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On Skinny Subsets of $\mathcal{P}_\kappa\lambda$

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0 Introduction

The purpose of this note/talk is to introduce the notion of skinniness and its variants, then discuss the following topics:

(1) Existence and non-existence of skinny stationary subsets of $\mathcal{P}_\kappa\lambda$

(2) Consequences of the existence of skinny stationary (and unbounded) subsets of $\mathcal{P}_\kappa\lambda$

All of the results are stated without proof. For their proofs, we cite original sources when possible.

Throughout this note, we let κ denote an uncountable regular cardinal, λ denote a cardinal $\geq \kappa$, and $\mathrm{NS}_{\kappa\lambda}$ denote the non-stationary ideal over $\mathcal{P}_\kappa\lambda$ ($:= \{x \subseteq \lambda \mid |x| < \kappa\}$). If X is a stationary subset of $\mathcal{P}_\kappa\lambda$, then we let $\mathrm{NS}_{\kappa\lambda} \upharpoonright X := \{Y \subseteq \mathcal{P}_\kappa\lambda \mid X \cap Y \in \mathrm{NS}_{\kappa\lambda}\}$. ($\mathrm{NS}_{\kappa\lambda} \upharpoonright X$ is the κ-complete normal ideal generated by $\mathrm{NS}_{\kappa\lambda}$ and $\{\mathcal{P}_\kappa\lambda - X\}$.)

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Notation. For any set x of ordinals, we define $\sup^*(x)$ by $\sup^*(x) = \sup(x)$ if $\sup(x) \notin x$. Let $\sup^*(x)$ be undefined if $\sup(x) \in x$. For $X \subseteq \mathcal{P}_\kappa\lambda$, we let

$$E_X := \{\sup^*(x) \mid x \in X\} \ .$$

Furthermore, for $\alpha \leq \lambda$, let

$$X^\alpha := \{x \in X \mid \sup^*(x) = \alpha\} \ .$$

Note that $E_X \subseteq E^\lambda_{<\kappa} \cup \{\lambda\}$ where $E^\lambda_{<\kappa} := \{\alpha < \lambda \mid \mathrm{cf}(\alpha) < \kappa\}$.

Now we present the notion of skinniness and its variants.

Definition. Let X be a subset of $\mathcal{P}_\kappa\lambda$ and μ be some cardinal.

(1) X is said to be *skinny* if $|X^\alpha| < |\mathcal{P}_\kappa\alpha|$ for every $\alpha \leq \lambda$.

(2) X is said to be *skinnier* if $|X^\alpha| \leq |\alpha|$ for every $\alpha \leq \lambda$.

(3) X is said to be *skinniest* if $|X^\alpha| \leq 1$ for every $\alpha \leq \lambda$.

(4) X is said to be *μ-skinny* if $|X^\alpha| < \mu$ for every $\alpha \leq \lambda$.

Note that X is skinniest if and only if it is 2-skinny. And if X is μ-skinny for some $\mu < \lambda$, then $X \cap \{x \in \mathcal{P}_\kappa\lambda \mid \sup(x) \geq \mu\}$ is skinnier.

For a regular λ, some large cardinal properties of ideals over $\mathcal{P}_\kappa\lambda$ can imply the existence of skinnier or skinniest stationary subsets of $\mathcal{P}_\kappa\lambda$:

Folklore (Solovay). Suppose λ is a regular cardinal and κ is λ-supercompact. Let U be a normal fine κ-complete ultrafilter over $\mathcal{P}_\kappa\lambda$. Then there is a skinniest $X \subseteq \mathcal{P}_\kappa\lambda$ with $X \in U$.

1 Generic Large Cardinal Properties of $\mathrm{NS}_{\kappa\lambda}$

The study of "generic large cardinal properties" of ideals such as saturation and precipitousness, especially of non-stationary ideals, has played an important role in set theory.

For example, Foreman-Magidor-Shelah's theorem [2] showing the consistency of $\aleph_2$-saturation of $\mathrm{NS}_{\aleph_1}$ from MM sparked a paradigm shift in the theory of large cardinals and descriptive set theory.

It turns out that, even for some singular λ, generic large cardinal properties of $\mathrm{NS}_{\kappa\lambda}$ imply the existence of skinny stationary subsets of $\mathcal{P}_\kappa\lambda$. As an example, we state the next result [4], [6].

Theorem 1. *Assume λ is either a strong limit cardinal or the successor of a cardinal δ with $\delta^{<\kappa} = 2^\delta$. Let X be a stationary subset of $\mathcal{P}_\kappa\lambda$.*

(1) If $\mathrm{NS}_{\kappa\lambda} \restriction X$ is precipitous, then X has a skinny stationary subset.

