数理解析研究所講究録
第2080巻 2018年 43-52

On an ODE related to the stationary problem of a reaction-diffusion equation on a thin domain

Toru Kan
Department of Mathematics, Tokyo Institute of Technology, Japan

1 Introduction

We are concerned with the boundary value problem

$$\begin{cases} u_{xx}+\lambda f(u)=0, & x\in(-1,1)\setminus\{0\}, \\ u_x=0, & x=-1,1, \\ u(-0)+au_x(-0)=u(+0)-au_x(+0), & \\ u_x(-0)=u_x(+0), & \end{cases} \tag{1.1}$$

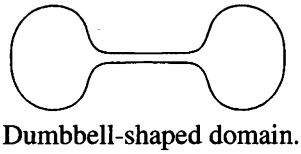
where $\lambda>0$ is a bifurcation parameter, $a>0$ is a fixed constant and the symbols -0 and $+0$ stand for the limits from the left and right, respectively. f is assumed to be a bistable nonlinearity (e.g., $f(u)=u-u^3$) and precise assumptions will be made later. Our interest is the solution structure of (1.1) in the bifurcation diagram.

1.1 Background

The motivation comes from a scalar reaction-diffusion equation

$$\begin{cases} u_t=\Delta u+\lambda f(u), & x\in\Omega, \\ \dfrac{\partial u}{\partial\nu}=0, & x\in\partial\Omega. \end{cases} \tag{1.2}$$

Here $\Omega\subset\mathbb{R}^N$ $(N\geq 2)$ is a bounded domain and $\partial/\partial\nu$ stands for the outward normal derivative. One of fundamental problems in the study of reaction-diffusion equations is the existence and stability of nonconstant stationary solutions, because stable nonconstant stationary solutions correspond to spatial patterns. In scalar equations, it is known that not only the properties of a nonlinearity, but also the shape of a domain is important for the existence of such solutions. In fact, it was shown by Casten and Holland [1] and Matano [5], independently, that (1.2) does not have any stable nonconstant stationary solutions if Ω is convex, and Matano [5] found that such a solution indeed exists if f is a bistable nonlinearity and Ω is a dumbbell-shaped domain which are chosen suitably.



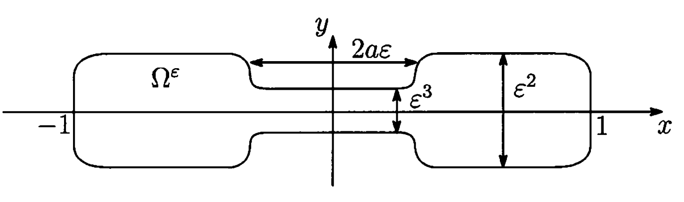
Dumbbell-shaped domain.

Now let us fix a bistable nonlinearity f and a dumbbell shaped domain Ω. Then (1.2) will have a stable nonconstant stationary solution if λ is not small, while (1.2) cannot have it if λ is too small. Therefore a stable nonconstant stationary solution must appear through a bifurcation. It is seen that a nonconstant stationary solution bifurcating from a constant solution is always unstable near the bifurcation point. This means that we need global information on the bifurcation diagram to know how stable nonconstant stationary solutions appear. One of the ways to overcome this difficulty is to focus on solutions which converge to a constant on each weight of the dumbbell as the thickness of the channel of the dumbbell tends to zero. This way enables us to reduce the equation (1.2) into a finite-dimensional equation which is much easier to handle. In this direction, Vegas [10] studied the bifurcations of stationary solutions. He obtained a reduced equation by applying the Lyapunov-Schmidt reduction and observed that stable nonconstant stationary solutions appear through a secondary bifurcation if both the nonlinearity and the domain are symmetric (see also [3]). Fang [2] and Morita [7] discussed the stability of solutions from the viewpoint of dynamical systems. They constructed a finite-dimensional invariant manifold which contain stationary solutions and revealed how stable nonconstant stationary solutions appear by analyzing the flows on the manifold (see also [6, 8]).

