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# Model completeness revisited

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**Abstract**

We review two theorems concerning the model completeness; the first one is the real numbers with the exponentiation and the second one is differential fields admitting quantifier elimination without being differentially closed.

## 1 Introduction

In model theory analysis of the properties of definable sets is very important. If a theory admits elimination of quantifiers, then definable sets are definable by quantifier free formulas. Hence definabl sets are easy to handle.

Classical examples of theories admitting elimination of quantifiers are

(1) the first-order theory of $(\mathbb{R}, +, -, \cdot, <, 0, 1)$,

(2) the first-order theory of $(\mathbb{C}, +, -, \cdot, 0, )$.

If the theory does not admit elimination of quantifiers, then the next best situation is to bound the complexity of defining formulas to existential formulas. Those theories are called model complete. In this note we discuss two topics concerning model completeness.

## 2 Model completeness

The notion of model completeness is introduced by Abraham Robinson;

- In 1950, at ICM, A. Robinson gave a proof to Hilbert Nullstellensatz as an application of model completeness.
- He also gave an alternative proof to Hilbert 17th Problem using model completeness.
- Non-standard analysis is also originated by him in early 1960's.

First recall the definition. $\mathcal{L}$ is a language, $T$ is a $\mathcal{L}$-theory.

**Definition 1** *The theory $T$ is called model complete if for all models $M$ of $T$,*

$$T \cup Diagram_0(M)$$

*is a complete $L(M)$-theory.*

Let $\mathcal{L}$ be a first-order language, $T$ is a $\mathcal{L}$-theory.

**Theorem 2** The following are equivalent;

1. $T$ is model complete;
2. for all models $M$ and $N$, if $M \subset N$ then $M \preccurlyeq N$;
3. for all models $M$ and $N$, if $M \subset N$ then $M \preccurlyeq_1 N$;
4. for every $\mathcal{L}$-formula $\varphi(x)$, there exists an existential $\mathcal{L}$-formula $\psi(x)$ such that $T \models \forall x\, (\varphi(x) \longleftrightarrow \psi(x))$

The following is clear.

**Theorem 3** *If $T$ admits quantifier elimination then $T$ is model complete.*

# 3 Infinitesimals and $T_{\exp}$

Tarski proved that $\mathrm{Th}(\mathbb{R}, +, -, \cdot, <, 0, 1)$ admits elimination of quantifiers, hence the theory is model complete. The structure $(\mathbb{R}, +, -, \cdot, < 0, 1)$ (denoted $\overline{\mathbb{R}}$) is a typical example of o-minimal structure, i.e., any definable subset of $\mathbb{R}$ is the union of a finite set and a finite union of intervals.

In 1991 A. Wilkie proved a theorem stating that $\mathrm{Th}(\mathbb{R}, +, -, \cdot, < .0, 1, \exp x)$ is model comlete which then implies the o-minimality of the theorem. The theorem was a major breakthrough for the subject at that time. His proof uses infinitesimals.

## 3.1 Model completeness of $T_{\exp}$

Let $T_{\exp}$ be the theory of $(\mathbb{R}, +, \cdots, <, 0, 1, \exp x)$. From the model completeness of $T_{\exp}$, we get the o-minimality of $T_{\exp}$.

- $T_{\exp}$ does not admit elimination of quantifiers.
- However $\mathrm{Th}(\mathbb{R}, +, \cdot, <, 0, 1, \exp, \log, f \in An)$ admits elimination of quantifiers, where $An$ denotes the set of all restricted real analytic functions, [vDMM].

