}(z_{1+})\} J_m\left(j_{mq}\frac{r}{r_0}\right) r dr = 0, \tag{18}$$

$$\int_0^{r_0} \left\{\frac{\partial \phi_{0m}}{\partial z}(z_{1-}) - \frac{\partial \phi_{1m}}{\partial z}(z_{1+})\right\} J_m\left(j_{mq}\frac{r}{r_1}\right) r dr = 0, \tag{19}$$

and

$$\int_{r_0}^{r_1} \frac{\partial \phi_{1m}}{\partial z}(z_{1+}) J_m\left(j_{mq}\frac{r}{r_1}\right) r dr = 0, \tag{20}$$

for $q = 0, 1, 2, \cdots$. Combining (19) with (20) we get

$$\int_0^{r_0} \frac{\partial \phi_{0m}}{\partial z} \partial z(z_{1-}) J_m\left(j_{mq}\frac{r}{r_1}\right) r dr - \int_0^{r_1} \frac{\partial \phi_{1m}}{\partial z}(z_{1+}) J_m\left(j_{mq}\frac{r}{r_1}\right) r dr = 0. \tag{21}$$

4 Nondimensionalization

Next we make the equations nondimensional by introducing the nondimensional parameters

$$\tilde{r} = \frac{r}{r_0}, \quad \hat{r} = \frac{r}{r_1}, \quad \tilde{z} = \frac{z}{r_0}, \quad \tilde{k} = kr_0, \quad \alpha = \frac{r_0}{r_1}. \tag{22}$$

Also, let

$$\tilde{k}_{0mn} = \mathrm{i}\sqrt{j_{mn}^2 - \tilde{k}}, \quad \tilde{k}_{1mn} = \sqrt{\tilde{k} - \alpha^2 j_{mn}^2}. \tag{23}$$

The three series solutions (6), (9), and (11) are now expressed in the forms

$$\phi_{1m} = \sum_{n=0}^{\infty} \left\{ C_{mn}\, \mathrm{e}^{\mathrm{i}\tilde{k}_{1mn}(\tilde{z}-\tilde{z}_1)} + D_{mn}\, \mathrm{e}^{-\mathrm{i}\tilde{k}_{1mn}(\tilde{z}-\tilde{z}_2)} \right\} \frac{1}{\tilde{k}_{1mn}} \frac{J_m\left(j_{mn}\alpha\tilde{r}\right)}{J_m^2\left(j_{mn}\right)} \frac{2j_{mn}^2}{j_{mn}^2 - m^2}, \tag{24}$$

$$\phi_{0m}^{-} = \sum_{n=0}^{\infty} A_{mn}\, \mathrm{e}^{-\mathrm{i}\tilde{k}_{0mn}(\tilde{z}-\tilde{z}_1)} \frac{J_m\left(j_{mn}\tilde{r}\right)}{J_m^2\left(j_{mn}\right)} \frac{2j_{mn}^2}{j_{mn}^2 - m^2}, \quad -\infty < \tilde{z} < \tilde{z}_1 < 0, \tag{25}$$

and

$$\phi_{0m}^{+} = \sum_{n=0}^{\infty} B_{mn}\, \mathrm{e}^{\mathrm{i}\tilde{k}_{0mn}(\tilde{z}-\tilde{z}_2)} \frac{J_m\left(j_{mn}\tilde{r}\right)}{J_m^2\left(j_{mn}\right)} \frac{2j_{mn}^2}{j_{mn}^2 - m^2}, \quad 0 < \tilde{z}_2 < \tilde{z} < \infty. \tag{26}$$

5 Determinant condition

The matching and boundary conditions (18) and (21), evaluated at $\tilde{z} = \tilde{z}_1$ and $\tilde{z} = \tilde{z}_2$, can with (24), (25), and (26) inserted be written

