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On a potential-well type result and global bounds of solutions for semilinear parabolic equation involving critical Sobolev exponent

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Abstract

In this note, we are concerned with the asymptotics of (mainly nonnegative) solutions for a semilinear parabolic equation involving critical Sobolev exponent. The existence of a potential-well structure is discussed. In addition, the existence of the global bounds for the Sobolev norm of time-global solutions and long-time asymptotics of them are given.

Keywords

critical Sobolev exponent, noncompact orbit, potential-well structure, ε-regularity, profile decomposition, global bounds in the Sobolev space, the LaSalle principle.

Contents

Organization of this note.

In this note, we are concerned with the behavior of solutions for semilinear parabolic equations involving critical Sobolev exponent.

In §1, we introduce our problem in §1.1 and state our main results in §1.2. §1.3 is devoted to the review of known facts and motivations for our main results.

In §2, a typical argument for the subcritical case will be given in §2.1 and the difficulty for the critical case will be clarified in §2.2.

In §3, preliminary facts used for proofs of main results are introduced. First we introduce in §3.1 a scale invariant structure of (P) with $p = 2^*$ which plays an important role in the anaylysis of the critical problem. §3.2 is devoted to the proof of the ε-regulatity type result and the profile decompoition of Gerárd and Jaffard will be reviewd in §3.3.

§4 is concerned with the proof of main results. In §4.1, a potential-well type result together with the structure of the space of initial data in the critical case, Theorem 1.1 and Theorem 1.2, are proved. The existence of global bounds for the Sobolev norm for time-global solutions, Theorem 1.3, and the asymptotics of such solutions, Theorem 1.4, are verified in §4.2

In the final section §5, we introduce some discussions on (P) with critical exponent. The first topic is the possibility for getting an extention of the theory of an abstract dynamical system. For a dynamical system with a compact orbit, the asymptotics will be clarified by the LaSalle principle. We discuss in §5.1 the possibilility to extend the LaSalle principle to dynamical systems with a noncompact orbit with the aid of an abstract version of the profile decomposition. In §5.2, some (basic) open problems for (P) in the critical case will be introduced.

In the appendix, we review the concavity argument of Payne-Sattinger-Levine for the reader's convenience.

Apart from Theorem 1.3 and Theorem 1.4, results presented in this note is already published and the author try to give a self-contained argument for them except for proofs of Theorem 1.3 and Theorem 1.4 in §1.2.2.

1 Problem, main results, known results and motivations

1.1 Problem and basic facts

1.1.1 Problem

Let $N \geq 3$, $\Omega \subset \mathbb{R}^N$ be a smooth domain and let $\dot{H}^1(\Omega)$ be a homogeneous Sobolev space defined as a closure of $C_0^\infty(\Omega)$ by the homogeneous Sobolev norm $\|\nabla \cdot\|_2$, where $\|\cdot\|_r$ denotes the standard L^r-norm. Let $2^* := \frac{2N}{N-2}$ be the critical Sobolev exponent of the Sobolev embedding $\dot{H}^1 \hookrightarrow L^p$. It is known that $\dot{H}^1 \hookrightarrow L^{2^*}$ is continuous but fails to be compact. We consider

$$
\text{(P)} \qquad \begin{cases} \partial_t u &= \Delta u + u|u|^{p-2} \quad \text{in} \quad \Omega \times (0, T_m), \\ u|_{t=0} &= u_0 \qquad\qquad\quad\;\; \text{in} \quad \Omega \end{cases}
$$

with the homogeneous Dirichlet boundary condition

$$
u = 0 \text{ on } \partial\Omega \times (0, T_m)
$$

if $\partial\Omega \neq \emptyset$, where $u_0 \in L^\infty \cap H^1$ for the sake of simplicity, T_m denotes the maximal existence time of the classical solution u of (P). A solution with $T_m = \infty$ is called as a time-global solution. In the main body of this note, we assume $p = 2^*$, $\Omega = \mathbb{R}^N$ and $u_0 \geq 0$.

We discuss in this note two topics concerning the asymptotic behavior of solutions of (P). The first topic is the existence of so called a "stable set" and an "unstable set" in $\dot{H}^1$. By using this fact, we can clarify the structure of the space of initial data. These results will be given in Theorem 1.1 and Theorem 1.2. The second topic is concerned with the validity of the following global bounds for time-global solutions u:

$$
\sup_{t>0} \|\nabla u(t)\|_2 < \infty. \tag{1}
$$

As is shown in §2.1 and in the proof of Theorem 1.4, the analysis of a bound of the form (1) is a first step for the analysis of the asymptotic behavior of a time-global solution u in the "energy space" $\dot{H}^1$. Note that by the decreasing property of the energy J_p along the orbit of u (see (8) below), (1) is equivalent to

$$
\sup_{t>0} \|u(t)\|_p < \infty. \tag{2}
$$

We will introduce an argument to establish the validity of (2) for the case where $p = 2^*$, $\Omega = \mathbb{R}^N$ and u is a nonnegative time-global solution of (P).

Based on this bound, the asymptotics of time-global nonnegative solutions are given, see Theorem 1.3 and Theorem 1.4.

1.1.2 Time-local existence of a solution

We review basic facts concering the time local existence of solutions of (P) which is needed in proving main results. For the proof of facts stated below, see e.g. Brezis-Cazenave [3], Ruf-Terraneo [45] and Weissler [54].

We consider the solution of (P) in the following sense:

$$u \in C^{2,1}(\mathbb{R}^N \times (0,T_m)) \cap C^1((0,T_m);L^2) \cap C([0,T_m);H^1). \tag{3}$$

The solution in this class is easily constructed. Indeed, since $u_0 \in L^\infty$, the existence of a classical solution of (P) is a standard fact and for $u_0 \in H^1$, a solution $u \in C^1((0,T_m);L^2) \cap C([0,T_m);H^1)$ is constructed.

Since u in the class (3) is a classical solution, it satisfies the blow-up alternative in L^∞-sense:

$$\text{if } T_m < \infty, \text{ then } \lim_{t \to T_m} \|u(t)\|_\infty = \infty. \tag{4}$$

It is also well known that this class of solution satisfies the integral equation

$$u(t) = e^{t\Delta}u_0 + \int_0^t ds e^{(t-s)\Delta} u(s)|u(s)|^{p-2} \tag{5}$$

associated with (P).

1.1.3 The energy structure

By multiplying $\partial_t u$ to both sides of (P) and integrating over $\mathbb{R}^N$, we (formally) obtain the energy equality

$$\|\partial_t u(t)\|_2^2 = -\frac{d}{dt}J_p(u(t)), \tag{6}$$

where J_p denotes the energy functional associated with (P) defined by

$$J_p(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u(t)\|_p^p.$$

It is known that solutions u of (P) satisfying (3) actually satisfy (6) for any $t \in (0,T_m)$.

In the main body of this note, we assume that $p = 2^*$, $\Omega = \mathbb{R}^N$ and u is a nonnegative time-global solution of (P). In this case, the concavity argument (this name comes from the concavity of a part $-\frac{1}{p}\|u\|_p^p$ in the energy functional) of Payne-Sattinger [44] and Levine [35] for bounded domains together with the comparison argument implies that

$$\lim_{t\to\infty} J_p(u(t)) \geq 0 \tag{7}$$

and, by (6) and (7), we have the existence of $d \geq 0$ satisfying

$$J_p(u_0) \geq J_p(u(t)) \downarrow d \text{ as } t \to \infty, \tag{8}$$

see §6 and Mizoguchi [39, Lemma 2.4].

Remark 1.1

The assumption of the nonnegativity of solutions is only used to assure (8), in other words, to exclude the existence of a solution satisfying

$$T_m = \infty \text{ and } \lim_{t\to\infty} J_p(u(t)) = -\infty. \tag{9}$$

For bounded Ω, we can exclude the existence of such solutions by the concavity argument, see §6. In an unbounded domain case, we can also exclude solutions satisfying (9) under the nonnegativity assumption by using the comparison argument together with the corresponding result in bounded domains. For sigh-changing solutions in an unbounded domains, the existence of a solution which satsifies (9) seems to be an open problem, see Open problem 5.1. As for the subcritical problem in unbounded domains, see e.g. Kavian [32], Mizoguchi-Yanagida [41] and Mizoguchi-Ninomiya-Yanagida [40]. ∎

1.2 Main results

The main results of this note consists of two parts.

The first part, Theorem 1.1 and Theorem 1.2, is concerned with the existence of so-called "potential-well structure" and the structure of the initial data space for (P) in $\mathbb{R}^N$ with $p = 2^*$, respectively. Theorem 1.1 gives an affirmative answer for conjectures (19), (20) and (21) below (for an "unstable set", we need the nonnengativity of solutions).

The second part, Theorem 1.3 and Theorem 1.4, give a time-global bounds for Sobolev norms of time-global solutions of (P) in $\mathbb{R}^N$ with $p = 2^*$ and its asymptotic behavior as time tends to infinity.

Known results and motivations for these problems will be discussed in §1.3 in detail. We start with the first part.

1.2.1 On the potential-well structure

Let

$$W_{2^*} := \left\{ w \in \dot{H}^1;\ -\|\nabla w\|_2^2 + \|w\|_{2^*}^{2^*} < 0,\ J(u) < \frac{1}{N} S^{\frac{N}{2}} \right\},$$
$$V_{2^*} := \left\{ w \in \dot{H}^1;\ -\|\nabla w\|_2^2 + \|w\|_{2^*}^{2^*} > 0,\ J(u) < \frac{1}{N} S^{\frac{N}{2}} \right\},$$

where $S := \inf_{w \in \dot{H}^1 \setminus \{0\}} \frac{\|\nabla w\|_2^2}{\|w\|_{2^*}^{2^*}} (> 0)$ denotes the best Sobolev constant. Note that W_{2^*} forms a neighborhood of the origin in $\dot{H}^1$ and V_{2^*} a neighborhood of the infinity in $\dot{H}^1$. W_{2^*} (resp. V_{2^*}) is called a stable set (resp. an unstable set).