(2) If $\mathrm{NS}_{\kappa\lambda} \restriction X$ is 2^λ-saturated, then there exists a club $C \subseteq \mathcal{P}_\kappa\lambda$ such that $C \cap X$ is skinny.

But the next result [4], [6] show that skinny stationary subsets of $\mathcal{P}_\kappa\lambda$ are hard to come by for singular λ.

Theorem 2 ((1) Matsubara–Shelah [4], (2) Matsubara–Usuba [6])**.**

(1) If λ is a strong limit singular cardinal $> \kappa$, then there is no skinny stationary subset of $\mathcal{P}_\kappa\lambda$.

(2) If λ is a singular cardinal $> \kappa$, then there is no skinnier stationary subset of $\mathcal{P}_\kappa\lambda$.

These results have the following consequences.

Corollary 3. *Let λ be a strong limit singular cardinal $> \kappa$. Then*

(1) $\mathrm{NS}_{\kappa\lambda}$ is nowhere precipitous (i.e. $\mathrm{NS}_{\kappa\lambda} \restriction X$ is not precipitous for every stationary $X \subseteq \mathcal{P}_\kappa\lambda$).

(2) $\mathrm{NS}_{\kappa\lambda}$ is nowhere 2^λ-saturated.

2 Combinatorial Principles

In this section we discuss relationship between the existence of skinny (skinnier, etc.) stationary sets and some combinatorial principles.

The existence of skinnier or skinniest stationary sets is related to Jensen's $\diamondsuit$ principle.

Definition. Let S be a stationary subset of $E^\lambda_{<\kappa}$ where λ is a regular cardinal $> \kappa$. We say that S *bears* a skinny (skinnier, skinniest, μ-skinny) stationary set if there is a skinny (skinnier, skinniest, μ-skinny, respectively) stationary $X \subseteq \mathcal{P}_\kappa\lambda$ with $E_X \subseteq S$.

In [6], we proved the following result:

Theorem 4. *Let λ be a regular cardinal $> 2^{<\kappa}$. Then the following are equivalent for a stationary $S \subseteq E^{\lambda}_{<\kappa}$:*

(i) $\diamondsuit_\lambda(S)$.

(ii) S bears a skinniest stationary subset of $\mathcal{P}_\kappa\lambda$, and $2^{<\lambda} = \lambda$.

(iii) S bears a skinnier stationary subset of $\mathcal{P}_\kappa\lambda$, and $2^{<\lambda} = \lambda$.

Note. The existence of a skinniest stationary subset of $\mathcal{P}_\kappa\lambda$ cannot imply $2^{<\lambda} = \lambda$. Starting with a skinniest stationary set, one can blow up 2^ω to violate $2^{<\lambda} = \lambda$ by Cohen forcing preserving stationarity of our skinniest set.

So if we assume $V = L$, then for each regular cardinal λ ($\geq \kappa$), every stationary subset of $E^{\lambda}_{<\kappa}$ bears a skinniest stationary subset of $\mathcal{P}_\kappa\lambda$. Actually we obtained a stronger result [5].

Theorem 5. *Assume $V = L$. If λ is a regular cardinal, then every stationary subset of $\mathcal{P}_\kappa\lambda$ has a skinniest stationary subset.*

From the above mentioned result about $\diamondsuit_\lambda$ together with Shelah's theorem on $\diamondsuit_\lambda$ [7], we obtain the following result [5].

Theorem 6. *Let λ be a cardinal with $2^\lambda = \lambda^+$. If $\max\{\kappa, \mathrm{cf}(\lambda)^+\} \geq \aleph_2$, then there is a skinniest stationary subset of $\mathcal{P}_\kappa\lambda^+$.*

Jensen's $\square$ principle has some implications [5] about the existence of skinniest stationary sets.

Theorem 7. *Suppose $\lambda = \kappa^{+n}$ for some $n < \omega$. If $\square_{\kappa^{+m}}$ holds for every $m < n$, then there exists a skinniest stationary subset of $\mathcal{P}_\kappa\lambda$.*

This theorem implies that non-existence of such a skinniest set has a strong consistency strength.