### 3.1.1 Quasipolynomials

**Definition 4** *$\mathbb{M}_0$ is a substructure of a model $\mathbb{M}$ of $T_{\exp}$.*
*A function $M^n$ to $M$*

$$(x_1, \cdots, x_n) \mapsto P(x_1, \cdots, x_n, e^{x_1}, \cdots, e^{x_n})$$

*where $P$ is a polynomial in $2n$ variables with coefficients in $M_0$ is called a quasipolynomial with coefficients in $M_0$.*

### 3.1.2 Key lemma

Suppose:

- $\mathbb{M}_0, \mathbb{M}$: models of $T_{\exp}$ with $\mathbb{M}_0 \subseteq \mathbb{M}$
- $M_0, M$ domains (i.e., underlying sets), respectively

In order to show the model completeness of $T_{\exp}$;

**Lemma 5** *it is sufficient to show that for any $F : M^n \to M$ is a quasipolynomial with coefficients in $M_0$, and $b \in M^n$ is a non-singular solution to $F(x) = 0$, i.e.,*

$$F(b) = 0 \text{ and } J_F(b) \neq 0$$

*then $b \in M_0^n$, where $J_F(b)$ is the Jacobian at $b$.*

### 3.1.3 Gavrielov's theorem

**Theorem 6 (Gabrielov)** *$\tilde{\mathbb{R}}$ is the reduct of $\mathbb{R}_{\mathrm{an}}$ such that for each resricted analytic function $\tilde{f}$ in the language, $\dfrac{\partial \tilde{f}}{\partial x_j}$ is also in the language for each $j$. Then $\mathrm{Th}(\tilde{\mathbb{R}})$ is model complete in the language $\mathcal{L}(\tilde{\mathbb{R}})$.*

**Corollary 7** $\mathrm{Th}(\mathbb{R}_{\mathrm{rexp}})$ is model complete, where rexp means that the exponentiation is restricted to (0, 1). It follows that $\mathrm{Th}(\mathbb{R}_{\mathrm{rexp}})$ is o-minimal.

### 3.1.4 Khovanski's theorem

**Theorem 8 (Khovanski)** *Let $f_1, \cdots, f_m$ be quasipolinomials from $\mathbb{R}^n$ to $\mathbb{R}$. Then the regular zero set of $f_1, \cdots, f_m$, i.e., $\{x \in \mathbb{R}^n : f_1(x) = \cdots = f_m(x) = 0\}$, is finite and can be bounded uniformly in the complexity of $f_i'$s.*

### 3.2 Outline of the proof of Lemma 5

We show that Lemma 5 is true by contradiction. Infinitesimals play an important role in the proof of model completeness of $T_{\exp}$.

Recall $\mathbb{M}_0 \subseteq \mathbb{M} \models T_{\exp}$. Let

$$Fin(\mathbb{M}) := \{a \in M : \exists N \in \mathbb{Q}(|a| < N)\},$$

$$\mu(\mathbb{M}) := \{a \in M : \forall q \in \mathbb{Q}_{>0}(|a| < q)\}.$$

$Fin(\mathbb{M})$ is a subring of $\mathbb{M}$ and $\mu(\mathbb{M})$ is the unique maximal ideal of $Fin(\mathbb{M})$. Hence $Fin(\mathbb{M})/\mu(\mathbb{M})$ is a field, called the residue field of $\mathbb{M}$. We define a valuation group $\langle \Gamma, <, +, 0\rangle$ of $\mathbb{M}$ and a valuation map $v : M \to M \setminus \{0\} \to \Gamma$ with the following property. For $a, b \in M \setminus \{0\}$,

(i) $\nu(a \cdot b) = \nu(a) + \nu(b)$;

(ii) $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$;

(iii) $\nu(a) = 0$ if and only if $a \in Fin(\mathbb{M}) \setminus \mu(\mathbb{M})$.

Properties of the valuation.

- Let $\varepsilon$ be an infinitesimal. Then, $\dfrac{1}{\varepsilon}$ is an infinite. Since $\varepsilon \cdot \dfrac{1}{\varepsilon} = 1$, we have $v\left(\dfrac{1}{\varepsilon}\right) = -v(\varepsilon)$.
- Let $\varepsilon$ be an infinitesimal, and $a \in Fin(\mathbb{M})$. Then $v(a\varepsilon) = v(a) + v(\varepsilon) = v(\varepsilon)$.

We also use the following inequality:

$$rank(\mathbb{M}) \geq resrank(\mathbb{M}) + \dim_{\mathbb{Q}}(\Gamma),$$

where $resrank(\mathbb{M})$ denotes the rank of residue field.