We have the following (see [26]):

Theorem 1.1

Let u be a solution of (P) *in* $\mathbb{R}^N$ *with* $p = 2^*$.
(a) *Assume that there exists* $t_0 \in [0, T_m)$ *such that* $u(t_0) \in W_{2^*}$. *Then* $T_m = \infty$ *and* $\|\nabla u(t)\|_2 \to 0$ *as* $t \to \infty$.
(b) *Suppose that, in addition,* u *is a nonnegative solution of* (P). *Assume that there exists* $t_0 \in [0, T_m)$ *such that* $u(t_0) \in V_{2^*}$. *Then* $T_m < \infty$.

Remark 1.2 (On the nonnegativity assumption in (b))

The nonnegativity assumption of solutions in above is only used to assure (8), and if we can establish (8) for sign-changing solutions, then we can remove the nonnegativity assumption from (b). See Remark 1.1 and Open problem 5.1. ∎

Remark 1.3 (On the blow-up of $\|\nabla u(t)\|_2$ in (b))

In the case (b), we have $\|u(t)\|_\infty \to \infty$ as $t \to T_m$ by the blow-up alternative (4). It is not known that whether $\|\nabla u(t)\|_2 \to \infty$ as $t \to T_m$ or not, as is stated in Open problem 5.5. ∎

We next clarify the fine structure of the initial data space by assuming the nonnegativity of solutions. This case the nonnegativity assumption is crucial since the proof needs a comparison principle in an essential way.

Theorem 1.2

For any nonnegative function $\varphi \in L^\infty \cap H^1$, *there exist* $0 < \underline{\lambda} \leq \overline{\lambda} < \infty$ *satisfying the following: let* u_λ *be a solution of* (P) *in* $\mathbb{R}^N$ *with* $p = 2^*$ *and initial data* $u_0 = \lambda\varphi$ *for* $\lambda > 0$, *then there hold*

(a) *if* $\lambda \in (0, \underline{\lambda})$, *then there exists* $t_0 \in [0, T_m)$ *such that* $u_\lambda(t_0) \in W_{2^*}$.
(b) *if* $\lambda \in (\overline{\lambda}, \infty)$, *then there exists* $t_0 \in [0, T_m)$ *such that* $u_\lambda(t_0) \in V_{2^*}$.
(c) *if* $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, *then the orbit of* $u_\lambda(t)$ *does not intersect with* $W_{2^*} \cup V_{2^*}$.

Remark 1.4 (On the regularity assumption on the profile function)
We can relax the regularity assumption on φ by using the parabolic regularity. ∎

Remark 1.5 (On the asymptotic behavior)
Hence we see that $u_\lambda(t) \to 0$ in $\dot{H}^1$ as $t \to \infty$ if $\lambda < \underline{\lambda}$ and $\|u_\lambda(t)\|_\infty \to \infty$ as $t \to T_m(< \infty)$ if $\lambda > \overline{\lambda}$ by combining Theorem 1.1 and Theorem 1.2. As for the solution u_λ with $\lambda \in [\underline{\lambda}, \overline{\lambda}]$, if φ is in addition radially symmetric, then there hold $T_m = \infty$ and

$$u_\lambda(\cdot, t) - \|u(t)\|_\infty U(\|u(t)\|_\infty^{\frac{2}{N-2}} \cdot) = o(1) \text{ in } \dot{H}^1 \tag{10}$$

as $t \to \infty$, where U is a Talenti function with $\|U\|_\infty = 1$ (which attains the best constant S in (17)). Moreover, we can prove $\underline{\lambda} = \overline{\lambda}$, i.e., the set of threshold modulus $[\underline{\lambda}, \overline{\lambda}]$ is a singleton, see [27]. For results on an asymptotic behavior of u_λ for $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ with a nonradial φ, see [23] and [24]. ∎

Remark 1.6 (The existence and the nonexistence of the potential-well structure with respect to p)
Note that if $p < 2^*$, then $\dot{H}^1$ cannot be embedded to L^p, which yields $S_p := \inf_{w \in \dot{H}^1 \setminus \{0\}} \frac{\|\nabla w\|_2^2}{\|w\|_p^p} = 0$. Hence there is no "potential well" structure in this case and the existence of the potential-well structure for the case $\Omega = \mathbb{R}^N$ only holds for the critical case. In spite of the fact $S_p = 0$, the similar result with Theorem 1.2 also holds in the subcritical case. This is based on a potential well structure for a "forward self-similarly transformed equation" of (P), see e.g. Kavian [32], Kawanago [33] and references therein. ∎

1.2.2 On global bounds for time-global solutions

By [27], we know that if u is a nonnegative, radially symmetric, time-global solution of (P) with $p = 2^*$ and $\Omega = \mathbb{R}^N$, then the time-global bounds (1) holds. We here show the validity of (1) for nonnegative global-in-time solution of (P) without the assumption of radial symmetry, and give an asymptotic behavior of them.

Theorem 1.3 (Global bounds for the critical case)

Let u be a nonnegative time-global solution of (P) *with $p = 2^*$ and $\Omega = \mathbb{R}^N$. Then there holds* $\sup_{t>0} \|\nabla u(t)\|_2 < \infty$. ∎

Remark 1.7 (For the general case)

In theorem above, the nonnegativity assumption of solutions is only used to assure (8), and if we establish (8) for sign-changing solutions, then we can remove the nonnegativity assumption, see also Remark 1.1. For (P) on general smooth domain Ω with $p = 2^*$, we have

$$\limsup_{t\to\infty} \|\nabla u(t)\|_2 < \infty$$

if

$$\liminf_{t\to\infty} \|\nabla u(t)\|_2 < \infty, \tag{11}$$

see [28]. Therefore, for an arbitrary time-global solution of u, we have either

$$\limsup_{t\to\infty} \|\nabla u(t)\|_2 < \infty$$

or

$$\lim_{t\to\infty} \|\nabla u(t)\|_2 = \infty.$$

For a bounded Ω, we always have (11) (see Corollary 2.1 below for the subcritical case whose proof is also valid for the critical case, see also Remark 2.5). For $\Omega = \mathbb{R}^N$ with $p = 2^*$, we have the alternative

$$\limsup_{t\to\infty} \|\nabla u(t)\|_2 < \infty$$

or

$$\lim_{t\to\infty} \|\nabla u(t)\|_2 = \infty \text{ and } \lim_{t\to\infty} J_{2^*}(u(t)) = -\infty. \tag{12}$$

since $\lim_{t\to\infty} J_{2^*}(u(t)) > -\infty$ implies (11) as is shown by the same argument for bounded Ω above (the conclusion of Lemma 2.2 (b) for the bounded domain case should be replaced by that of Proposition 3.2 in the case $\Omega = \mathbb{R}^N$). The existence of a sign-changing solution u satsifying (12) is an open problem, see Open problem 5.1. ∎

Remark 1.8 (An extension of a class of initial data)

We can considerably enlarge the admissible class of initial datum, see e.g. Brezis-Cazenave [3] and Ruf-Terraneo [45]. ∎

Based on Theorem 1.3, we can clarify the following asymptotics of time-global solutions of (P) which are bounded in $\dot{H}^1$. For a Banach space X and for $A \subset X$, let $\mathrm{dist}_X(u, A) := \inf_{v \in A} \|u - v\|_X$.

Theorem 1.4 (Asymptotics for the critical case)
Let a time-global solution u of (P) *with $p = 2^*$ and $\Omega = \mathbb{R}^N$ satisfies*

$$\sup_{t>0} \|\nabla u(t)\|_2 < \infty. \tag{13}$$

Let $E_\infty(u_0)$ be a set defined by

$$\begin{aligned} &E_\infty(u_0) \\ := &\left\{ \sum_{j=1}^{n} (\lambda^j)^{\frac{N-2}{2}} \varphi^j(\lambda^j(\cdot - y^j));\ \varphi^j \text{ is a stationary solution of (P)}, \right. \\ &\left. (\lambda^j)_{j=1}^n \subset \mathbb{R}_+,\ (y^j)_{j=1}^n \subset \mathbb{R}^N,\ n \in \mathbb{N} \cup \{0\} \text{ with } \sum_{j=1}^{n} J_{2^*}(\psi^j) \le J_{2^*}(u_0) \right\}. \end{aligned}$$

Then there holds

$$\mathrm{dist}_{L^{2^*}}(u(t), E_\infty(u_0)) \to 0 \tag{14}$$

as $t \to \infty$ (Note that all ψ^j may be trivial).