3 Non-existence of skinny stationary sets

It is clear that, for each regular cardinal $\lambda \geq \kappa$, $\lambda^{<\kappa} = 2^{<\kappa} \cdot |X|$ holds for every unbounded subset X of $\mathcal{P}_\kappa\lambda$. Hence if there is a γ^+-skinny unbounded subset of $\mathcal{P}_\kappa\lambda$, then we have $\lambda^{<\kappa} \leq 2^{<\kappa} \cdot \gamma \cdot \lambda$. In particular, if there exists a λ^+-skinny unbounded subset of $\mathcal{P}_\kappa\lambda$, then $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$ holds. Now we state the result relating the existence of skinny unbounded subsets and the Singular Cardinal Hypothesis (SCH), which asserts $2^\delta = \delta^+$ for every singular strong limit cardinal δ. Using Silver's theorem on SCH [8], the following result was proven in [5]:

Theorem 8. *Suppose there exists a λ^+-skinny unbounded subset of $\mathcal{P}_\kappa\lambda$ for every regular $\lambda \geq \kappa$. Then* SCH *holds above κ.*

The following is an immediate corollary.

Corollary 9. *If* SCH *fails at some singular strong limit cardinal δ, then, for every uncountable regular cardinal $\kappa < \delta$, there must be some regular λ such that $\kappa < \lambda$ where $\mathcal{P}_\kappa\lambda$ has no skinnier unbounded subset.*

Furthermore, it is easy to see that the non-existence of a skinniest unbounded subset of $\mathcal{P}_\kappa\lambda$ for regular cardinals κ and λ with $\aleph_2 \leq \kappa \leq \lambda$ is a large cardinal property. The next result is a consequence of Jensen's Covering Theorem and Theorem 5.

Theorem 10. *If $0^\sharp$ does not exist, then there exists a skinniest unbounded subset of $\mathcal{P}_\kappa\lambda$ for every regular cardinals κ and λ with $\aleph_2 \leq \kappa \leq \lambda$.*

We have mentioned that the failure of SCH implies the non-existence of skinnier unbounded subset of $\mathcal{P}_\kappa\lambda$ for some regular cardinal λ. It turns out, even under GCH, there may be a regular cardinal λ for which $\mathcal{P}_\kappa\lambda$ has no skinnier stationary subsets. Starting with a sufficiently strong large cardinal λ, using Radin forcing, Woodin [1] built a model of GCH in which $\diamondsuit_\lambda$ fails and λ remains inaccessible. So in his model, there are no skinnier stationary subsets of $\mathcal{P}_\kappa\lambda$ for every regular uncountable $\kappa < \lambda$.

As for the case where λ is a successor cardinal, by Theorem 6, if $2^\lambda = \lambda^+$, then the only possibility for the non-existence of skinnier stationary subsets of $\mathcal{P}_\kappa\lambda$ to occur is the case where $\kappa = \aleph_1$ and $\lambda = \delta^+$ with $\mathrm{cf}(\delta) = \omega$. Gitik and Rinot [3] built a model of $\neg\diamondsuit_{\aleph_{\omega+1}}(S)$ for some stationary $S \subseteq E^{\aleph_{\omega+1}}_\omega$ together with GCH. So in this model, S bears no skinnier stationary subset of $\mathcal{P}_{\aleph_1}\aleph_{\omega+1}$.

We proved the following more general theorem [5] which tells us that the set of stationary subsets of $E^\lambda_{<\kappa}$ not bearing skinnier stationary set can be "dense" in the collection of all stationary subsets of $E^\lambda_{<\kappa}$.

Theorem 11. *Let κ, μ and λ be uncountable regular cardinals with $\kappa \leq \mu < \lambda$. Suppose $2^{<\mu} = \mu$. Then there is a poset $\mathbb{P}$ satisfying the following:*

(i) $\mathbb{P}$ *has the* μ^+*-c.c. and adds no new sequence of ordinals of length* $< \mu$. *(So* $\mathbb{P}$ *preserves cofinalities.)*

(ii) *In* $V^{\mathbb{P}}$, *there is a sequence* $\langle S_\delta \mid \delta < \mu \rangle$ *of subsets of* $E^\lambda_{<\kappa}$ *such that* $\bigcup_{\delta<\mu} S_\delta = E^\lambda_{<\kappa}$ *and* S_δ *bears no* μ*-skinny stationary subsets of* $\mathcal{P}_\kappa\lambda$ *for any* $\delta < \mu$.

First we force our sequence $\langle S_\delta \mid \delta < \mu \rangle$ of subsets of $E^\lambda_{<\kappa}$ with certain desirable properties. Then we perform a $<\mu$-support iteration of some "club shooting" posets of length 2^λ, making all of S_δ $(\delta < \mu)$ of our sequence bear no μ-skinny stationary subsets of $\mathcal{P}_\kappa\lambda$.

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