(1) First we argue that it is enough to show Lemma 5 in $T_{\text{rexp}}$ where rexp denotes the exponential function restricted to $(0, 1)$. It is known that $T_{\text{rexp}}$ is model complete.

(2) We then argue that it is enough to work with $b$ with coordinates $b_1, \cdots, b_n$ such that all of $b_i$ are in $Fin(M)$.

(3) We use infinitesimals i.e., elements in $\mu(M)$ to reduce to the previous case. (here we use valuation theory and the property of "independence" of infinitesimals)

## 4 Differential fields

All fields are of characteristic zero. The notion of differential fields was introduced by Ritt in the thirties.

**Definition 9** $(K, \delta)$ is a differential field if $K$ is a field and $\delta$ is a derivation on $K$, i.e., for $x, y \in K$

- $\delta(x + y) = \delta(x) + \delta(y)$,
- $\delta(xy) = \delta(x)y + x\delta(y)$.

- $C = \{x \in K : \delta(x) = 0\}$, the field of constants.
- Language of differential fields $= \{+, -, \cdot, \delta, 0, 1\}$

### 4.1 Differentially closed fields

Historically it took for a while before reaching the right definition of differentially closed fields.

**Definition 10** *Let $(K, \delta)$ be a differential field. The polynomial ring*

$$K[X_1, \cdots, X_n, \delta(X_1), \cdots, \delta(X_n), \cdots, \delta^m(X_1), \cdots, \delta^m(X_n), \cdots]$$

*is called the ring of differential polynomials over $K$, and denoted $K\{X_1, \cdots, X_n\}$.*

**Definition 11** *A differentially closed field $K$ is differentially closed if, for any $f, g \in K\{X\}$ with $g$ be nonzero,* $\operatorname{ord}(f) > \operatorname{ord}(g)$, *there is $a \in K$ such that $f(a) = 0$ and $g(a) \neq 0$.*

## 4.2 Existentially closed differential fields

**Definition 12** *A differentially closed field $K$ is existentially closed, if for any $f_1, \cdots f_m \in K\{X_1, \cdots, X_n\}$ there is a differental field $L \supset K$ containing a solution to the system of differentail eqauations $f_1 = \cdots = f_m = 0$, there is already a solution in $K$.*

## 4.3 Basic properties (1)

**Theorem 13** Let $K$ be a differential field. TFAE.

1. $K$ is differentially closed.
2. $K$ is existentailly closed.
3. $K$ is algebraically closed and for every irreducible algebraic variety $V \subseteq K^n$, if $W$ is an irreducible subvarity of $V^{(1)}$ such that the projection of $W$ onto $V$ is Zariski dense in $V$ and $U$ is a Zariski open subset of $V$, then $(x, \delta(x)) \in U$ for some $x \in V$.

## 4.4 Basic properties (2)

DCF is the theory of differentially closed fields.

**Theorem 14** *DCF is $\omega$-stable, admits QE (hence model complete), complete.*

**Proposition 15** *Let $K$ be a differentail field.*

$$K \text{ is existentially closed} \iff K \text{ is differentially closed.}$$

**Lemma 16** *$K, L$ are $\omega$-saturated models of DCF. Assume*

- *$\overline{a} \in K, \overline{b} \in L$, and $k = \mathbb{Q}\langle\overline{a}\rangle, \ell = \mathbb{Q}\langle\overline{b}\rangle$.*
- *$\sigma : k \to \ell$ is an iso such that $\sigma(\overline{a}) = \overline{b}$.*

*Then, for all $\alpha \in K$, there is an extension of $\sigma$ to an iso $\sigma^*$ from $k\langle\alpha\rangle$ into $L$.*

## 4.5 DCF admits QE

Here we show that DCF admits elimination of quantifiers.
**Proof:** Suffices to show: assume

- $K, L \models DCF$,
- $k \subseteq K,\ k \subseteq L,\ \overline{a} \in k,\ b \in K$,
- $\varphi(v, \overline{w})$ is quantifier free, and
- $K \models \varphi(b, \overline{a})$.

