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Small stable theories with the tree property

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Let T be a complete theory in a countable first order language. A non-isolated type $p \in S(T)$ is said to have the tree property, if there are realizations $\bar{a}, \bar{b}, \bar{c} \models p$ such that $\mathrm{tp}(\bar{b}\bar{c}/\bar{a})$ is isolated and $\bar{b} \downarrow \bar{c}$ (see [1] for the definition). It is known that any stable Ehrenfeucht theory has a type with the tree property, and moreover that any type with the tree property has infinite weight ([1]). Using the Hrushovski amalgamation construction ([2]), Herwig constructed a small stable theory with a type of weight ω ([1]), but it does not have a type with the tree type. In this short note, we show that there is a small stable theory with the tree property.

Notation 1 $M, N, \ldots$ will denote L-structures, and $A, B, \ldots$ subsets of structures. Elements of structures will be denoted by $a, b, \ldots$, and finite tuples of elements by $\bar{a}, \bar{b}, \ldots$. If members of the tuple $\bar{a}$ come from A we sometimes write $\bar{a} \in A$. $A \subset_{\mathrm{fin}} B$ means that A is a finite subset of B. AB means $A \cup B$. $\mathrm{tp}(\bar{a}/A)$ denotes a type of $\bar{a}$ over A. $S(A)$ denotes the set of all types over A and $S(T)$ means $S(\emptyset)$. The set of all algebraic elements over A in M is denoted by $\mathrm{acl}_M(A)$. $B \downarrow_A C$ means that B and C are independent over A in the sense of forking.

Definition 2 Let L_0 be a language consisting of a binary relation R. Here, a directed graph means an L_0-structure (A, R^A), where $R^A = \{ab \in A : A \models R(a, b)\}$, satisfying that

- $A \models \forall x \forall y [R(x, y) \to \neg R(y, x)]$;
- $A \models \forall x \forall y [R(x, y) \to x \neq y]$.

Let $\mathbf{K}_0$ be a class of the finite directed graphs.

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For $A \in \mathbf{K}_0$, a predimension of A is defined by

$$\delta(A) = |A| - \alpha|R^A|,$$

where $\alpha \in (0, 1]$. For $A, B \in \mathbf{K}_0$, $\delta(B \cup A) - \delta(A)$ is denoted by $\delta(B/A)$. For $A \subset B \in \mathbf{K}_0$, A is said to be closed in B (written $A \leq B$), if

$$\delta(X/A) \geq 0 \text{ for any } X \subset B - A.$$

Definition 3 For a function $f : \omega \to \mathbb{R}^+$, let $\mathbf{K}_f$ be a class of finite L_0-structures A satisfying that

- $A \in \mathbf{K}_0$;
- $\delta(A') > f(|A'|)$ for any $A' \subset A$.

For $A, B, C \in \mathbf{K}_0$ with $A = B \cap C$, B and C is said to be free over A (written $B \perp_A C$), if

$$R^{B \cup C} = R^B \cup R^C.$$

When $B \perp_A C$, a graph $B \cup C$ is called the free amalgam of B and C over A, and written $B \oplus_A C$.

Note 4 By the similar argument as in the Hrushovski construction [2], we can take an irrational $\alpha \in (0, 1]$ and a function $f : \omega \to \mathbb{R}^+$ such that

- f is unbounded;
- $(\mathbf{K}_f, \leq)$ has the free amalgamation property, i.e., whenever $A \leq B \in \mathbf{K}_f$ and $A \leq C \in \mathbf{K}_f$ then $B \oplus_A C \in \mathbf{K}_f$.

Moreover, we can assume that

- $f(0) = 0$ and $f(1) = 1$;
- $\alpha < 1/2$.

Note 5 From Since $f0) = 0$ it follows that $\mathrm{acl}(\emptyset) = \emptyset$. Moreover, from $f(1) = 1$ it follows that any 1-element is closed. If $abc \in \mathbf{K}_0$ satisfies $R(a, b) \wedge R(a, c)$, then we have $bc \leq abc$.