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left\{ \delta_{qn} A_{mn} - \mathcal{F}_{1mqn} C_{mn} - \mathcal{F}_{1mqn} \mathrm{e}^{\mathrm{i}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)} D_{mn} \right\} = 0, \\
&\sum_{n=0}^{\infty} \left\{ \delta_{qn} B_{mn} - \mathcal{F}_{1mqn} \mathrm{e}^{\mathrm{i}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)} C_{mn} - \mathcal{F}_{1mqn} D_{mn} \right\} = 0, \\
&\sum_{n=0}^{\infty} \left\{ \mathcal{F}_{2mqn} A_{mn} + \delta_{qn} C_{mn} - \delta_{qn} \mathrm{e}^{\mathrm{i}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)} D_{mn} \right\} = 0, \\
&\sum_{n=0}^{\infty} \left\{ \mathcal{F}_{2mqn} B_{mn} - \delta_{qn} \mathrm{e}^{\mathrm{i}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)} C_{mn} + \delta_{qn} D_{mn} \right\} = 0,
\end{aligned} \tag{27}$$

for $q = 0, 1, \cdots$, where δ_{qn} is Kronecker's delta and

$$\begin{aligned}
\mathcal{F}_{1mqn} \;=\; & \frac{2\alpha^2 j_{mn}^2}{j_{mn}^2 - m^2} \frac{\tilde{k}_{1mn}^{-1}}{\alpha^2 j_{mn}^2 - j_{mq}^2} \frac{1}{J_m^2\left(j_{mn}\right)} \\
& \times \left\{ \alpha j_{mn} J_{m+1}(\alpha j_{mn}) J_m(j_{mq}) - J_m(\alpha j_{mn}) J_{m+1}(j_{mq}) \right\},
\end{aligned} \tag{28}$$

$$\begin{aligned}
\mathcal{F}_{2mqn} \;=\; & \frac{2\alpha^2 j_{mn}^2}{j_{mn}^2 - m^2} \frac{\tilde{k}_{0mn}}{j_{mn}^2 - \alpha^2 j_{mq}^2} \frac{1}{J_m^2\left(j_{mn}\right)} \\
& \times \left\{ j_{mn} J_{m+1}(j_{mn}) J_m(\alpha j_{mq}) - \alpha J_m(j_{mn}) J_{m+1}(\alpha j_{mq}) \right\}.
\end{aligned} \tag{29}$$

The functions (28) and (29) have been obtained by employing the relations [9, Ch. V]

$$\int_0^1 J_m\left(j_{mn}\tilde{r}\right) J_m\left(j_{mq}\tilde{r}\right) \tilde{r} d\tilde{r} = \frac{1}{2}\frac{\delta_{qn}}{j_{mq}^2}\left[j_{mn}^2 - m^2\right]\left\{J_m\left(j_{mn}\right)\right\}^2, \tag{30}$$

and

$$\int_0^1 J_m\left(j_{mn}\alpha\hat{r}\right) J_m\left(j_{mq}\hat{r}\right) \hat{r} d\hat{r} = \frac{1}{j_{mn}^2 - j_{mq}^2} \tag{31}$$
$$\times\Big\{\alpha j_{mn} J_m\left(j_{mq}\right) J_{m+1}\left(\alpha j_{mn}\right) - j_{mq} J_m\left(\alpha j_{mn}\right) J_{m+1}\left(j_{mq}\right)\Big\}.$$

Written in matrix form, (27) can be expressed as

$$\mathbf{Ab} = \mathbf{0}, \tag{32}$$

where $\mathbf{b} = \{A_{m0}\, B_{m0}\, C_{m0}\, D_{m0}\, A_{m1}\, B_{m1}\, C_{m1}\, D_{m1}\, \ldots\}^T$. The elements of the matrix $\mathbf{A}$ are denoted by a_{qn}, $q, n = 0, 1, \cdots$. The condition for the existence of a trapped mode at a given value of the wavenumber $\tilde{k}$ is that the infinite determinant

$$\Delta = \det \mathbf{A} = |a_{qn}|_0^\infty = 0. \tag{33}$$

In terms of the nondimensional variables a trapped mode may exist in mode (m, n) in the wavenumber range