The equation (1.1) is another type of reduced equation. Indeed, for the domain Ω^ε shown in the figure below, (1.1) is obtained as the limiting equation of the stationary problem

$$\begin{cases} \Delta u + \lambda f(u) = 0, & x \in \Omega^\varepsilon, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega^\varepsilon \end{cases} \tag{1.3}$$

as $\varepsilon \to 0$. The width and the length of the channel of Ω^ε are ε^3 and $2a\varepsilon$, respectively, and the widths of the weights are ε^2. The domain Ω^ε converges to the interval $(-1, 1)$ while the channel shrinks to the origin as $\varepsilon \to 0$. The limiting equation (1.1) is relatively easier to handle than (1.3) and, in comparison with finite-dimensional reductions, it would have much information on the solution structure of (1.3).



Thin tubular dumbbell-shaped domain.

Let us formally derive the equation (1.1). We only focus on the derivation of the last two conditions. Assume that $u^\varepsilon = u^\varepsilon(x, y)$ is a solution of (1.3) with the property that $u^\varepsilon(x, y) = O(1)$ as $\varepsilon \to 0$. Since the domain is thin in the vertical direction, we can expect that u^ε is approximated by some function depending only on x, that is, $u^\varepsilon(x, y) = v^\varepsilon(x) + o(1)$ for some v^ε as $\varepsilon \to 0$. Since the length of the channel is $O(\varepsilon)$, it is reasonable to assume that $v_x^\varepsilon(x) = O(1/\varepsilon)$ and $u_x^\varepsilon(x,y) = v_x^\varepsilon(x)+o(1/\varepsilon)$ if (x,y) lies on the channel, and $v_x^\varepsilon(x) = O(1)$ and $u_x^\varepsilon(x,y) = v_x^\varepsilon(x)+o(1)$ otherwise.. As shown in the figure below, we take vertical line segments Γ_-^ε, Γ_+^ε and Γ_0 lying on the left weight, right weight and channel. x_-^ε, x_+^ε and x_0 denote the positions of Γ_-^ε, Γ_+^ε and Γ_0, and x_-^ε and x_+^ε are assumed to converge to 0 as $\varepsilon \to 0$. Integrating both sides of (1.3) over R^ε which is surrounded by Γ_+^ε, Γ_0 and $\partial\Omega^\varepsilon$, we have

$$\begin{aligned} 0 &= \int_{R^\varepsilon} (\Delta u^\varepsilon + \lambda f(u^\varepsilon))dx = \int_{\Gamma_+^\varepsilon} u_x^\varepsilon dx - \int_{\Gamma_0} u_x^\varepsilon dx + \lambda \int_{R^\varepsilon} f(u^\varepsilon)dx \\ &= v_x^\varepsilon(x_+^\varepsilon)\varepsilon^2 - v_x^\varepsilon(x_0)\varepsilon^3 + o(\varepsilon^2). \end{aligned} \tag{1.4}$$

In a similar manner, integrating over the region surrounded by Γ_-^ε, Γ_0 and $\partial\Omega^\varepsilon$ gives

$$0 = -v_x^\varepsilon(x_-^\varepsilon)\varepsilon^2 + v_x^\varepsilon(x_0)\varepsilon^3 + o(\varepsilon^2). \tag{1.5}$$

Adding (1.4) to (1.5) leads to

$$v_x^\varepsilon(x_-^\varepsilon) = v_x^\varepsilon(x_+^\varepsilon) + o(1),$$

and hence we have the last condition in (1.1) by letting $\varepsilon \to 0$.

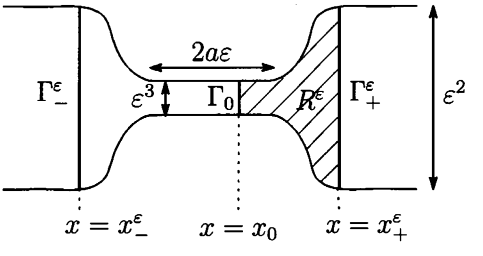
To get the other condition, we subtract (1.4) from (1.5). Then we have

$$v_x^\varepsilon(x_0) = \frac{v_x^\varepsilon(x_+^\varepsilon) + v_x^\varepsilon(x_-^\varepsilon) + o(1)}{2\varepsilon},$$