Definition 6 Let L consist of L_0 and countably many unary predicates $U_0, U_1, \ldots$. Let $\mathbf{K}$ be a class of finite L-structures A satisfying

- $A \in \mathbf{K}_f$;
- $U_0^A \subset U_1^A \subset \ldots$;
- For any $a, b \in A$ and $i \in \omega$, if $A \models R(a,b) \wedge U_i(b)$, then there is a $j < i$ with $A \models U_j(a)$.

Let $\overline{\mathbf{K}}$ be a class of (possibly infinite) L-structures N satisfying $A \in \mathbf{K}$ for any $A \subset_{\text{fin}} N$. For $A, B \in \overline{\mathbf{K}}$ with $A \subset B$, $A \leq B$ is defined by

$$A \cap X \leq B \cap X \text{ for every } X \subset_{\text{fin}} B.$$

For A, M with $A \subset M$, the closure of A in M is

$$\bigcap \{B : A \subset B \leq M\},$$

and it will be written $\mathrm{cl}_M(A)$.

Definition 7 A countable L-structure M is said to be a $(\mathbf{K}, \leq)$-generic structure, if it satisfies

1. if $M \in \overline{\mathbf{K}}$;
2. if $A \leq M$ and $A \leq B \in \mathbf{K}$, then there is a $B' \cong_A B$ with $B \leq M$;
3. if $A \subset_{\text{fin}} M$, then $\mathrm{cl}_M(A)$ is finite.

Since $(\mathbf{K}_f, \leq)$ has the free amalgamation property, so is $(\mathbf{K}, \leq)$. Therefore, there is a unique generic structure for $(\mathbf{K}, \leq)$.

In what follows, let M be the generic structure, and $\mathcal{M}$ a big model.

Notation 8 Let $\Sigma(x) = \{\neg U_i(x) : i \in \omega\}$;

For $n \geq 3$, $a_1 \ldots a_n \in \mathbf{K}$ is called n-cycle, if it satisfies $R(a_1, a_2) \wedge R(a_2, a_3) \wedge \cdots \wedge R(a_n, a_1)$.

Note 9 For the generic M, let $M_1 = \Sigma^M$ and $M_0 = M - M_1$. Then

1. M_0 has no cycles;
2. For each $n \geq 3$ and $a \in M_1$, there is an n-cycle containing a.

Proof: (1) Suppose that there would be a cycle. Then we can take a, b in the cycle such that $\models U_i(a) \wedge U_j(b) \wedge R(a,b)$ for some i, j with $j < i$. A contradiction.
(2) By Note 5, we have $a \leq M$. Take any cycle S with $a \in S$ and $\models \Sigma(b)$ for any $b \in S$. Then it can be seen that $a \leq S \in \mathbf{K}$. By genericity of M, we can assume that $S \subset M$.

Notation 10 Let $\pi(x) = \exists y \exists z \exists w[R(x,y) \wedge R(y,z) \wedge R(z,w) \wedge R(w,x)]$. For $A \subset \mathcal{M}$, let

- $p_1^A = \Sigma^A \cap \pi^A$;
- $p_2^A = \Sigma^A \cap (\neg\pi)^A$;
- $p_0^A = A - (p_1^A \cup p_2^A)$.

Note 11 It can be seen that $p_1^A = \pi^A$.

Note 12 Since f is unbounded, it can be seen that for a finite $A \subset \mathcal{M}$, $A \leq \mathcal{M}$ is definable, i.e., there is a formula $\theta(X) \in \mathrm{tp}(A)$ such that $\models \theta(A')$ implies $A' \leq \mathcal{M}$.

Notation 13 Let $\bar{a} \in \mathcal{M}$. Then

- let $\theta_{\bar{a}}(\bar{x})$ be a formula expressing that $\bar{x}$ is closed;
- let $\psi_{\bar{a}}(\bar{x})$ be a formula expressing that $\bar{x}$ and $\bar{a}$ are L_0-isomorphic;
- for $n \in \omega$, let $\alpha_{\bar{a}}^n(\bar{x}) = \bigwedge\{U_i(x_k)^{\text{if } \models U_i(a_k)} : i \leq n, a_k \in \bar{a} = a_0 ... a_m\}$;
- let $\beta_{\bar{a}}(\bar{x}) = \bigwedge\{\pi(x_k)^{\text{if } \models \pi(a_k)} : a_k \in \bar{a} = a_1 ... a_m\}$.