$$\alpha j_{mn} < \tilde{k} < j_{mn}. \tag{34}$$

6 Symmetric and antisymmetric solutions

It is seen directly from (27) that two types of solutions are possible: symmetric solutions, $\phi(z) = \phi(-z)$, and antisymmetric solutions, $\phi(z) = -\phi(-z)$. By symmetric solutions, $A_{mn} = B_{mn}$, and $C_{mn} = D_{mn}$, for any m, n. By antisymmetric solutions, $A_{mn} = -B_{mn}$, and $C_{mn} = -D_{mn}$, for any m, n. In both cases, (27) reduces to the following system of just two equations:

$$\sum_{n=0}^{\infty}\Big\{\delta_{qn} A_{mn} - \mathcal{F}_{1mqn} C_{mn}\Big(1 \pm e^{i\tilde{k}_{1mn}(\tilde{z}_2 - \tilde{z}_1)}\Big)\Big\} = 0, \tag{35}$$
$$\sum_{n=0}^{\infty}\Big\{\mathcal{F}_{2mqn} A_{mn} + \delta_{qn} C_{mn}\Big(1 \mp e^{i\tilde{k}_{1mn}(\tilde{z}_2 - \tilde{z}_1)}\Big)\Big\} = 0,$$

$q = 0, 1, \cdots$. In each of these two equations the upper sign (before the exponential function) gives the symmetric solution, while the lower sign gives the antisymmetric solution.

7 The case of a shallow cavity

Here we consider the case where $\alpha \lessapprox 1$, that is to say, α is slightly smaller than one. Use will be made of the expansion [1, p. 363]

$$J_m(\alpha\tilde{r}j_{mn}) = \alpha^m \sum_{p=0}^{\infty} \frac{1}{p!}(-1)^p(\alpha^2 - 1)^p J_{m+p}(\tilde{r}j_{mn}). \tag{36}$$

Let $\epsilon = 1 - \alpha$. For $0 < \epsilon \ll 1$ we can write

$$J_m(\alpha \tilde{r} j_{mn}) = \alpha^m J_m(\tilde{r} j_{mn}) + O(\epsilon). \tag{37}$$

The functions $\mathcal{F}_{1mqn}$ and $\mathcal{F}_{2mqn}$, defined by (28) and (29), then reduce to

$$\mathcal{F}_{1mqn} \approx \delta_{qn} \alpha^{m+2} \tilde{k}_{1mn}^{-1}, \quad \mathcal{F}_{2mqn} \approx \delta_{qn} \alpha^m \tilde{k}_{0mn}. \tag{38}$$

A comparison between the 'full' functions $\mathcal{F}_{1mqn}$ and $\mathcal{F}_{2mqn}$ and their asymptotic approximations (38) is shown in figure 3. The graphs are obtained with $m = 1$, $q = n = 0$, and with $\tilde{k} = \frac{1}{2} j_{10}(\alpha + 1)$. As would be expected, the agreements are very good when α is close to one, i.e. when $\epsilon = 1 - \alpha$ is small, and less good otherwise. The trends are similar for any values of m, q, n, and $\tilde{k}$.

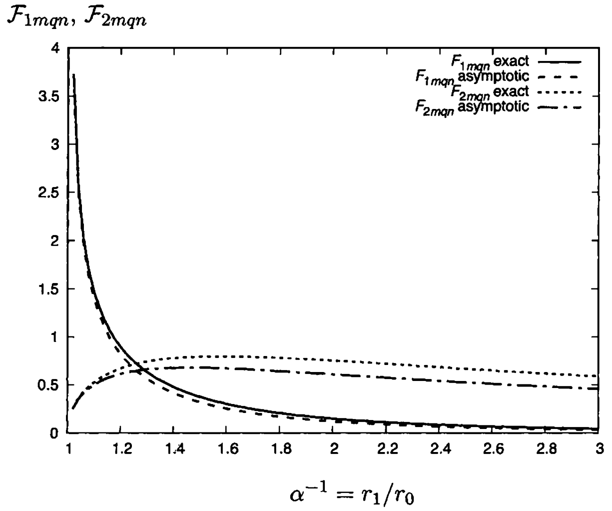

Figure 3: Comparisons between the functions $\mathcal{F}_{1mqn}$ and $\mathcal{F}_{2mqn}$, defined by (28) and (29), and their asymptotic approximations, defined by (38).