Remark 1.9 (For nonnegative solutions)
If u is a nonnegative solution of (P), then the assumption of Theorem 1.4 holds by virtue of Theorem 1.3. In this case, ψ^j in the definition of $E_\infty(u_0)$ can be taken as a nonnegative function and identical for any j, and the convergence in (14) can be improved to that in $\dot{H}^1$ by using a "quantization of the energy limit", see [28]. It is not clear whether we can improve the convergence in (14) to $\dot{H}^1$ for sign-changing case, see Open problem 5.2. ∎

Remark 1.10 (Meaning of the asymptotics in the critical case)
We here discuss the intuitive meaning of the result in Theorem 1.4, see also §4.2.2, the proof of Proposition 4.4. Let u be a time-global solution with (13). From Theorem 1.4 and the proof of Proposition 4.4, we see that for any time sequence (t_n) with $t_n \to \infty$, there exists a subsequence (denoted by the same symbol) of (u_n), $(\lambda_n^j) \subset \mathbb{R}_+$, $(y_n^j) \subset \mathbb{R}^N$ and a sequence of

stationary solutions $(\psi_n^j) \subset \dot{H}^1$ of (P), where $j = 1, \cdots$,, such that

$$u(\cdot, t_n) - \sum_{j=1}^{l} (\lambda_n^j)^{\frac{N-2}{2}} \varphi_n^j(\lambda_n^j(\cdot - y_n^j)) := r_n^l, \tag{15}$$

$$\lim_{l\to\infty} \lim_{n\to\infty} \|r_n^l\|_{2^*} = 0. \tag{16}$$

Note that (P) is invariant under the spatial translations, i.e., if $u(x,t)$ satisfies (P), then $u(x-y,t)$ also satisfies (P) with initial $u_0(x-y)$ for any $y \in \mathbb{R}^N$. Also, (P) has a scale invariance under $u(x,t) \mapsto \mu^{\frac{2}{p-2}} u(\mu x, \mu^2 t)$, where $\mu \in \mathbb{R}_+$, see Proposition 3.1 below. The peculiarity of the critical case $p = 2^*$ is that, only in this case, the energy function J is also invariant under the scaling above. In other words, only in the critical case, the evolution equation structure and the variatioinal strucure are both invariant under the scaling. The relation (16) says that time-global solutions behave like as a superposition of rescaled stationary solutions by reflecting this invariance. This behavior is out of the scope of "the absorbtion to a set of equilibrium", a postulate (24) below in the subcritical case. In §5.1, we will try to interpret the result of Theorem 1.4 as "the absorbtion of a set of extended equilibrium". ∎

1.3 Known results and motivation for main results

In this subsection, we review known facts and motivate main results.

1.3.1 On the potential-well structure

Here we review known facts concerning Theorem 1.1 and Theorem 1.2. We start by reviewing the "potential-well structure" which motivates results like in Theorem 1.1.

The Sobolev inequality The first important thing is the Sobolev inequality. Indeed, the inequality of the following type is called as the Sobolev inequality:

$$S_p\|u\|_p^2 \leq \|\nabla u\|_2^2, \quad u \in \dot{H}^1(\Omega),$$

where

$$S_p := \inf_{u\in \dot{H}^1\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_p^2} \tag{17}$$

is called as the best Sobolev constant. It is well-known that $S_p > 0$ for bounded Ω with $p \in [1, 2^*]$, or $\Omega = \mathbb{R}^N$ with $p = 2^*$.

The Nehari manifold, the stable set and the unstable set Let

$$J_p(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p,$$

the energy functional associated with (P). Take any $\varphi \in \dot{H}^1$. Then it is easy to see that a function

$$f(\lambda) := J_p(\lambda\varphi) = \frac{\lambda^2}{2}\|\nabla\varphi\|_2^2 - \frac{\lambda^p}{p}\|\varphi\|_p^p$$

attains its maximum uniquley at some $\overline{\lambda} > 0$ and $w := \overline{\lambda}\varphi$ satisfies

$$\|\nabla w\|_2^2 - \|w\|_p^p = 0.$$

Hence every ray emanating from the origin in $\dot{H}^1$ intersects with

$$N := \{w \in \dot{H}^1 \setminus \{0\};\ \|\nabla w\|_2^2 - \|w\|_p^p = 0\}$$

at a unique point. The manifold N is called as a Nehari manifold.

A stationary solution u of (P) satisfies $-\Delta u = u|u|^{p-2}$. Then multiplying u to the both sides and integration over $\mathbb{R}^N$, we have $\|\nabla u\|_2^2 = \|u\|_p^p$. Hence N contains all the stationary solutions of (P).

It is known that, under the assumption on Ω and p which allow the Sobolev inequality, there holds

$$\inf_{u\in N} J_p(u) = \left(\frac{1}{2} - \frac{1}{p}\right) S_p^{\frac{p}{p-2}} =: d_p, \tag{18}$$

where S_p is the best Sobolev constant defined by (17). The value d_p is called a "potential depth", also as a "mountain pass value" or "ground state energy" of J_p, see e.g. Willem [53].

Now let

$$\begin{aligned} W_p &:= \{w \in \dot{H}^1;\ -\|\nabla w\|_2^2 + \|w\|_p^p < 0,\ J_p(u) < d_p\}, \\ V_p &:= \{w \in \dot{H}^1;\ -\|\nabla w\|_2^2 + \|w\|_p^p > 0,\ J_p(u) < d_p\}. \end{aligned}$$

Then W_p forms a neighborhood of the origin in $\dot{H}^1$ and V_p a neighborhood of the infinity in $\dot{H}^1$. W_p (resp. V_p) is called as a stable set (resp. an unstable set). By considering the level set structure of J_p and the decreasing property (8) of $J_p(u(t))$, we can expect that

$$W_p \text{ and } V_p \text{ are invariant sets of a flow associated with (P).} \tag{19}$$

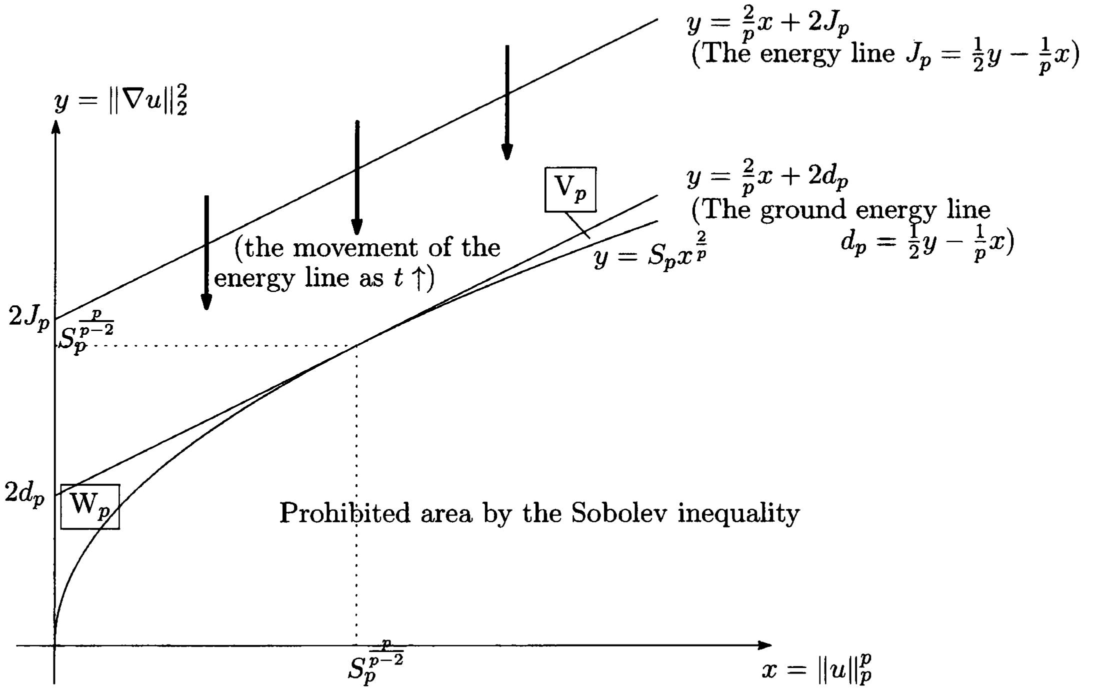
Moreover, since W_p is a neighborhood of the origin,

$$\text{an orbit which intersects with } W_p \text{ may exist globally in time and tends to 0} \tag{20}$$

while

$$\text{an orbit enters } V_p \text{ may blow up in finite time} \tag{21}$$

since V_p forms a neighborhood of the infinity. These situation can be drawn in a picture which is first introduced by Ôtani [43].



Known results and perspectives The verification of (19) and (20) is started by Payne-Sattinger [44] for hyperbolic equations and by Levine [35] for pababolic equations. Later a vast amount of works are done and, among of them, Ikehata-Suzuki [24] proved (19) and (20) are indeed true for the subcritical and bounded domain case. Actually, as for the stable set W_p, it is rather easy to see the invariance of W_p as is shown in the proof of Proposition 4.1. Then we see that

$$\sup_{t<T_m} \|\nabla u(t)\|_2 < \infty,$$

since it is easy to see that

$$\sup_{w\in W_p} \|\nabla w\|_2^2 < S_p^{\frac{p}{p-2}}$$

by the picture above. In the subcritical and bounded case, we can conclude $T_m = \infty$ from this relation since in this case we have

$$\lim_{t\uparrow T_m} \|\nabla u(t)\|_2 = \infty \text{ if } T_m < \infty, \tag{22}$$

see Lemma 2.1 and Lemma 5.1. On the other hand, in the critical case, (22) does not hold in general. This is proved by Schweyer [46] (see Open problem 5.5 of this note). This result says that we cannot rely on the argument given above to obtain Theorem 1.1. In this note, we overcome this difficulty to introduce Proposition 3.3, the ε-regularity, which claims

$$\text{if } \sup_{t\in[0,T_m)} \|\nabla u(t)\|_2 < S_{2^*}^{\frac{2^*}{2^*-2}}, \text{ then } T_m = \infty.$$

By using the potential-well structure together with the comparison argument, Lions [37] and Cazenave-Lions [5] give a similar result as in Theorem 1.4 again for the subcritical and bounded domain case. In this case, the orbit associated with time-global solution is compact by virtue of the compactness of the Sobolev embedding and one can prove that

$$\text{every global-in-time solution converges to a stationary solution.} \tag{23}$$

This together with the nonnegativity assumption yields the set of threshlod modulus $[\underline{\lambda}, \overline{\lambda}]$ as is given in Theorem 1.1 is a singleton.

Once we have established Theorem 1.1, then the proof of Theorem 1.2 is not so much difficult. The analysis of the structure of the set consists of threshold modulus is different from the one sketched above, see Remark 1.5 and [27], and is not treated in this note.