Notation 14 For $\bar{a} \leq \mathcal{M}$, let

$$\mathrm{qftp}^*(\bar{a}) = \{\theta_{\bar{a}}(\bar{x})\} \cup \{\psi_{\bar{a}}(\bar{x})\} \cup \{\alpha_{\bar{a}}^n(\bar{x})\}_{n \in \omega} \cup \{\beta_{\bar{a}}(\bar{x})\}$$

Lemma 15 Let $\bar{a}, \bar{a}' \leq \mathcal{M}$. If $\mathrm{qftp}^*(\bar{a}) = \mathrm{qftp}^*(\bar{a}')$ then $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{a})$.

Proof. It is enough to show that, for any finite A, A', B with $\mathrm{qftp}^*(A) = \mathrm{qftp}^*(A')$ and $A \le B \le \mathcal{M}$, there is $B' \le \mathcal{M}$ with $\mathrm{qftp}^*(A'B') = \mathrm{qftp}^*(AB)$. Assume otherwise. Then

$$\{\psi_{BA}(YA')\} \cup \{\alpha_B^n(Y)\}_{n\in\omega} \cup \{\beta_B(Y)\} \cup \{\theta_{AB}(A'Y)\}$$

is inconsistent. Let $\Gamma(X) = \mathrm{tp}(A')$. Then

$$\Gamma(X) \cup \{\psi_{BA}(YX)\} \cup \{\alpha_B^n(Y)\}_{n\in\omega} \cup \{\beta_B(Y)\} \cup \{\theta_{AB}(XY)\}$$

is inconsistent. By compactness, there are $\varphi(X) \in \Gamma(X)$ and $n \in \omega$ such that

$$\varphi(X) \wedge \psi_{BA}(YX) \wedge \alpha_B^n(Y) \wedge \beta_B(Y) \wedge \theta_{AB}(XY)$$

is inconsistent. We can assume that $\models \forall X[\varphi(X) \to (\psi_A(X) \wedge \theta_A(X))]$. Let $\gamma(XY)$ denote the formula above. Take $A^* \subset M$ with $M \models \varphi(A^*)$. Note that

$$\models \neg\exists Y \gamma(A^*Y).$$

On the other hand, since p_2^B has no cycles, we can take $B^* \in \mathbf{K}$ and an L_0-isomorphism $\sigma : BA \to B^*A^*$ satisfying

- $\sigma(A) = A^*$;
- for any $b \in p_0^B \cup p_1^B$ and $i \in \omega$, $B \models U_i(b)$ iff $B^* \models U_i(\sigma(b))$;
- for any $b \in p_2^B$, $n < \sup\{i \in \omega : B^* \models U_i(\sigma(b))\} < \omega$.

Since $A^* \le B^* \in \mathbf{K}$ and $A^* \le \mathcal{M}$, by genericity of M, we can assume that $B^* \le M$. Then it can be seen that $M \models \gamma(A^*B^*)$. A contradiction.

Corollary 16 T is small.

For $\bar{a} \in \mathcal{M}$, a dimension of $\bar{a}$ is defined by $d(\bar{a}) = \delta(\mathrm{cl}(\bar{a}))$. For $\bar{a}, \bar{b} \in \mathcal{M}$, $d(\bar{a}/\bar{b})$ will denote $d(\bar{a}\bar{b}) - d(\bar{b})$. For a (possibly infinite) $B \subset \mathcal{M}$, we define $d(\bar{a}/B) = \inf\{d(\bar{a}/B_0) : B_0 \subset_{\mathrm{fin}} B\}$.

Fact 17 ([3]) Let $A, B, C \subset \mathcal{M}$ with $A = B \cap C$ and $B, C \le \mathcal{M}$. Then the following are equivalent:

1. $B \perp_A C$ and $BC \le \mathcal{M}$;
2. $d(B/C) = d(B/A)$.

Lemma 18 T is stable.