Employing (38) the system (27) reduces to

$$\begin{aligned}
&\sum_{n=0}^{\infty} \delta_{qn} \left\{ A_{mn} - \frac{\alpha^{m+2}}{\tilde{k}_{1mn}} C_{mn} - \frac{\alpha^{m+2}}{\tilde{k}_{1mn}} e^{i\tilde{k}_{1mn}(\tilde{z}_2 - \tilde{z}_1)} D_{mn} \right\} = 0, \\
&\sum_{n=0}^{\infty} \delta_{qn} \left\{ B_{mn} - \frac{\alpha^{m+2}}{\tilde{k}_{1mn}} e^{i\tilde{k}_{1mn}(\tilde{z}_2 - \tilde{z}_1)} C_{mn} - \frac{\alpha^{m+2}}{\tilde{k}_{1mn}} D_{mn} \right\} = 0, \\
&\sum_{n=0}^{\infty} \delta_{qn} \left\{ \alpha^m \tilde{k}_{0mn} A_{mn} + C_{mn} - e^{i\tilde{k}_{1mn}(\tilde{z}_2 - \tilde{z}_1)} D_{mn} \right\} = 0, \\
&\sum_{n=0}^{\infty} \delta_{qn} \left\{ \alpha^m \tilde{k}_{0mn} B_{mn} - e^{i\tilde{k}_{1mn}(\tilde{z}_2 - \tilde{z}_1)} C_{mn} + D_{mn} \right\} = 0,
\end{aligned} \tag{39}$$

$q = 0, 1, \cdots$. That is to say, the determinant decouples into four-rowed minors [4, p. 20], i.e., blocks of size 4×4, placed on a diagonal, and with zeros elsewhere. The (determinant of the) minor no. n is evaluated as

$$\Delta_n = 2\frac{\mathrm{e}^{\mathrm{i}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)}}{\tilde{k}_{1mn}}\Big[2\tilde{k}_{0mn}\tilde{k}_{1mn}\alpha^{2(m+1)}\cos(\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)) \\ -\mathrm{i}\left(\tilde{k}_{0mn}^2\alpha^{4(m+1)}+\tilde{k}_{1mn}^2\right)\sin(\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1))\Big]. \tag{40}$$

The original determinant is evaluated as $\Delta = \prod_{n=0}^{\infty}\Delta_n$.

Consider first the case where $\tilde{k} \gtrapprox \alpha j_{mn}$ ($\tilde{k}$ is just slightly larger than αj_{mn}), specifically let $\delta^2 = \tilde{k}^2 - \alpha^2 j_{mn}^2$, $0 < \delta^2 \ll 1$. Then $\tilde{k}_{1mn} = \delta$ and $\tilde{k}_{0mn} \approx \mathrm{i}j_{mn}(1-\alpha^2)^{\frac{1}{2}}$. Inserting this into (40) gives

$$\begin{aligned}\frac{1}{2}\frac{\tilde{k}_{1mn}}{\mathrm{e}^{\mathrm{i}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)}}\Delta_n &\approx \mathrm{i}\Big[2\delta j_{mn}(1-\alpha^2)^{\frac{1}{2}}\alpha^{2(m+1)}\cos(\delta(\tilde{z}_2-\tilde{z}_1)) \\ &\quad -\left(-j_{mn}^2\alpha^{4(m+1)}(1-\alpha^2)+\delta^2\right)\sin(\delta(\tilde{z}_2-\tilde{z}_1))\Big] \\ &\approx \mathrm{i}\delta\Big[2j_{mn}(1-\alpha^2)^{\frac{1}{2}}\alpha^{2(m+1)}+j_{mn}^2\alpha^{4(m+1)}(1-\alpha^2)(\tilde{z}_2-\tilde{z}_1)\Big].\end{aligned} \tag{41}$$