1.3.2 On global bounds for time-global solutions

The investigation of the existence of global bounds of the form (1) is initiated in Ôtani [43] in the setting of an abstract evolution equation theory governed by subdifferential operators. The systematic analysis of the asymptotics of time-global solutions for abstract nonlinear parabolic equation is introduced by e.g. Henry [22].

Subcritical case For a subcritical problem on a bounded domain, i.e., problem (P) with $p < 2^*$ and bounded Ω, Ôtani [43] obtained (1) for p in the subcritical range. Later, more detailed analysis was done, see e.g. Cazenave-Lions [5], Giga [17], Fila [12], Ikehata-Suzuki [24] and references therein. All these works are concerned with the subcritical case and it is proved that every (time-global) solution has a time-global bounds (1). Also, based on this global bounds, it is proved that

$$\text{every time-global solution is attracted to a set of stationary solutions,} \tag{24}$$

see e.g. Cazenave-Haraux [4, §9] and references therein. We also discuss in this note how to obtain this fact, see Proposition 3.1 below. As for a subcritical problem on the entire domain, see e.g. Kavian [32], Kawanago [33] and references therein. See e.g. Cortázar-del Pino-Elgueta [7], Feireisl-Petzeltová [11], Chill-Jendoubi [6] and references therein for (P) with a linear term.

Critical case There is not so much result on the case $p = 2^*$, a critical problem. As for the asymptotics of time-global solution, it is pointed out in Ni-Sacks-Tavantzis [42] that (P) with bounded domain admits a time-global *weak* solution which is unbounded in L^∞-sense. Since the solution treated in [42] is a weak one, it is not clear whether the solution blows-up in finite time or not in a classical sense. Later, it is proved in Galaktionov-Vazquez [15] that these solutions are indeed time-global in the classical sense under the assumtion of radial symmetry and nonnegativity of solutions. The precise asymptotics of these solutions are given in [27] which is described as

$$u(\cdot,t) - \|u(t)\|_\infty U(\|u(t)\|_\infty^{\frac{2}{N-2}}\cdot) = o(1) \text{ in } \dot{H}^1 \tag{25}$$

as $t \to \infty$, where U is a unique nonnegative nontrivial stationary solution of (P) (in $\mathbb{R}^N$) with $\|U\|_\infty = 1$ (U is called a Talenti function, see [48] and e.g. [47, §I]). This results shows that the solution u behaves like a scaling of a nontrivial stationary solution of (P) in the long-time asymptotics. Since $\dot{H}^1$-norm is invariant under the scaling appeared in (25) (see Propoition 3.1 below), we have

$$\|\nabla u(t)\|_2^2 = \|\nabla(\|u(t)\|_\infty U(\|u(t)\|_\infty^{\frac{2}{N-2}}\cdot))\|_2 + o(1) = \|\nabla U\|_2^2 + o(1) \tag{26}$$

as $t \to \infty$, thus (1) holds for this solution. Based on this fact, it is proved in [27] that the time-global bounds (1) is true for any time-global, radially symmetric and nonnegative solution u of (P) in ball or $\mathbb{R}^N$. For the validity of (1) for another case, see e.g. [26] and references therein.

The asymptotics (25) suggests that the general asymptotic behavior in the critical case is not so simple as in the subcritical case (24). Indeed, for (P) on a ball, it is proved in [27] that there holds $\|u(t)\|_\infty \to \infty$ as $t \to \infty$, hence a solution in (25) concentrates at the origin as $t \to \infty$ while the Sobolev norm is bounded (26). Observe that this u does not converges to any function in the strong $\dot{H}^1$-topology, since $u(t) \rightharpoonup 0$ as $t \to \infty$ in $\dot{H}^1$ (this comes from $u(x,t) \to 0$ a.e. x as $t \to \infty$ by (25)) while $\|\nabla u(t)\|_2^2 \not\to 0$ which is obvious from (26). Hence, in the critical case, some time-global solution exhibit different behavior from the absorbtion to a set of stationary solution and the validity of (1) for general time-global solution is an open problem so far.

We claim in this note that, in spite of these evidences which indicate the difference between the subcritical and the critical case, general nonnegative time-global solution of (P) with $p = 2^*$ and $\Omega = \mathbb{R}^N$ satisfy (1) (Theorem 1.3). Moreover, we will clarify the fact that, different from the subcritical case, time-global solutions behave like a finite number of superposition of rescaled and translated starionary solutions (Theorem 1.4) as is implied by the asymptotics (25) for the radially symmetric case.

2 Backgrounds: methods valid for the subcritical case are not applicable for the critial case

2.1 Problem with subcritical and bounded domain: a compact case

In order to motivate Theore 1.3 and Theorem 1.4 and to clarify the difficulty in the critical case further, let us review the argument for the subcritical problem in a bounded domain. We always assume

$$p < 2^* \text{ and } \Omega \text{ is a bounded domain in } \mathbb{R}^N \tag{1}$$

in this subsection unless stated.

Proposition 2.1 (Bounds and asymptotics in the subcritical and the bounded domain case)

Let us assume (1) *and let u be a time-global solution of (P) with $p < 2^*$. Then we have the following:*

(a) *There holds* $\sup_{t>0} \|\nabla u(t)\|_2 < \infty$.
(b) *Let* $E(u_0)$ *be a set defined by*

$$E(u_0) := \{\varphi;\ \varphi \textit{ is a stationary solution of } (\mathrm{P})\}.$$

Then there holds $\mathrm{dist}_{L^p}(u(t), E(u_0)) \to 0$ *as* $t \to \infty$.

Remark 2.1 (On the applicability of the argument for subcritical and bounded domain case to the critical case)
The heart of the proof of proposition above is that

- for (a): control of an oscillation of $\|u(t)\|_p$. The following proof of this needs the subcriticality of the nonlinearity (6).
- for (b): compactness of the embedding $\dot{H}^1 \hookrightarrow L^p$. This needs the subcriticality of the nonlinearity and the boundedness of the domain.

Hence the following proof of Proposition 2.1 cannot be applied to our original problem, i.e., problem with critical exponent and on the entire domain. See §2.2 for more detail. ▮

Remark 2.2 (The convergence in $\dot{H}^1$)
In (b), it is not hard to extend the convergence in $\dot{H}^1$, say, by using L^∞-global bounds (see e.g. Cazenave-Lions [5], Giga [17]). The verification needs more analysis and we omit here for the simplicity. Note that there exists a time-global solution without L^∞-global bounds in the critical case, see (25), see also Remark 1.9 and Open problem 5.2. ▮

2.1.1 Prelimnaries

Here we introduce preliminary facts for the proof of Proposition 2.1.

Evolution equation aspects First we recall that the energy equality (6) holds, hence the decreasing property of J_p along the orbit of u is assured. Lemma 6.1 implies that the limit of the energy along a time-global solution (in a bounded domain) is nonnengative, hence we obtain

$$J_p(u_0) \geq J_p(u(t)) \downarrow d \text{ as } t \to \infty \tag{2}$$

with $d \geq 0$.

The proof of Proposition 2.1 (a) heavily relies on the uniform dependence of a local existence time of solution of (P) on L^p-norm of the initial data.

Lemma 2.1 (Non-oscillation theorem for $\|u(t)\|_p$ in the subcritical case)
For any $M > 0$, there exists $T(M) > 0$ which satisfies the following: for any solution u of (P) *with u_0 satisfying $\|u_0\|_p \leq M$,*

$$\|u(t)\|_p \leq 2\|u_0\|_p \textit{ for } t \leq T(M) \tag{3}$$

holds.

Proof of Lemma 2.1.
Taking L^p-norm of (5), we see

$$\|u(t)\|_p \leq \|u_0\|_p + \int_0^t ds \|e^{(t-s)\Delta}u^{p-1}\|_p.$$

Recall the decay estimates of $e^{t\Delta}$ (see e.g. Giga-Giga-Saal [18, §1.1.2]):

$$\|e^{t\Delta}\varphi\|_r \leq \frac{C}{t^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}}\|\varphi\|_q, \tag{4}$$

where $1 \leq q \leq r \leq \infty$ and $\varphi \in L^q$. By using L^p-$L^{\frac{p}{p-1}}$ estimate of $e^{t\Delta}$ above, we have

$$\int_0^t ds\|e^{(t-s)\Delta}u^{p-1}\|_p \leq \int_0^t ds \frac{C}{(t-s)^{\frac{N}{2}\left(\frac{p-1}{p}-\frac{1}{p}\right)}}\|u\|_p^{p-1}. \tag{5}$$

The convergence of s-integral needs $\delta := 1 - \frac{N}{2}\left(\frac{p-1}{p} - \frac{1}{p}\right) > 0$, which is (somewhat remarkably) equivalent to

$$p < \frac{2N}{N-2}(= 2^*). \tag{6}$$

By Combining these two relations, we have

$$\|u(t)\|_p \leq \|u_0\|_p + Ct^\delta \max_{s\in[0,t]} \|u(s)\|_p^{p-1}.$$

Thus for $\max_{s\in[0,t]} \|u(t)\|_p =: M_p(t)$, we obtain

$$M_p(t) \leq \|u_0\|_p + Ct^\delta M_p(t)^{p-1}. \tag{7}$$

Now suppose that $M_p(t)$ reached the twice of the L^p-norm of initial data:

$$M_p(t) = 2\|u_0\|_p. \tag{8}$$

Then by (7), we see that

$$2\|u_0\|_p \leq \|u_0\|_p + Ct^\delta(2\|u_0\|_p)^{p-1},$$

which is equivalent to

$$T(\|u_0\|_p) := \left(\frac{1}{2^{p-1}C\|u_0\|_p^{p-2}}\right)^{\frac{1}{\delta}} \leq t.$$