which yields

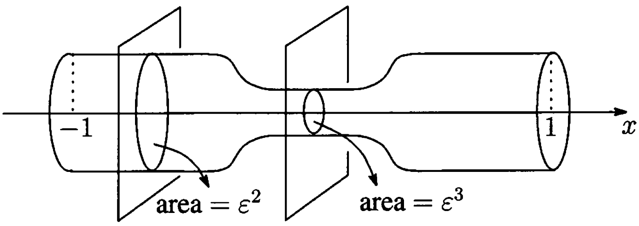
$$v^\varepsilon(x_+^\varepsilon) - v^\varepsilon(x_-^\varepsilon) = \int_{x_-^\varepsilon}^{x_+^\varepsilon} v_x^\varepsilon dx = \int_{-a\varepsilon}^{a\varepsilon} v_x^\varepsilon dx + o(1) = a(v_x^\varepsilon(x_+^\varepsilon) + v_x^\varepsilon(x_-^\varepsilon)) + o(1).$$

Letting $\varepsilon \to 0$, we obtain the desired condition.



Enlarged view of the channel.

We can also deal with higher-dimensional domains. In the higher-dimensional case, width should be replaced with cross sectional area.



Thin tubular dumbbell-shaped domain in a higher-dimensional space.

1.2 Main result

To state our main result, we specify the assumptions on f and introduce notation. Here and subsequently, we always assume the following.

$$\begin{cases} f \in C^2(\mathbb{R}), \\ f(-1) = f(0) = f(1) = 0,\ f'(-1) < 0,\ f'(0) > 0,\ f'(1) < 0, \\ (u - u^3)f(u) > 0 \quad \text{for } u \in \mathbb{R} \setminus \{-1, 0, 1\}, \\ f(u) = -f(-u) \quad \text{for } u \in [-1, 1]. \end{cases}$$

In order to obtain the detailed properties of solutions, we will additionally make one of the following assumptions if needed.

$$\frac{f(u)}{u} > f'(u) \quad \text{for } u \in (-1, 1) \setminus \{0\}, \tag{1.6}$$

$$u\frac{d}{du}\left\{\frac{f'(u)}{f(u)^3}\left(\int_0^u f(s)ds\right)^{\frac{3}{2}}\right\} < 0 \quad \text{for } u \in (-1, 1) \setminus \{0\}. \tag{1.7}$$

One can show that (1.7) implies (1.6) and that the cubic nonlinearity $f(u) = u - u^3$ satisfies all of the above assumptions.

Let u be a solution of (1.1) and consider the linearized eigenvalue problem

$$\begin{cases} \varphi_{xx} + \lambda f'(u)\varphi = \mu\varphi, & x \in (-1, 1) \setminus \{0\}, \\ \varphi_x = 0, & x = -1, 1, \\ \varphi(-0) + a\varphi_x(-0) = \varphi(+0) - a\varphi_x(+0), & \\ \varphi_x(-0) = \varphi_x(+0). & \end{cases}$$

It is known that eigenvalues are real and simple, and accumulate only at $-\infty$. We denote the k-th largest eigenvalue by $\mu_k(u)$ and say that u is nondegenerate if $\mu_k(u) \neq 0$ for all $k = 1, 2, \ldots$. The number of positive eigenvalues is called the Morse index of u and is denoted by $i(u)$.

Our main result is the following.

Theorem 1. *Bifurcation points on the trivial branch* $\{(\lambda, u); u \equiv 0\}$ *are given by* $p_k^0 := (\lambda_k^0, 0)$, $k = 1, 2, \ldots$, *where*