Then $L \models \exists v \varphi(v, \overline{a})$.

WLOG, assume 1) $K, L$ are $\omega$-saturated, 2) $k$ is the diff. field generated by $\overline{a}$. By the previous lemma we can find $\beta \in L$ such that $k\langle b\rangle \cong k\langle\beta\rangle$. Thus $L \models \varphi(\beta, \overline{a})$. So, $L \models \exists v \varphi(v, \overline{a})$. ■

Let $F$ be an infinite field. The following theorem tells us that the field $F$ being algebraically closed and the theory of $F$ admitting quantifier elimination are equivalent:

**Theorem 17** *If $F$ is algebraically closed, then the theory of $F$ admits elimination of quantifiers. Moreover if the theory of $(F, +, \cdot, 0, 1)$ admits elimination of quantifiers, then the field $F$ is algebraically closed.*

For differential fields, being differentially closed implies its theory admits qunantifier elimination. So it is vary natural to consider the following question.

**Question 18** *Suppose the theory of a differential field $K$ admits QE. Is it necesary that the differential field $K$ is differentially closed.*

## 4.6 Need to understand strongly minimal sets.

**Definition 19** *A definable set $X$ is strongly minimal if for any definable set $Y$, either $X \cap Y$ is finite, or $X - Y$ is finite.*

Study of geometric properies of strongly minimal sets in differential fields is the key.

## 4.7 QE, but not differentially closed

First we need.

**Proposition 20 (Prop 2.1 of HI)**
- *$C$ is a curve of genus $\geq 1$.*
- *$X$ is a strongly minimal subset of $C$ with induced trivial geometry on $X$. (called strictly minimal)*
- *$Y_b$ is a definable family of Kolchin-closed definable sets of finite differential order.*
  *THEN: The set of $b$ such that $X$ is orthogonal to the generic type of $Y_b$ is definable.*

## 4.8 $T(X)$ admmits QE, but not DCF.

By the previous Proposition we may find a curve $C$ and a set $X$ defined over $\mathbb{Q}$. Suppose $X$ is defined by $\varphi(v)$. We define the theory $T(X)$ consisting the theory of differential fields, admitting elinimation of quantifiers but is not differentially closed.

**Definition 21 (Definition of $T(X)$)** *The universal part $T_\forall$ of $T(X)$ consists of*

- *DF of charcteristic 0,*
- *$X$ has no solution, i.e., $\neg\exists x\, \varphi(x)$, and*
- *$X$ has no solution even in the algebraic closure.*

*The rest of $T(X)$ consists of the first-order data making $T_\forall$ to admit QE.*

## 4.9 First-order data making $T_\forall$ to admit QE

**Definition 22 (Definition of $T(X)$ continued)**
- *for each $U, W$ differential-algebraic varieties over $\mathbb{Q}$, $\pi : U \to W$ such that $U_b := \pi^{-1}(b)$ has finite differential order for each $b \in W$.*
  *$W_1 = \{b : U_b \text{ is irreducible}\}$ is a quantifier-free definable set in DCF*
- *Let $p_b$ denote the generic type of $U_b$ for $b \in W_1$.*
- *$W_2 := \{b \in W_1 : p_b \text{ is orthogonaltoX}\}$ is also quantifier-free definable by $\theta(w)$*
- *AXIOM: $\forall w \in W\Big(\theta(w) \to \exists u \in U(\pi(u) = w)\Big)$*

## 4.10 Proof goes as follows:

- $T_\forall$ is consitent and any model of $T_\forall$ can be extended to a model of $T(X)$. Hence $T$ is consistent.
- By definition of $T(X)$, $T_\forall$ admits QE.
- $T(X)$ is a model completion of $T_\forall$.
- It follows that $T(X)$ admits QE.
- By definition, models of $T(X)$ are not differentially closed.

## 4.11 Zariski geometry and Geometric Mordell-Lang conjecture

Strongly minimal sets in DCF form Zariski geometries and those Zariski geometries are used for the alternative solution to the geometric Mordell-Lang conjecture for characteristic zero case.

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