This together with (8) yields

$$\|u(t)\|_p \leq 2\|u_0\|_p \text{ if } t \leq T(\|u_0\|_p)$$

by taking a contraposition. ∎

Variational aspects The proof of Proposition 2.1 heavily relies on the variational aspect of an energy funcitonal J_p along the orbit of u:

Lemma 2.2 (Palais-Smale analysis along the orbit)
Let (t_n) be a time sequecne satsifying $\partial_t u(t_n) \to 0$ in L^2 as $n \to \infty$. Then the follwing holds.
(a) *$(u(t_n))$ satisfies $\|(dJ_p)_{u_n}\|_{(\dot{H}^1)^*} \to 0$ as $n \to \infty$.*
(b) *There holds $\|\nabla u(t_n)\|_2^2 = \|u(t_n)\|_p^p + o(1) = \frac{d}{\frac{1}{2}-\frac{1}{p}} + o(1)$ as $n \to \infty$.*
(c) *Then there exists a stationary solution $u \in \dot{H}^1$ of* (P) *and a subsequence of $(u(t_n))$ such that $u(t_n) \to u$ in $\dot{H}^1$ as $n \to \infty$.*

Remark 2.3 (Terminology from the the variationl analysis)
Let J be a C^1-functional on a Banach space X and let $(u_n) \subset X$. Then

- (u_n) is said to be a "Palais-Smale sequence of J at level d" if

$$\|(dJ)_{u_n}\|_{X^*} \to 0, \ J(u_n) \to d \text{ as } n \to \infty.$$

- J is said to satisfy a "Palais-Smale condition at level d" if *every Palais-Smale sequence of J at level d contains a strongly convergent subsequence.*

In these terminology, Lemma 2.2 says that $(u(t_n))$ is a Palais-Smale sequence of J_p at level d if (t_n) is a time-sequence satisfying $\|\partial_t u(t_n)\|_2 = o(1)$ as $n \to \infty$. Moreover, the proof below is essentially the same for the verification of the validity of the Palais-Smale condition at level d for J_p in the subcritical and bounded domain case, see e.g. Willem [53, §1]. Lemma 2.2 implies the analysis of the asymptptic behavior of a time-global solution of (P) is closely related with the variational analysis of the energy functional J_p. ∎

Proof of Lemma 2.2.

Let (t_n) be a sequence which satisfies

$$\partial_t u(t_n) \to 0 \text{ in } L^2 \tag{9}$$

as $n \to \infty$.

(a) First we show that $(u(t_n))$ satisfies $\|(dJ_p)_{u_n}\|_{(\dot{H}^1)^*} \to 0$ as $n \to \infty$. Now observe that, for any $\varphi \in \dot{H}^1$, we have

$$\left|\int (\Delta u(t_n) + u(t_n)|u(t_n)|^{p-2})\varphi\right| = \left|\int \partial_t u(t_n)\varphi\right| \leq \|\partial_t u(t_n)\|_2 \|\varphi\|_2. \tag{10}$$

Note that the Poincaré inequality holds since Ω is a bounded domain. Thus we obtain

$$(10) \leq C\|\partial_t u(t_n)\|_2 \|\nabla\varphi\|_2 \tag{11}$$

and we see

$$\begin{aligned}
& \|(dJ_p)_{u(t_n)}\|_{(\dot{H}^1(\Omega))^*} \\
= & \sup_{\varphi\in\dot{H}^1(\Omega),\ \|\nabla\varphi\|_2=1} \left|\int \left(-\nabla u(t_n)\nabla\varphi + u(t_n)|u(t_n)|^{p-2}\varphi\right)\right| \\
= & \sup_{\varphi\in\dot{H}^1(\Omega),\ \|\nabla\varphi\|_2=1} \left|\int (\Delta u(t_n) + u(t_n)|u(t_n)|^{p-2})\varphi\right| \\
\leq & \sup_{\varphi\in\dot{H}^1(\Omega),\ \|\nabla\varphi\|_2=1} C\|\partial_t u(t_n)\|_2 \|\nabla\varphi\|_2 \\
= & C\|\partial_t u(t_n)\|_2 = o(1)
\end{aligned}$$

as $n \to \infty$, where we have used (9) in the last line. This implies $\|(dJ_p)_{u_n}\|_{(\dot{H}^1)^*} \to 0$ as $n \to \infty$.

(b) By $\|(dJ_p)_{u_n}\|_{(\dot{H}^1)^*} \to 0$ as $n \to \infty$, we see that $(dJ_p)_{u_n}(\frac{u(t_n)}{\|\nabla u(t_n)\|_2}) \to 0$ as $n \to \infty$, i.e.,

$$\|\nabla u(t_n)\|_2^2 - \|u(t_n)\|_p^p = o(1)\|\nabla u(t_n)\|_2 \tag{12}$$

as $n \to \infty$. Moreover, by the decreasing property of the energy (2), we also see that

$$\frac{1}{2}\|\nabla u(t_n)\|_2^2 - \frac{1}{p}\|u(t_n)\|_p^p = J_p(u(t_n)) = d + o(1)$$

as $n \to \infty$. From these relations, we easily obtain

$$\left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u(t_n)\|_2^2 = d + o(1)\|\nabla u(t_n)\|_2 + o(1),$$

which yields the boundedness of $\|\nabla u(t_n)\|_2$. By using this bounds together with (12), we have

$$\|\nabla u(t_n)\|_2^2 = \|u(t_n)\|_p^p + o(1) \tag{13}$$

as $n \to \infty$. This relation and (2) leads

$$\begin{aligned} d + o(1) &= J_p(u(t_n)) = \frac{1}{2}\|\nabla u(t_n)\|_2^2 - \frac{1}{p}\|u(t_n)\|_p^p \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u(t_n)\|_2^2 + o(1), \end{aligned}$$

hence

$$\|\nabla u(t_n)\|_2^2 = \frac{d}{\frac{1}{2} - \frac{1}{p}} + o(1)$$

as $n \to \infty$. This relation and (13) yields the conclusion for $\|u(t_n)\|_p^p$.
(c) By (b) and the compactness of $\dot{H}^1 \hookrightarrow L^p$, we see that

$$u(t_n) \rightharpoonup u \text{ weakly in } \dot{H}^1 \text{ and strongly in } L^p \tag{14}$$

for some $u \in \dot{H}^1$ along a subsequence. Particularly, by $\|(dJ)_{u(t_n)}\|_{(\dot{H}^1)^*} \to 0$ as $n \to \infty$, we have $(dJ)_{u(t_n)}(u) \to 0$ as $n \to \infty$, i.e.,

$$\int \nabla u(t_n) \nabla u - \int |u(t_n)|^{p-2} u(t_n) u = o(1).$$

This relation yields

$$\|\nabla u\|_2^2 = \|u\|_p^p \tag{15}$$

since (14) implies

$$\int \nabla u(t_n) \nabla u = \int \nabla u \nabla u + o(1), \quad \int |u(t_n)|^{p-2} u(t_n) u = \int |u|^p + o(1).$$

Hence by (b), (14) and (15), we have

$$\|\nabla u(t_n)\|_2^2 = \|u(t_n)\|_p^p + o(1) = \|u\|_p^p + o(1) = \|\nabla u\|_2^2 + o(1)$$

and from this relation together with (14) we obtain

$$u(t_n) \to u \text{ strongly in } \dot{H}^1$$

as $n \to \infty$.

Now we take any $\varphi \in \dot{H}^1$. From the convergence above togther with the Sobolev embedding, we see that

$$\int \nabla u(t_n)\nabla\varphi = \int \nabla u\nabla\varphi + o(1), \tag{16}$$

$$\int u(t_n)|u(t_n)|^{p-2}\varphi = \int u|u|^{p-2}\varphi + o(1) \tag{17}$$

as $n \to \infty$. Observe that the assertion (a) implies

$$\left| -\int \nabla u(t_n)\nabla\varphi + \int u(t_n)|u(t_n)|^{p-2}\varphi \right| = |(dJ_p)_{u(t_n)}(\varphi)| = o(1)$$

as $n \to \infty$, which together with (16) and (17) yields

$$\int \nabla u\nabla\varphi = \int u|u|^{p-2}\varphi,$$

i.e., u is a weak stationary solution of (P). The standard elliptic regularity says that u is a classical stationary solution of (P), see e.g. [47, Appendix B]. This completes the proof. ∎

Remark 2.4 (The validity for the critical case I)

Note that, for a bounded domain Ω, proofs above for Lemma 2.2 (a) and (b) hold true also for $p = 2^*$. In the proof of Lemma 2.2 (c), the compactness of of the Sobolev embedding is used to obtain (14) (the strong convergence in L^p) and this is the only place we need the subcriticality of p in the proof of Lemma 2.2. ∎

The following partial result on the bounds for L^p-norm immediately follows from the Lemma above:

Corollary 2.1 (Liminf is finite)

There holds $\liminf_{t\to\infty} \|u(t)\|_p < \infty$.

Proof of Corollary 2.1.

First we claim that

$$\text{there exists a time sequecne satsifying } \partial_t u(t_n) \to 0 \text{ in } L^2. \tag{18}$$

Indeed, by (2), there exists $t_n \to \infty$ such that

$$\left.\frac{d}{dt} J_p(u(t))\right|_{t=t_n} = o(1)$$

as $n \to \infty$ (since otherwise $J_p(u(t)) \to -\infty$ as $t \to \infty$). Observe that the energy equality (6) yields

$$\left.\frac{d}{dt} J_p(u(t))\right|_{t=t_n} = -\|\partial_t u(t_n)\|_2^2.$$

By combining these relatios, we have

$$\|\partial_t u(t_n)\|_2^2 = o(1)$$

as $n \to \infty$, whence follows (18).