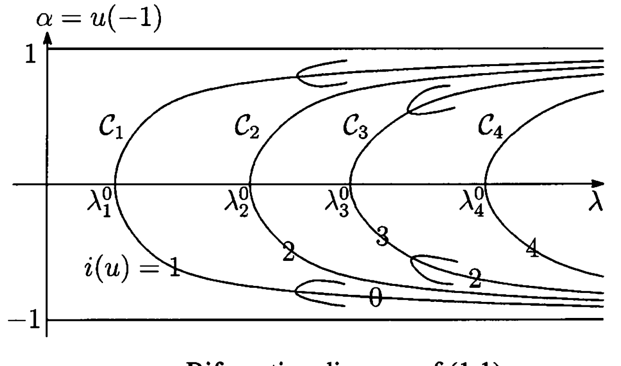
$$\lambda_k^0 := \begin{cases} \left(\dfrac{z_{(k-1)/2}}{f'(0)}\right)^2 & \text{if } k \text{ is odd,} \\ \left(\dfrac{k\pi}{2f'(0)}\right)^2 & \text{if } k \text{ is even} \end{cases} \tag{1.8}$$

and z_l *is a unique root of the equation* $az \tan z = 1$ *in* $(l\pi, (l + 1/2)\pi)$. *From each point* p_k^0, *a solution branch* $C_k = \{(\lambda_k(\alpha), u_k(\cdot; \alpha))\}_{\alpha \in (-1,1)}$ *emanates and it has the following properties.*

(i) $u_k(-1; \alpha) = (-1)^k u_k(1; \alpha) = \alpha$, $u_k(x; \alpha) = (-1)^k u_k(-x; \alpha) = -u_k(x; -\alpha)$, $\lambda_k(0) = \lambda_k^0$, $\lambda_k(\alpha) = \lambda_k(-\alpha)$ *and* $\lambda_k(\alpha) \to \infty$ *as* $\alpha \to \pm 1$.

(ii) Assume that k *is even and that* (1.6) *holds. Then* $u_k(\cdot; \alpha)$ *is nondegenerate and* $i(u_k(\cdot; \alpha)) = k$ *for all* $\alpha \in (-1, 1) \setminus \{0\}$. *In particular, there is no bifurcation point on* $C_k \setminus \{p_k^0\}$.

(iii) Assume that k *is odd and that* (1.7) *holds. Then there exists* $\alpha_k^* \in (0, 1)$ *such that* $p_k^+ := (\lambda_k(\alpha_k^*), u_k(\cdot; \alpha_k^*))$ *and* $p_k^- := (\lambda_k(-\alpha_k^*), u_k(\cdot; -\alpha_k^*))$ *are bifurcation points from which solution branches emanate. Furthermore,* $u_k(\cdot; \alpha)$ *is nondegenerate if* $0 < |\alpha| \neq \alpha_k^*$, $i(u_k(\cdot; \alpha)) = k$ *if* $|\alpha| < \alpha_k^*$ *and* $i(u_k(\cdot; \alpha)) = k - 1$ *if* $\alpha_k^* \leq |\alpha| < 1$. *In particular, there is no bifurcation point on* $C_k \setminus \{p_k^0, p_k^+, p_k^-\}$.



Bifurcation diagram of (1.1).

The remaining sections are devoted to discussing how to analyze (1.1).

2 The shooting method

We analyze (1.1) by the shooting method. For $\alpha_1 \in (-1, 1)$ and $\alpha_2 \in (-1, 1)$, we define $u_1 = u_1(x; \alpha^-)$ and $u_2 = u_2(x; \alpha_2)$ to be solutions of the equation $u_{xx} + \lambda f(u) = 0$ with the initial conditions $u_1(-1) = \alpha_1$, $(u_1)_x(-1) = 0$, $u_2(1) = \alpha_2$ and $(u_2)_x(1) = 0$. Then we see that (1.1) admits a solution u satisfying $u(-1) = \alpha_1$ and $u(1) = \alpha_2$ if and only if the pair (α_1, α_2) satisfies

$$\begin{cases} u_1(0;\alpha_1) + a(u_1)_x(0;\alpha_1) = u_2(0;\alpha_2) - a(u_2)_x(0;\alpha_2), \\ \qquad\qquad (u_1)_x(0;\alpha_1) = (u_2)_x(0;\alpha_2), \end{cases} \tag{2.1}$$

and if (2.1) holds, then the solution u is given by

$$u(x) = \begin{cases} u_1(x;\alpha_1) & (x \in [-1,0]), \\ u_2(x;\alpha_2) & (x \in (0,1]). \end{cases} \tag{2.2}$$