Proof. Take λ with $\lambda^{\aleph_0} = \lambda$. Take any $B \le \mathcal{M}$ with $|B| \le \lambda$. We want to show that $|S(B)| \le \lambda$. Take any $\bar{e} \in \mathcal{M} - B$. By the definition of the dimension d, there is a countable $A \le B$ with

$$d(\bar{e}/B) = d(\bar{e}/A) \text{ and } \mathrm{cl}(\bar{e}A) \cap B = A.$$

Take any $\bar{e}' \models \mathrm{tp}(\bar{e}/A)$ with

$$d(\bar{e}'/B) = d(\bar{e}'/A) \text{ and } \mathrm{cl}(\bar{e}'A) \cap B = A.$$

Since $\mathrm{tp}(\bar{e}/A) = \mathrm{tp}(\bar{e}'/A)$, we have

$$\mathrm{qftp}^*(\mathrm{cl}(\bar{e}A)/A) = \mathrm{qftp}^*(\mathrm{cl}(\bar{e}'A)/A).$$

On the other hand, by Fact 17, we have

$$\mathrm{cl}(\bar{e}A) \perp_A B \text{ and } \mathrm{cl}(\bar{e}'A) \perp_A B.$$

Therefore we have

$$\mathrm{qftp}^*(\mathrm{cl}(\bar{e}A)/B) = \mathrm{qftp}^*(\mathrm{cl}(\bar{e}'A)/B).$$

Again by Fact 17,

$$\mathrm{cl}(\bar{e}A)B, \mathrm{cl}(\bar{e}'A)B \le \mathcal{M}.$$

Then we have $\mathrm{tp}(\bar{e}/B) = \mathrm{tp}(\bar{e}'/B)$. Therefore any type over B is determined by a type of over some countable $A \subset B$. Hence

$$|S(B) \le \lambda^{\aleph_0} \times 2^{\aleph_0} = \lambda.$$

Therefore T is λ-stable.

Fact 19 ([3]) Let $A, B, C \subset_{\text{fin}} \mathcal{M}$ with $A = B \cap C$, $\mathrm{acl}(A) = A$ and $B, C \le \mathcal{M}$. If $B \perp_A C$ and $BC \le \mathcal{M}$ then $B \downarrow_A C$.

Lemma 20 T has a type with the tree property.

Proof. Take any $a \in \mathcal{M}$ with $\models \Sigma(a)$ and $\models \neg\pi(a)$. Since any 1-element is closed, we have

$$\Sigma(x) \cup \{\neg\pi(x)\} \vdash \mathrm{qftp}^*(a) \vdash \mathrm{tp}(a).$$

Let $p(x) = \mathrm{tp}(a)$. We want to show that p has the tree property. By compactness, we can take $b, c \models p$ with $\models R(a,b) \wedge R(a,c) \wedge \neg R(b,c)$ and $abc \leq \mathcal{M}$. First, we show that

$$b \downarrow c.$$

Note that $\mathrm{acl}(\emptyset) = \emptyset$. Since $\delta(a/bc) = 1 - 2\alpha > 0$, by Note 5, we have

$$bc \leq abc \leq \mathcal{M}.$$

Note that

$$b \perp c.$$

By Fact 19, we have $b \downarrow c$. Next, we show that

$$\mathrm{tp}(bc/a) \text{ is isolated.}$$

Let

$$\varphi(yz, a) = R(a,y) \wedge R(a,z) \wedge \neg\pi(y) \wedge \neg\pi(z) \wedge \neg R(y,z) \wedge \theta_{abc}(ayz).$$

Take any $b'c' \models \varphi$. Since $\models \Sigma(a)$ and $\models R(a,b') \wedge R(a,c')$, we have

$$\models \Sigma(b') \text{ and } \models \Sigma(c').$$

SInce $\models \varphi(b'c', a)$, we have

$$\models \neg R(b',c') \wedge \neg\pi(b') \wedge \neg\pi(c') \text{ and } ab'c' \leq M.$$

Then

$$\mathrm{qftp}^*(abc) = \mathrm{qftp}^*(ab'c').$$

By Lemma 15, we have

$$\mathrm{tp}(b'c'/a) = \mathrm{tp}(bc/a).$$

It follows that $\mathrm{tp}(bc/a)$ is isolated.

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