The assertion (18) and Lemma 2.2 (b) yields

$$\|\nabla u(t_n)\|_2^2 = \|u(t_n)\|_p^p + o(1) = \frac{d}{\frac{1}{2} - \frac{1}{p}} + o(1),$$

as $n \to \infty$, which implies the conclusion. ∎

Remark 2.5 (The validity for the critical case II)

Note that the proof of Corollary 2.1 is based on (2), Lemma 2.2 (a) and (b) and they hold true also for the critical case, see Remark 2.4 and Remark 6.1. Thus the conclusion of Corollary 2.1 is true for (P) on bounded Ω with $p = 2^*$. ∎

Hence in order to prove the existence of a global bound for $\|u(\cdot)\|_p$, it is enough to exclude the possibility of an oscillation of $\|u(\cdot)\|_p$. This is done by using Lemma 2.1.

2.1.2 The existence of global bounds for Soboev norm: proof of Proposition 2.1 (a)

Now we show $\sup_{t>0} \|u(t)\|_p < \infty$. Assume on the contrary $\limsup_{t\to\infty} \|u(t)\|_p < \infty$, namely, there exists (τ_n) such that $\tau_n \to \infty$ and

$$\|u(\tau_n)\|_p \to \infty \tag{19}$$

as $n \to \infty$. Let $T(\cdot)$ be a function appeared in Lemma 2.1 and let $u_n(s) := u(\tau_n + s)$, where $s \in [-T(M), 0]$ with $M := 2\left(\frac{d}{\frac{1}{2}-\frac{1}{p}}\right)^{\frac{1}{p}}$. By the energy equality (6) and the decreasing property of the energy (2) with a finite energy limit $d \geq 0$, we have

$$\begin{aligned}
\int_{-T(M)}^{0} ds\|\partial_s u_n(s)\|_2^2 &= \int_{\tau_n - T(M)}^{\tau_n} dt\|\partial_t u(t)\|_2^2 \\
&= J_p(u(\tau_n - T(M))) - J_p(u(\tau_n)) = d - d + o(1) \\
&= o(1).
\end{aligned}$$

Hence, for a.e. $s \in [-\frac{T(M)}{2}, 0]$, there holds

$$\partial_t u(\tau_n + s) \to 0 \text{ in } L^2.$$

Take such $s \in [-\frac{T(M)}{2}, 0]$ and let $t_n := \tau_n + s$. Then the above relation says that (t_n) satisfies $\partial_t u(t_n) \to 0$ in L^2, i.e., the assumption of Lemma 2.2 (b) holds, which yields Hence we have

$$\|u(t_n)\|_p^p = \frac{d}{\frac{1}{2}-\frac{1}{p}} + o(1) \leq 2\left(\frac{d}{\frac{1}{2}-\frac{1}{p}}\right)^{\frac{1}{p}} = M \tag{20}$$

for large n. Hence by regarding $u(t_n)$ as an initial data and applying Lemma 2.1, we see that

$$\|u(t_n + \sigma)\|_p \leq 2\|u(t_n)\|_p, \quad \sigma \in [0, T(M)]$$

holds. Recall that $\tau_n = t_n - s$ and $s \in [-\frac{T(M)}{2}, 0]$. Hence we can put $\sigma := -s$ in the relation above, and thus we have

$$\|u(\tau_n)\|_p \leq 2\|u(t_n)\|_p. \tag{21}$$

Relations (19), (20) and (21) yield a contradiction.

2.1.3 The asymptotic behavior of time-global solutions: proof of Proposition 2.1 (b)

Let us assume that, on the contrary, the conclusion does not hold. Then there exists a time sequecne (t_n) with $t_n \to \infty$ as $n \to \infty$ and $\varepsilon > 0$ satsifying

$$\mathrm{dist}_{L^p}(u(t_n), E(u_0)) \geq \varepsilon. \tag{22}$$

Let $u_n(s) := u(t_n + s)$ for $s \in [0, 1]$.

Step 1. Identification of the limit I. Existence.

Note that $(u_n(0)) \subset \dot{H}^1$ is bounded by Proposition 2.1 (a) (note that $u_n(0) = u(t_n)$). Hence by

$$\text{the compactness of } \dot{H}^1 \hookrightarrow L^p, \tag{23}$$

which is assured by the assumption of the subcriticality and the boundedness of the domain, we see that, passing to a subsequence if necessary,

$$u_n(0) \to u(0) \text{ strongly in } L^p \tag{24}$$

as $n \to \infty$, where $u(0)$ is some element of $\dot{H}^1$.

Step 2. Identification of the limit II. Stationary solution.

We show that $u(0)$ is a stationary solution of (P).

By the energy equality (6) and the decreasing property (2) with finite d, we have

$$\begin{aligned} \int_0^1 ds \|\partial_s u_n(s)\|_2^2 &= \int_{t_n}^{t_n+1} dt \|\partial_t u(t)\|_2^2 = J_p(u(t_n)) - J_p(u(t_n + 1)) \\ &= d - d + o(1) = o(1). \end{aligned} \tag{25}$$

Hence, for a.e. $s \in [0, 1]$, there holds

$$\partial_t u(t_n + s) \to 0 \text{ in } L^2.$$

Take such $s \in [0, 1]$. Then the above relation says that $(t_n + s)$ satisfies $\partial_t u(t_n + s) \to 0$ in L^2. Hence by the subcriticality asuumption together with Lemma 2.2 (c), we see that

$$u(t_n + s) \to u(s) \text{ strongly in } \dot{H}^1 \tag{26}$$

as $n \to \infty$, where

$$u(s) \text{ is a stationary solution of (P).} \tag{27}$$

We prove $u(s) = u(0)$. Indeed, we have

$$\begin{aligned}\|u(s) - u(0)\|_2 &\leq \|u(s) - u(t_n + s)\|_2 + \|u(t_n + s) - u(t_n)\|_2 \\ &\quad + \|u(t_n) - u(0)\|_2 \\ &= o(1)\end{aligned}$$

since (24), (26) together with the boundedness of Ω and

$$\|u(t_n + s) - u(t_n)\|_2 \leq \int_0^s d\sigma \, \|\partial_\sigma u_n(\sigma)\|_2 \leq \sqrt{s}\sqrt{\int_0^1 ds \|\partial_s u_n(s)\|_2^2} = o(1)$$

by (25). This relation and (27) yields

$$u(0) \text{ in (24) is a stationary solution.} \tag{28}$$

By the decreasing property of the energy (2), we also have

$$J(u(0)) \leq J(u_0).$$

This relation together with the result above says $u(0) \in E(u_0)$. Hence (24) yields, along a subsequence,

$$\mathrm{dist}_{L^p}(u(t_n), E(u_0)) \to 0$$

as $n \to \infty$, contradicting (22). ∎

2.2 Difficulty in the critical and the whole domain case

As is already mentioned in Remark 2.1, the heart of the proof of Proposition 2.1 is that the control an oscillation of $\|u(t)\|_p$ for (a) and the compactness of the embedding $\dot{H}^1 \hookrightarrow L^p$ for (b). The former needs the subcriticality of the nonlinearity (6) while the latter the subcriticality of the nonlinearity and the boundedness of the domain to assure (23).

In our original problem (P), i.e., the entire domain and the critical case, an argument given in the previous subsection confronts several difficulties. In this subsection, we discuss this difficulty.

Difficulty comes from the noncompactness of the Sobolev embedding First we try to obtain

$$\liminf_{t \to \infty} \|u(t)\|_{2^*} < \infty, \tag{29}$$

a part of the assetion of Theorem 1.3, which is proved in Corollary 2.1 for the subcritical, bounded domain case. Note that the proof of Corollary 2.1 for obtaining (29) proceed in the following steps:

- **Step 1.** Proof of the existence of a time sequence (t_n) such that $\|\partial_t u(t_n)\|_{L^2} = o(1)$ as $n \to \infty$.
- **Step 2.** Proof of the fact that $\|\partial_t u(t_n)\|_{L^2} = o(1)$ as $n \to \infty$ can be sharpened to $\|\partial_t u(t_n)\|_{(\dot{H}^1)^*} = o(1)$ as $n \to \infty$ (Lemma 2.2 (a)).

Recall that **Step 1** requires an energy decreasing property (2) with a finite energy limit d. The decreasing property of $J_p(u(t))$ immediately follows from the energy equality (6) while the proof of the finiteness of the limit d given in Lemma 6.1 requires the boundedness of the domain. As is stated in Remark 6.1, it seems difficult to extend the proof of Lemma 6.1 to general unbounded domains.

Also, **Step 2** needs the Poincaré inequality (11), hence this step cannot be cleared for general unbounded domains. Of course for several unbounded domains such as infinite strip-like domains the Poincaré inequality holds and **Step 2** may be cleared for such domains .

Secondly, we try to show Theorem 1.4, the asymptotics of time-global solutions, which is proved in Propoition 2.1 (b) for the subcritical, bounded domain case. Recall that the verification of Proposition 2.1 (b), particularly (28), heavily relies on Lemma 2.2 (c) which needs the compactness of the Sobolev embedding as is observed in the proof of it.

These observations show that it is difficult to prove Theorem 1.3 and Theorem 1.4 by using the direct extension of the argument for Proposition 2.1, the subcriticl and the bounded domain case.