To get the representation of the solutions u_1 and u_2, we introduce a solution $(U, V) = (U(x;\alpha), V(x;\alpha))$ of the initial value problem

$$\begin{cases} U_x = V, \quad V_x = -f(U), \qquad x \in \mathbb{R}, \\ (U(0;\alpha), V(0;\alpha) = (\alpha, 0), \end{cases}$$

where $\alpha \in (-1,1)$. Then we have

$$\begin{aligned} u_j(x;\alpha_j) &= U\left(\sqrt{\lambda}((-1)^{j-1}x+1);\alpha_j\right), \\ (u_j)_x(x;\alpha_j) &= (-1)^{j-1}\sqrt{\lambda}V\left(\sqrt{\lambda}((-1)^{j-1}x+1);\alpha_j\right) \end{aligned} \tag{2.3}$$

for $j = 1, 2$. Set $F(u) = 2\int_0^u f(s)ds$ and $\beta_0 := \sqrt{F(-1)} = \sqrt{F(1)}$, and define a function $G : (-\beta_0, \beta_0) \to (-1,1)$ to be the inverse of the function

$$u \mapsto v = \begin{cases} \sqrt{F(u)} & (u \in (-1,0]), \\ -\sqrt{F(u)} & (u \in (0,1)). \end{cases}$$

One can check that $G \in C^2((-\beta_0,\beta_0)) \cap C^3((-\beta_0,\beta_0) \setminus \{0\})$. By putting $\alpha = G(\beta)$ and $U = G(W)$, the periodic orbit determined by $V^2 + F(U) = F(\alpha)$ transforms into the circle given by $V^2 + W^2 = \beta^2$. Hence (U, V) must be of the form

$$(U, V) = (G(W), V) = (G(\beta\cos\Theta), -\beta\sin\Theta). \tag{2.4}$$

Since $U_x = V$, we see that $\Theta = \Theta(x;\beta)$ is determined by $\int_0^\Theta G'(\beta\cos\tau)d\tau = x$. The representation formula as in (2.4) is helpful in the analysis of time maps (for instance, see [9, 4]).

Substituting (2.3) and (2.4) into (2.1), we obtain the equation

$$(P(\beta_1;\lambda), Q(\beta_1;\lambda)) = (P(\beta_2;\lambda), -Q(\beta_2;\lambda)), \tag{2.5}$$

where

$$P(\beta;\lambda) := G(\beta\cos\theta(\beta;\lambda)) - a\sqrt{\lambda}\beta\sin\theta(\beta;\lambda),$$
$$Q(\beta;\lambda) := -\beta\sin\theta(\beta;\lambda),$$
$$\theta(\beta;\lambda) := \Theta(\sqrt{\lambda};\beta).$$

Each pair $(\beta_1,\beta_2) \in (-\beta_0,\beta_0)^2$ which solves (2.5) corresponds to a solution u of (1.1) with $u \not\equiv -1, 1$ and, from (2.2) and (2.4), the correspondence is given by

$$u(x) = \begin{cases} G\left(\beta_1\cos\Theta(\sqrt{\lambda}(x+1);\beta_1)\right) & (x \in [-1,0]), \\ G\left(\beta_2\cos\Theta(\sqrt{\lambda}(-x+1);\beta_2)\right) & (x \in (0,1]). \end{cases} \tag{2.6}$$

We note that the constant solution $u = 0$ corresponds to the pair $(\beta_1,\beta_2) = (0,0)$ and θ is determined by the relation

$$\int_0^\theta G'(\beta\cos\tau)d\tau = \sqrt{\lambda}. \tag{2.7}$$

In what follows, we examine the solution structure of (2.5) in the $\lambda\beta_1\beta_2$-space. By the correspondence (2.6), discussions in the next two sections can be translated to the claims in Theorem 1.

3 Primary branches

Let us consider solutions of (2.5) bifurcating from the trivial branch $L^0 := \{(\lambda,\beta_1,\beta_2);\ \beta_1 = \beta_2 = 0\}$. The theory of local bifurcations leads to the following fact.