Consider next the case where $\tilde{k} \lessapprox j_{mn}$ ($\tilde{k}$ is just slightly smaller than j_{mn}), specifically let $\delta^2 = j_{mn}^2 - \tilde{k}^2$, $0 < \delta^2 \ll 1$. Then $\tilde{k}_{0mn} = \mathrm{i}\delta$ and $\tilde{k}_{1mn} \approx j_{mn}(1-\alpha^2)^{\frac{1}{2}}$. Inserting this into (40) gives

$$\begin{aligned}\frac{1}{2}\frac{\tilde{k}_{1mn}}{\mathrm{e}^{\mathrm{i}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)}}\Delta_n &\approx \mathrm{i}\Big[2\delta j_{mn}(1-\alpha^2)^{\frac{1}{2}}\alpha^{2(m+1)}\cos(j_{mn}\sqrt{1-\alpha^2}(\tilde{z}_2-\tilde{z}_1)) \\ &\quad -\left(-\delta^2\alpha^{4(m+1)}+j_{mn}^2(1-\alpha^2)\right)\sin(j_{mn}\sqrt{1-\alpha^2}(\tilde{z}_2-\tilde{z}_1))\Big] \\ &\approx \mathrm{i}\Big[2\delta j_{mn}(1-\alpha^2)^{\frac{1}{2}}\alpha^{2(m+1)} \\ &\quad -\left(-\delta^2\alpha^{4(m+1)}+j_{mn}^2(1-\alpha^2)\right)(j_{mn}\sqrt{1-\alpha^2}(\tilde{z}_2-\tilde{z}_1)\Big] \\ &\approx -\mathrm{i}j_{mn}^3(1-\alpha^2)^{\frac{3}{2}}(\tilde{z}_2-\tilde{z}_1).\end{aligned} \tag{42}$$

In the first case, (41) clearly shows that $-\mathrm{i}\frac{1}{2}\frac{\tilde{k}_{1mn}}{\mathrm{e}^{\mathrm{i}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)}}\Delta_n$ is real and > 0 for $\tilde{k} \gtrapprox \alpha j_{mn}$, while in the second case, (42) shows that $-\mathrm{i}\frac{1}{2}\frac{\tilde{k}_{1mn}}{\mathrm{e}^{\mathrm{i}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)}}\Delta_n$ is real and < 0 for $\tilde{k} \lessapprox j_{mn}$. It is noted, also, that neither $\mathrm{e}^{\mathrm{i}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)}$ nor $\tilde{k}_{1mn}$ is zero in $\alpha j_{mn} < \tilde{k} < j_{mn}$. Thus Δ_n (and thus Δ) has, at least, one zero for $\alpha j_{mn} < \tilde{k} < j_{mn}$. To locate one or more of these zeros, set $\Delta_n = 0$ and let $\tilde{k} = \tilde{s}j_{mn}$. Using (40), the expression for $\Delta_n = 0$ can be written as

$$\tan\left((\tilde{z}_2-\tilde{z}_1)j_{mn}(\tilde{s}^2-\alpha^2)^{\frac{1}{2}}\right) = \alpha^{2(m+1)}\frac{(1-\tilde{s}^2)^{\frac{1}{2}}(\tilde{s}^2-\alpha^2)^{\frac{1}{2}}}{\alpha^{4(m+1)}(\tilde{s}^2-1)+(\tilde{s}^2-\alpha^2)}. \tag{43}$$

Assuming that $\tilde{s} \approx \alpha \lessapprox 1$ we can use that $\tan(\cdots) \approx (\cdots)$. Using also that $(1-\tilde{s}^2)^{\frac{1}{2}} \approx 1 - \frac{1}{2}\tilde{s}$, (43) gives that

$$\tilde{s} = \left\{\frac{\alpha^{2(m+1)}+(\tilde{z}_2-\tilde{z}_1)j_{mn}\left(\alpha^2+\alpha^{4(m+1)}\right)}{\frac{1}{2}\alpha^{2(m+1)}+(\tilde{z}_2-\tilde{z}_1)j_{mn}\left(1+\alpha^{4(m+1)}\right)}\right\}^{\frac{1}{2}}, \tag{44}$$

or equivalently

$$\tilde{s}^2 = 1 + \frac{\frac{1}{2}\alpha^{2(m+1)}-(\tilde{z}_2-\tilde{z}_1)j_{mn}\left(1-\alpha^2\right)}{\frac{1}{2}\alpha^{2(m+1)}+(\tilde{z}_2-\tilde{z}_1)j_{mn}\left(1+\alpha^{4(m+1)}\right)}. \tag{45}$$