Difficulty comes from the control of the oscillation Moreover, even if we can get (29), we have to prove

$$\limsup_{t\to\infty} \|u(t)\|_p < \infty$$

to obtain Theorem 1.3. This is heavily related with the "prevention of the oscillation of $\|u(t)\|_p$" in t which is given in Lemma 2.1 by using the standard decay estimate of the heat kernel $e^{t\Delta}$ for the subcritical case. Here recall that the proof of Lemma 2.1 again requires (6), the subcriticality in an essential way. Observe that this time the subcriticality is needed to assure the convergence of s-integral in (5) and seems difficult to extend to the critical case. Indeed, Lemma 2.1 says that the local existence time $T(\cdot)$ can be taken uniformly for the bounded set of initial data in L^p. In the critical case, this is not proved (see Open problem 5.5) and the local existence time can be taken uniformly only for a "compact" set of an initial data in

L^{2^*}, see e.g. Brezis-Cazenave [3] and Ruf-Terraneo [45]. This indicates the possibility of the existence of a finite time blow-up solution which develops a singularity in the L^∞-sense by keeping L^{2^*}-norm bounded but losing a compactness in L^{2^*} at T_m. A solution which have the asymptotics above is indeed constructed by Schweyer [46]. Also, in several heat flows associated with the "critical" geometric functionals such as harmonic map heat flow admits this kind of blow-up phenomena which is called a "bubbling", see Open problem 5.5.

These suggests that the assumption of the subcriticality in the proof of Lemma 2.1 (a) (hence in that of Proposition 2.1 (a)) above is an essential one and it is not clear how to get the boundedness of $\|u(t)\|_p$ in the critical case.

Strategy for the critical and the entire domain case The consideration above shows that the direct extension of the argument which is valid for the subcritical case to the critical case is not so obvious, thus the existence of a global bound for the critical problem remains an open problem for a certain period of time.

In the following, we abondon the standard argument above which relies on the subcriticality and the compactness of the Sobolev embedding, and rely on a different approach based on the scaling argument together with the "profile decomposition", a compactness device which gives the detailed information for the lack of the compactness in the critical case. Based on it, we prove Proposition 4.6, a substitute of Lemma 2.1 in the critical case and clarify the asymptotic behavior of time-global solutions. Consequently, we have Theorem 1.3 and Theorem 1.4.

3 Preliminaries

We introduce preliminary facts which will be needed in the proof of Theorem 1.1, Theorem 1.3 and Theorem 1.4.

3.1 Scaling invariance and the existence of a balanced time sequence

In this subsection, we check the invariance property of (P) on $\mathbb{R}^N$ with $p = 2^*$ and J_{2^*} under the scaling with x, t and u, and introduce the existence of time sequence (t_n) satsifying

$$\|\nabla u(t_n)\|_2^2 = \|u(t_n)\|_{2^*}^{2^*} + o(1) \tag{1}$$

as $n \to \infty$.

Let u be a solution of (P) and let $\mu > 0$. For any $x_0 \in \mathbb{R}^N$ and $t_0 > 0$, let

$$y := \mu(x - x_0), \quad s := \mu^2(t - t_0), \quad \mu^{\frac{2}{p-2}} u_{\mu,x_0}(y, s) := u(x, t). \tag{2}$$

Then it is easy to see that

Proposition 3.1 (Scale invariance)
Let $\delta > 0$. Then u_{μ,x_0} satisfies

$$\partial_s u_{\mu,x_0} = \Delta_y u_{\mu,x_0} + u_{\mu,x_0}|u_{\mu,x_0}|^{p-2} \ in\ \mathbb{R}^N \times [0, \delta]$$

if and only if u satisfies

$$\partial_t u = \Delta_x u + u|u|^{p-2} \ in\ \mathbb{R}^N \times \left[t_0, t_0 + \frac{\delta}{\mu^2}\right].$$

Moreover, we have

$$\begin{aligned} &\mu^{\frac{N-2}{p-2}(2^*-p)} \int_0^\delta \|\partial_s u_{\mu,x_0}\|_2^2 ds = \int_{t_0}^{t_0+\frac{\delta}{\mu^2}} \|\partial_t u\|_2^2 dt, \\ &\mu^{\frac{N-2}{p-2}(2^*-p)} \|\nabla u_{\mu,x_0}(s)\|_2 = \|\nabla u(t)\|_2, \\ &\mu^{\frac{N-2}{p-2}\left(\frac{2}{N-2}(r-p)+2^*-p\right)} \|u_{\mu,x_0}(s)\|_r = \|u(t)\|_r. \end{aligned}$$

Remark 3.1 (The peculiarity of the critical problem)
The proposition above says that the problem (P) is always invariant under (2). The important feature of the critical case is that only in this case, the energy structure, i.e., $L^2(I; L^2)$, $\dot{H}^1$ and L^p-norms, is also invariant, i.e., there hold

$$\begin{aligned} &\int_0^\delta \|\partial_s u_{\mu,x_0}\|_2^2 ds = \int_{t_0}^{t_0+\frac{\delta}{\mu^2}} \|\partial_t u\|_2^2 dt, \\ &\|\nabla u_{\mu,x_0}(s)\|_2 = \|\nabla u(t)\|_2, \\ &\|u_{\mu,x_0}(s)\|_{2^*} = \|u(t)\|_{2^*}, \\ &(\|u_{\mu,x_0}(s)\|_2 = \mu\|u(t)\|_2). \end{aligned}$$

This is one of the origin of the noncompactness for the evolution and the variational structure assocated with (P) with $p = 2^*$. ∎

Proposition 3.2 (Existence of a balanced time sequence [26])

Let u be a nonnegative time-global solution of (P) *with $p = 2^*$ and $\Omega = \mathbb{R}^N$. Then there exists $t_n \to \infty$ such that $\|\nabla u(t_n)\|_2^2 - \|u(t_n)\|_p^p = o(1)$ as $n \to \infty$.*

Remark 3.2 (On the nonnegativity assumption)

The nonnegativity assumption of solutions is only used to assure (8), and if we can establish (8) for sign-changing solutions, then we can remove the nonnegativity assumption. See Remark 1.1 and Open problem 5.1. ▮

Remark 3.3 (The existence of Palais-Smale sequences in the orbit is delicate in an unbounded domain)

The conclusion of the Proposition above is the same as in Lemma 2.2 (b). As is already mentioned in §2.2, the proof for Lemma 2.2 (b) uses the subcriticality of the nonlinearity and the boundedness of the domain in an essential way. In the proof below, we use the scale invariance assured by Proposition 3.1 together with the existence of the energy limit (8) instead of the subcriticality of the nonlinearity and the boundedness of the domain. In this Proposition, we do not know whether $(u(t_n))$ is a Palais-Smale sequecne or not, while we know it in Lemma 2.2, see Open problem 5.2. The problem whether $u(t_n)$ is a Palais-Smale or not is equivalent to the control of the behavior of L^2-norm of $(u(t_n))$ as is observed from the following proof, see also Open problem 5.3. Moreover, the existence of a time sequence (t_n) satisfying the conclusion of Proposition above seems an open problem if we consider an "essentially unbounded domain" without sacle invariance and Poincaré inequality. ▮

Proof of Proposition 3.2.

Let $\tau_n \to \infty$ be a sequence such that

$$\lim_{n\to\infty} \|u(\tau_n)\|_2 = \limsup_{t\to\infty} \|u(t)\|_2 (\leq \infty).$$

We define $\lambda_n > 0$ by

$$\lambda_n^2 := \frac{1}{\|u(\tau_n)\|_2^2} \tag{3}$$

and define y, s, u_n by $y := \lambda_n x$, $s := \lambda_n^2(t - \tau_n)$ and $u_n(y, s) := \lambda_n^{\frac{N-2}{2}} u(x, t)$. Observe that

$$\|u_n(0)\|_2^2 = \lambda_n^2 \|u(\tau_n)\|_2^2 = 1 \tag{4}$$

by Proposition 3.1 and (3). Then by Proposition 3.1, (6) and (8), there holds

$$\begin{aligned}\int_0^\delta ds\|\partial_s u_n\|_2^2 &= -J_{2^*}(u_n(\delta)) + J_{2^*}(u_n(0)) \\ &= -J_{2^*}\left(u\left(\tau_n + \frac{\delta}{\lambda_n^2}\right)\right) + J_{2^*}(u(\tau_n)) \\ &= -d + d + o(1) = o(1) \qquad (5)\end{aligned}$$

as $n \to \infty$ for any $\delta > 0$, thus

$$\|u_n(\sigma) - u_n(0)\|_2 \le \int_0^\sigma \|\partial_s u_n(s)\|_2 ds \le \sqrt{\delta}\left(\int_0^\delta \|\partial_s u_n(s)\|_2^2 ds\right)^{\frac{1}{2}} = o(1)$$

as $n \to \infty$, uniformly in $\sigma \in [0, \delta]$. This relation together with (4) yields

$$\|u_n(\sigma)\|_2^2 \le 2\|u_n(0)\|_2^2 = 2, \quad \sigma \in [0, \delta]$$

for large n. Again by (5), we can find $\eta \in [0, \delta]$ such that

$$\|\partial_s u_n(\eta)\|_2 = o(1), \qquad (6)$$

as $n \to \infty$, passing to a subsequence if necessary. Since u_n satisfies (P) due to Proposition 3.1, by multplying u_n to (P) and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned}\left|-\|\nabla u_n(\eta)\|_2^2 + \|u_n(\eta)\|_{2^*}^{2^*}\right| &\le \left|\int \partial_s u_n(\eta) u_n(\eta)\right| \\ &\le \|\partial_s u_n(\eta)\|_2 \|u_n(\eta)\|_2 = o(1) \qquad (7)\end{aligned}$$

as $\to \infty$, where we used (6) in the last line. Let $t_n := \tau_n + \frac{\eta}{\lambda_n^2}$. Then from (7) and Proposition 3.1, we obtain

$$\|\nabla u(t_n)\|_2^2 = \|\nabla u_n(\eta)\|_2^2 = \|u_n(\eta)\|_{2^*}^{2^*} + o(1) = \|u(t_n)\|_{2^*}^{2^*} + o(1),$$

which implies the conclusion. ∎

3.2 The ε-regularity

Here we prove a kind of ε-regularity result. Let

$$S := \inf_{u \in \dot{H}^1 \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2},$$

the best constant of the Sobolev embedding $\dot{H}^1 \hookrightarrow L^{2^*}$.