Lemma 2. *Bifurcation points on L^0 are given by $q_k^0 := (\lambda_k^0, 0, 0)$, $k = 1, 2, \ldots$, where λ_k^0 is defined by* (1.8)*. For each k, solutions of* (2.5) *in a neighborhood of q_k^0 consist only of two branches which meet at q_k^0 transversally.*

We denote by $\mathcal{A}_k$ the (local) solution branch emanating from q_k^0. As is seen below, a point $(\lambda,\beta_1,\beta_2)$ on $\mathcal{A}_k$ satisfies $\beta_1 = -\beta_2$ if k is odd and $\beta_1 = \beta_2$ if k is even. Hence the corresponding solution of (1.1) is odd if k is odd and is even if k is even.

Let us first find a solution (β_1,β_2) of (2.5) satisfying $\beta_1 = -\beta_2$. To this end we put $(\beta_1,\beta_2) = (\beta,-\beta)$ and substitute this into (2.5). Then, since $P(\beta;\lambda) = -P(-\beta;\lambda)$ and $Q(\beta;\lambda) = -Q(-\beta;\lambda)$ are automatically satisfied, we see that (2.5) is reduced to the equation $P(\beta;\lambda) = 0$. Instead of investigating this equation directly, we first solve the equation

$$G(\beta\cos\phi) - a\left(\int_0^\phi G'(\beta\cos\tau)d\tau\right)\beta\sin\phi = 0$$

with respect to ϕ. One can show that for each $\beta \in (-\beta_0,\beta_0)\setminus\{0\}$ and $l = 0, 1, \ldots$, this equation has a unique solution $\phi_l(\beta)$ in $(l\pi, (l+1/2)\pi)$, and ϕ_l becomes continuous if we define $\phi_l(0) := z_l$, where z_l is a unique root of the equation $az\tan z = 1$ in $(l\pi, (l+1/2)\pi)$. From (2.7), we see that $\beta \in (-\beta_0, \beta_0)$ solves $P(\beta;\lambda) = 0$ if and only if $\theta(\beta;\lambda) = \phi_l(\beta)$ for some l. We thus obtain solution branches $\mathcal{B}_l^o$, $l = 0, 1, \dots$ given by

$$\mathcal{B}_l^o := \{(\lambda_l^o(\beta), \beta, -\beta)\}_{\beta\in(-\beta_0,\beta_0)}, \quad \lambda_l^o(\beta) := \left(\int_0^{\phi_l(\beta)} G'(\beta\cos\tau)d\tau\right)^2.$$

Since $\lambda_l^o(0) = (\phi_l(0)G'(0))^2 = (z_l/f'(0))^2 = \lambda_{2l+1}^0$, we see that $\mathcal{B}_l^o$ bifurcates from q_{2l+1}^0. By Lemma 2, we have $\mathcal{B}_l^o \cap \mathcal{N} = \mathcal{A}_{2l+1}$ for some neighborhood $\mathcal{N}$ of q_{2l+1}^0.

We look for a solution (β_1, β_2) with $\beta_1 = \beta_2 = \beta \neq 0$. In this case (2.5) is reduced to the equation $Q(\beta;\lambda) = 0$, which is equivalent to $\theta(\beta;\lambda) = l\pi$, $l = 1, 2, \dots$. Therefore we have solution branches

$$\mathcal{B}_l^e := \{(\lambda_l^e(\beta), \beta, \beta)\}_{\beta\in(-\beta_0,\beta_0)}, \ l = 1, 2, \dots, \quad \lambda_l^e(\beta) := \left(\int_0^{l\pi} G'(\beta\cos\tau)d\tau\right)^2.$$

$\mathcal{B}_l^e$ bifurcates from the point $(G'(0)l\pi, 0, 0) = q_{2l}^0$, and hence $\mathcal{B}_l^e \cap \mathcal{N} = \mathcal{A}_{2l}$ for some neighborhood $\mathcal{N}$ of q_{2l}^0.