In order to satisfy the condition $\alpha j_{mn} < \tilde{k} < j_{mn}$ it is necessary that $\alpha < \tilde{s} < 1$. It is clearly possible to choose a cavity length $\tilde{z}_2 - \tilde{z}_1$ such that this condition is satisfied. In particular,

$\tilde{s} < 1$ if $\tilde{z}_2 - \tilde{z}_1 > \alpha^{2(m+1)}/\{(1-\alpha^2)j_{mn}\}$. (It is noted that the lowest possible value of j_{mn} is $j_{10} = 1.8412\cdots$.)

We finally discuss the limit $\alpha \to 1$, where the cavity disappears altogether. In this limit (13) degenerates into a single point, $\tilde{k} = j_{mn}$. Then, from (23) we obtain $\tilde{k}_{0mn} = \tilde{k}_{1mn} = 0$. Thus there is no wave motion and no meaningful solution to the problem, as there shouldn't be either.

In the following we check (45) in this limit. As $\alpha \to 1$, $\tilde{k} \to \tilde{k}_{\max}$, that is, $\tilde{k} \to j_{mn}$, the cutoff frequency of the pipes. This means that $\tilde{s} \to 1$ in (43). Then, in (44) and (45), the square root function $(1-\tilde{s}^2)^{\frac{1}{2}}$ needs to be approximated not for $\tilde{s}$ small but for $1-\tilde{s}^2$ small. Here use can be made of Lanczos's power expansion of $y = \sqrt{x}$ in terms of Chebyshev polynomials [3, p. 485]. The first approximation is simply $y_1 = x$. Using this approximation in (43), (45) takes the form

$$\tilde{s}^2 \approx 1 - \frac{(\tilde{z}_2 - \tilde{z}_1)j_{mn}\left(1-\alpha^2\right)}{\alpha^{2(m+1)} + (\tilde{z}_2 - \tilde{z}_1)j_{mn}\left(1+\alpha^{4(m+1)}\right)}, \tag{46}$$

which correctly gives that $\tilde{s} \to 1$, and thus that $\tilde{k} \to \tilde{k}_{\max} = j_{mn}$, as $\alpha \to 1$.

8 Symmetric and antisymmetric solutions in the case of a shallow cavity

Employing the shallow cavity approximation (38) to the two equations (35), they reduce to

$$\sum_{n=0}^{\infty} \delta_{qn}\left\{A_{mn} - \frac{\alpha^{m+2}}{\tilde{k}_{1mn}}C_{mn}\left(1 \pm e^{i\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)}\right)\right\} = 0, \tag{47}$$

$$\sum_{n=0}^{\infty} \delta_{qn}\left\{\alpha^m \tilde{k}_{0mn}A_{mn} + C_{mn}\left(1 \mp e^{i\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)}\right)\right\} = 0.$$

Here the upper signs correspond to the solution symmetric about the r axis, while the lower signs correspond to the antisymmetric solution. The determinant of the nth minor is

$$\Delta_n = 1 \mp e^{i\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)} + \alpha^{2(m+1)}\frac{\tilde{k}_{0mn}}{\tilde{k}_{1mn}}\left(1 \pm e^{i\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)}\right). \tag{48}$$

We consider first the symmetric case, that is, the upper signs in (48). With these signs (48) can be written as

$$\begin{aligned}\Delta_n = \quad & 2e^{i\frac{1}{2}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)} \\ & \times\left\{\alpha^{2(m+1)}\frac{\tilde{k}_{0mn}}{\tilde{k}_{1mn}}\cos\left(\frac{1}{2}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)\right) - i\sin\left(\frac{1}{2}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)\right)\right\}.\end{aligned} \tag{49}$$