4 Spectral property of solutions

4.1 Secondary bifurcation

The implicit function theorem implies that if $(\lambda^*, \beta_1^*, \beta_2^*)$ is a bifurcation point, then the quantity

$$D(\lambda, \beta_1, \beta_2) := \det \begin{pmatrix} P_\beta(\beta_1;\lambda) & -P_\beta(\beta_2;\lambda) \\ Q_\beta(\beta_1;\lambda) & Q_\beta(\beta_2;\lambda) \end{pmatrix}$$

must vanish at $(\lambda^*, \beta_1^*, \beta_2^*)$. This fact and the following lemma give the nonexistence of bifurcation points on the even-numbered branch $\mathcal{B}_l^e \setminus \{q_{2l}^0\}$.

Lemma 3. *If* (1.6) *holds, then* $D(\lambda_l^e(\beta), \beta, \beta) \neq 0$ *for all* $\beta \in (-\beta_0, \beta_0) \setminus \{0\}$.

Let us discuss bifurcation points on the odd-numbered branch $\mathcal{B}_l^o$. We can show that if δ is small enough, then

$$D(\lambda_l^o(\delta), \delta, -\delta) < 0, \quad D(\lambda_l^o(\beta_0 - \delta), \beta_0 - \delta, -(\beta_0 - \delta)) > 0.$$

Hence $D(\lambda_l^o(\beta_l^*), \beta_l^*, -\beta_l^*) = D(\lambda_l^o(-\beta_l^*), -\beta_l^*, \beta_l^*) = 0$ for some $\beta_l^* \in (0, \beta_0)$ (note that $D(\lambda_l^o(\beta), \beta, -\beta)$ is even with respect to β). If we can additionally show that the derivative

$$E(\beta) := \frac{d}{d\beta}(D(\lambda_l^o(\beta), \beta, -\beta))$$

does not vanish at $\beta = \beta_l^*$, then the theory of local bifurcations implies that $q_l^+ := (\lambda_l^o(\beta_l^*), \beta_l^*, -\beta_l^*)$ and $q_l^- := (\lambda_l^o(-\beta_l^*), -\beta_l^*, \beta_l^*)$ are indeed bifurcation points and solutions in a neighborhood of q_l^+ (resp. q_l^-) consist of two branches which intersect at q_l^+ (resp. q_l^-) transversally. The most difficult point in our study is to check the transversality condition $E(\beta_l^*) \neq 0$. This can be proved under the assumption (1.7).

Lemma 4. *Assume that* (1.7) *holds and that* $D(\lambda_l^o(\beta^*), \beta^*, -\beta^*) = 0$ *for some* $\beta^* \in (0, \beta_0)$. *Then there holds* $E(\beta^*) > 0$. *In particular,* $\beta = \beta_l^*$ *is a unique solution of the equation* $D(\lambda_l^o(\beta), \beta, -\beta) = 0$ *in* $(0, \beta_0)$.

4.2 The Morse index

Finally, we study the Morse index of solutions. We begin with the eigenvalues of the constant solution 0.

Lemma 5. *If* $\lambda = \lambda_k^0$, *then* $\mu_k(0) = 0$.

The following lemma gives the basic estimates of the Morse index.

Lemma 6. *Suppose* (1.6). *Let* k *be a nonnegative integer and let* $u \not\equiv 0$ *be a solution of* (1.1) *which vanishes at* k *points in* $(-1, 1) \setminus \{0\}$. *Then* $\mu_{k+2}(u) < 0$ *if* $u(-0)u(+0) < 0$ *and* $\mu_{k+1}(u) < 0$ *if* $u(-0)u(+0) > 0$.

The nondegeneracy of solutions is determined by D.

Lemma 7. *Let* u *be a solution of* (1.1) *and let* (β_1, β_2) *be the corresponding solution of* (2.5). *Then* u *is nondegenerate if and only if* $D(\lambda, \beta_1, \beta_2) \neq 0$.

From the above lemmas, we can obtain the Morse index for solutions on $\mathcal{B}_l^e$. Let u_β^e be a solution of (1.1) corresponding to a point $(\lambda_l^e(\beta), \beta, \beta) \in \mathcal{B}_l^e$ and assume that (1.6) holds. Then Lemmas 5 and 7 imply that $\mu_{2l}(u_\beta^e)|_{\beta=0} > \mu_{2l+1}(u_\beta^e)|_{\beta=0} = 0$ and $\mu_{2l}(u_\beta^e) \neq 0$. We know that $\mu_{2l-1}(u_\beta^e)$ is continuous in β, and hence $\mu_{2l}(u_\beta^e) > 0$. Since u_β^e has $2l$ zeros and $u_\beta^e(-0)u_\beta^e(+0) = \beta^2 > 0$ if $\beta \neq 0$, we see from Lemma 6 that $\mu_{2l+1}(u_\beta^e) < 0$ for $\beta \neq 0$. Therefore $i(u_\beta^e) = 2l$.

Let u_β^o be a solution which corresponds to $(\lambda_l^o(\beta), \beta, -\beta) \in \mathcal{B}_l^o$ and suppose (1.7). To determine the Morse index of u_β^o, we need to know the relation between $\mu_{2l+1}(u_\beta^o)$ and $E(\beta)$ at $\beta = \beta_l^*$.

Lemma 8. *There holds*

$$E(\beta_l^*) \left. \frac{d}{d\beta} \mu_{2l+1}(u_\beta^o) \right|_{\beta=\beta_l^*} < 0.$$

It is seen that u_β^o has $2l$ zeros and $u_\beta^o(-0)u_\beta^o(+0) < 0$ if $\beta \neq 0$, and so Lemma 6 yields $\mu_{2l+2}(u_\beta^o) < 0$ for $\beta \neq 0$. By Lemmas 4, 5, 7 and 8, we have

$$\mu_{2l+1}(u_\beta^o) \begin{cases} > 0 & \text{if } |\beta| < \beta_l^*, \\ = 0 & \text{if } |\beta| = \beta_l^*, \\ < 0 & \text{if } \beta_l^* < |\beta| < 1. \end{cases}$$

This together with Lemmas 4 and 7 shows that $\mu_{2l}(u^\circ_\beta) > 0$. We thus conclude that $i(u^\circ_\beta) = 2l + 1$ if $|\beta| < \beta^*_l$ and $i(u^\circ_\beta) = 2l$ if $\beta^*_l \leq |\beta| < 1$.

References

[1] R. G. Casten and C. J. Holland, *Instability results for reaction diffusion equations with Neumann boundary conditions*, J. Differential Equations **27** (1978), no. 2, 266–273.

[2] Q. Fang, *Asymptotic behavior and domain-dependency of solutions to a class of reaction-diffusion systems with large diffusion coefficients*, Hiroshima Math. J. **20** (1990), no. 3, 549–571.

[3] J. K. Hale and J. Vegas, *A nonlinear parabolic equation with varying domain*, Arch. Rational Mech. Anal. **86** (1984), no. 2, 99–123.

[4] P. Korman, *Global solution branches and exact multiplicity of solutions for two point boundary value problems*, Handbook of differential equations: ordinary differential equations, Vol. III, 547–606, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2006.

[5] H. Matano, *Asymptotic behavior and stability of solutions of semilinear diffusion equations*, Publ. Res. Inst. Math. Sci. **15** (1979), no. 2, 401–454.

[6] M. Mimura, S. Ei and Q. Fang, *Effect of domain-shape on coexistence problems in a competition-diffusion system*, J. Math. Biol. **29** (1991), no. 3, 219–237.

[7] Y. Morita, *Reaction-diffusion systems in nonconvex domains: invariant manifold and reduced form*, J. Dynam. Differential Equations **2** (1990), no. 1, 69–115.

[8] Y. Morita and S. Jimbo, *Ordinary differential equations (ODEs) on inertial manifolds for reaction-diffusion systems in a singularly perturbed domain with several thin channels*, J. Dynam. Differential Equations **4** (1992), no. 1, 65–93.

[9] R. Schaaf, *Global Solution Branches of Two Point Boundary Value Problems*, Lecture Notes in Mathematics, Vol. 1458, Springer-Verlag, Berlin, 1990.

[10] J. Vegas, *Bifurcations caused by perturbing the domain in an elliptic equation*, J. Differential Equations **48** (1983), no. 2, 189–226.