Consider first the case where $\tilde{k} \gtrapprox \alpha j_{mn}$ ($\tilde{k}$ is just slightly larger than αj_{mn}), as done in the previous section. Then $\tilde{k}_{1mn} = \delta$ and $\tilde{k}_{0mn} \approx i j_{mn}(1-\alpha^2)^{\frac{1}{2}}$. Inserting these approximations into (49) it can be written

$$\Delta_n \approx 2\alpha^{2(m+1)}j_{mn}\sqrt{1-\alpha^2}\left\{-\frac{1}{2}(\tilde{z}_2-\tilde{z}_1) + i\delta^{-1}\right\}. \tag{50}$$

Consider next the case where $\tilde{k} \lessapprox j_{mn}$ ($\tilde{k}$ is just slightly smaller than j_{mn}). Then $\tilde{k}_{0mn} = i\delta$ and $\tilde{k}_{1mn} \approx j_{mn}(1-\alpha^2)^{\frac{1}{2}}$. Inserting these expressions into (49) gives

$$\Delta_n \approx \frac{1}{2}j_{mn}^2\left(1-\alpha^2\right)(\tilde{z}_2-\tilde{z}_1)^2 - i j_{mn}\sqrt{1-\alpha^2}(\tilde{z}_2-\tilde{z}_1). \tag{51}$$

Thus both the real part and the imaginary part change signs. This shows that a symmetric trapped mode exists in the domain $\alpha j_{mn} < \tilde{k} < j_{mn}$.

Consider next the antisymmetric case, that is, the lower signs in (48). Then (48) can be written

$$\begin{aligned} \Delta_n &= 2\mathrm{e}^{\mathrm{i}\frac{1}{2}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)} \\ &\times \left\{ \cos\left(\frac{1}{2}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)\right) - \mathrm{i}\alpha^{2(m+1)}\frac{\tilde{k}_{0mn}}{\tilde{k}_{1mn}} \sin\left(\frac{1}{2}\tilde{k}_{1mn}(\tilde{z}_2-\tilde{z}_1)\right) \right\}. \end{aligned} \tag{52}$$

In the case where $\tilde{k} \gtrapprox \alpha j_{mn}$ (52) can be approximated as

$$\Delta_n \approx 2 + \alpha^{2(m+1)} j_{mn}\sqrt{1-\alpha^2}(\tilde{z}_2-\tilde{z}_1) + \mathrm{i}O(\delta). \tag{53}$$

In the case where $\tilde{k} \lessapprox j_{mn}$ we get

$$\Delta_n \approx 2 + \mathrm{i} j_{mn}\sqrt{1-\alpha^2}(\tilde{z}_2-\tilde{z}_1). \tag{54}$$

The positive real parts in both (53) and (54) indicate that there is no antisymmetric trapped mode in the case of a shallow cavity.

9 Conclusion

In this paper we have carried out an analytical study of acoustic trapped modes in a cylindrical expansion chamber, connected with two semi-infinite pipes that are waveguides. The problem has been solved by employing the method of separation of variables, giving Fourier-Bessel expansion-type of solutions in each of the three subregions left pipe, expansion chamber, and right pipe. The solutions for each of these three regions have been linked via boundary- and matching-conditions, giving an infinite determinant for which the roots correspond to trapped modes. For the case of a shallow cavity – i.e. the cavity radius is only slightly larger than the pipe radius – it has been found that the infinite determinant decouples into four-rowed minors (sub-determinants of dimension 4×4) placed along a diagonal, enabling analytical evaluation into a simple expression. Asymptotic results predicting the existence of trapped modes have been given in this limit for which there, with low values of the circumferential mode number m, is just one trapped mode in the allowable wave number domain, $k_{\min} < k < k_{\max}$. It is shown that this mode is symmetric about a radial axis in the center of the cavity.

While we have restricted the analytical studies of the present paper to the case of a shallow cavity, there is another interesting limit case that may be studied analytically, namely that of a narrow cavity ($\tilde{z}_2 - \tilde{z}_1$ small). It may be suspected that a deep, narrow cavity will, in effect, approach a three-dimensional Helmholtz resonator, that is, a three-dimensional version of the first problem studied in [6]. This may be of interest to examine in future work.

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