Proposition 3.3 (The ε-regularity)
If $\sup_{t\in[0,T_m)}\|u(t)\|_{2^*}^{2^*} < S^{\frac{N}{2}}$, *then* $T_m = \infty$.

Observe that if $\sup_{t\in[0,T_m)}\|\nabla u(t)\|_2^2 < S^{\frac{N}{2}}$, then the assumption of the Proposition above holds by virtue of the Sobolev inequality.

Remark 3.4 (The meaning of Proposition 3.3)
Recall that S is attained by a function

$$U(x) := \left[\frac{\sqrt{N(N-2)}}{1+|x|^2}\right]^{\frac{N-2}{2}}$$

and also by $U_{\mu,y}(x) := \mu^{\frac{N-2}{2}}U(\mu(x-y))$ with $\mu > 0$ and $y \in \mathbb{R}^N$ due to the scale invariace of $\dot{H}^1$ and L^{2^*}-norms. These are only minimizers for S, see [48] and also e.g. [47, p.178]. The function $U_{\mu,y}$ is called the Talenti function and it is easy to see that

$$\|\nabla U_{\mu,y}\|_2^2 = \|U_{\mu,y}\|_{2^*}^{2^*} = S^{\frac{N}{2}}.$$

The proposition above says that if the norm of u cannot obtain enough quantity to exceed that of minimizers $U_{\mu,y}$, then u cannot provide enough norm to develop a singularity which is a rescaling of $U_{\mu,y}$, consequenlty, u is time-global. This type of result is called "ε-regularity" and known to hold in a various kinds of critical heat flows. ∎

Proof of Proposition 3.3.
Asume on the contrary that $T_m < \infty$ though

$$\sup_{t\in[0,T_m)}\|u(t)\|_{2^*}^{2^*} < S^{\frac{2^*}{2^*-2}}(= S^{\frac{N}{2}}). \tag{8}$$

Note that in this case we have

$$\sup_{t\in[0,T_m)}\|\nabla u(t)\|_2^2 \le 2J_{2^*}(u_0) + \frac{2}{2^*}\sup_{t\in[0,T_m)}\|u(t)\|_{2^*}^{2^*} < \infty. \tag{9}$$

Step 0. Finding blow-up sequence.
By the blow-up alternative (4), we see that $\|u(t)\|_\infty \to \infty$ as $t \to T_m$ and we can find a sequence (t_n) satisfying $t_n \to T_m$ and

$$\sup_{t\le t_n}\|u(t)\|_\infty = \|u(t_n)\|_\infty \to \infty \tag{10}$$

as $n \to \infty$. Now take $(x_n) \subset \mathbb{R}^N$ such that

$$|u(x_n, t_n)| \geq \frac{1}{2}\|u(t_n)\|_\infty. \tag{11}$$

Step 1. Construction of a rescaled solution sequence.

Let us introduce an family of rescaled function $u_n(y,s)$ of $u(x,t)$ by

$$\lambda_n^{\frac{N-2}{2}} u_n(y,s) = u(x,t), \tag{12}$$

where

$$\lambda_n^{\frac{N-2}{2}} := \|u(t_n)\|_\infty, \quad y := \lambda_n(x - x_n), \quad s := \lambda_n^2(t - t_n). \tag{13}$$

Then it is obvious from (10), (11), (12) and (13) that

$$\|u_n(s)\|_\infty \leq \|u_n(0)\|_\infty = 1, \; s \in [-1,0], \quad |u_n(0,0)| \geq \frac{1}{2}. \tag{14}$$

Note that by Proposition 3.1 and (5), u_n satisfies

$$u_n(s) = e^{t\Delta}u_n(0) + \int_0^s d\sigma e^{(s-\sigma)\Delta}u_n(\sigma)|u_n(\sigma)|^{2^*-2}.$$

Taking L^∞-norm of both sides of this relation and using L^∞-L^∞ estimate of $e^{t\Delta}$ (see (4)), we see

$$\begin{aligned} \|u_n(s)\|_\infty &\leq \|u_n(0)\|_\infty + \int_0^s d\sigma \|e^{(s-\sigma)\Delta}u_n^{2^*-1}\|_\infty \\ &\leq \|u_n(0)\|_\infty + s \sup_{\sigma\in[0,s]} \|u_n(\sigma)\|_\infty^{2^*-1}. \end{aligned}$$

Then for $\max_{\sigma\in[0,s]} \|u_n(\sigma)\|_\infty =: M_{n,\infty}(s)$, we obtain

$$M_{n,\infty}(s) \leq \|u_n(0)\|_\infty + sM_{n,\infty}(s)^{2^*-1}. \tag{15}$$

Now suppose that $M_{n,\infty}(s)$ reached the twice of the L^∞-norm of initial data:

$$M_{n,\infty}(s) = 2\|u_n(0)\|_\infty. \tag{16}$$

Then by (15), we see that

$$2\|u_n(0)\|_\infty \leq \|u_n(0)\|_\infty + s(2\|u_n(0)\|_\infty)^{2^*-1},$$

which is equivalent to

$$T_\infty(\|u_n(0)\|_\infty) := \frac{1}{2^{2^*-1}\|u_0\|_\infty^{2^*-2}} \leq s.$$

This together with (16) yields (by taking a contraposition)

there holds $\|u_n(s)\|_\infty \leq 2\|u_n(0)\|_\infty$ if $s \leq T_\infty(\|u_n(0)\|_\infty) =: \delta$.

Consequently, cobining this relation with (14), we have

$$\sup_{s\in[-1,\delta]} \|u_n(s)\|_\infty \leq 2 \tag{17}$$

for any n.

Step 2. Convergence of a rescaled solution sequence

Now the L^p-regularity theory of parabolic operators (see e.g. [36, p.172, Theorem 7.13]) implies that (u_n) is a bounded sequence in $W^{2,1}_{p,\mathrm{loc}}((-1,\delta]\times\mathbb{R}^N)$ for sufficeintly large p. Then we see that (u_n) is a bounded sequence in $C^{0,\gamma;0,\frac{\gamma}{2}}$ for any $\gamma \in (0,1)$, since $W^{2,1}_{p,\mathrm{loc}} \hookrightarrow C^{0,\gamma;0,\frac{\gamma}{2}}$ if $1-\frac{N+2}{p} > 0$ and $\gamma \in (0,1)$, see e.g. [34, p.80, Lemma 3.3]. Then by the standard parabolic regularity, for $\beta \in (0,1)$, we can find a function $u \in C^{2,\beta;1,\frac{\beta}{2}}_{\mathrm{loc}}(\mathbb{R}^N \times [-1,\delta])$ such that

$$u_n \to u \text{ in } C^{2,\beta;1,\frac{\beta}{2}}_{\mathrm{loc}}(\mathbb{R}^N \times [-1,\delta]) \tag{18}$$

Particularly, this convergence and (14) lead

$$\frac{1}{2} \leq |u_n(0,0)| = |u(0,0)| + o(1) \tag{19}$$

as $n \to \infty$.

Step 3. Properties of a limit function I. Time-independence.

It follows from Porposition 3.1 that u_n is a solution of (P):

$$\partial_s u_n(s) = \Delta u_n + u_n|u_n|^{2^*-2} \text{ in } \mathbb{R}^N \times [a,b], \tag{20}$$

where $-\infty < a < b < \infty$, hence the energy equality like (6) is satisfied:

$$\int_a^b dx\|\partial_s u_n(s)\|_2^2 ds = -J_{2^*}(u_n(b)) + J_{2^*}(u_n(a)). \tag{21}$$

Note that the scale invariance of the energy functional (Proposition 3.1) and the decreasing property of the energy functional (7) for the original function u yield

$$\begin{aligned} -J_{2^*}(u_n(b)) + J_{2^*}(u_n(a)) &= -J_{2^*}\left(u\left(t_n + \frac{b}{\lambda_n^2}\right)\right) + J_{2^*}\left(u\left(t_n + \frac{a}{\lambda_n^2}\right)\right) \\ &= -d + d + o(1) = o(1) \end{aligned}$$

as $n \to \infty$. Thus by combining these two relations we have, for $-\infty < a < b < \infty$,

$$\int_a^b dx \|\partial_s u_n(s)\|_2^2 ds = o(1) \tag{22}$$

as $n \to \infty$.

Now take $K \subset \mathbb{R}^N$ be a compact set and set

$$\|u\|_{2,K}^2 := \int_K |u|^2$$

and take any s_1, $s_2 \in [-1, \delta]$. Then there holds

$$\begin{aligned} \|u(s_1) - u(s_2)\|_{2,K} &\leq \|u(s_1) - u_n(s_1)\|_{2,K} + \|u_n(s_1) - u_n(s_2)\|_{2,K} \\ &\quad + \|u_n(s_2) - u(s_2)\|_{2,K} \\ &=: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}). \end{aligned} \tag{23}$$

The convergence (18) leads

$$(\mathrm{I}) = o(1) \text{ and } (\mathrm{III}) = o(1). \tag{24}$$

Moreover, by virtue of (22), we see

$$\|u_n(s_1) - u_n(s_2)\|_2 \leq \int_{s_1}^{s_2} \|\partial_s u_n(s)\|_2 ds \leq \sqrt{\delta + 1} \left(\int_{-1}^{\delta} \|\partial_s u_n(s)\|_2^2 ds \right)^{\frac{1}{2}} = o(1)$$

as $n \to \infty$. Combining this fact with (19), (23) and (24), we have

$$u \text{ is a time-independent nontrivial function.} \tag{25}$$

Now (9) and Proposition 3.1 imply

$$\|\nabla u_n(s)\|_2^2 = \left\| \nabla u \left(t_n + \frac{s}{\lambda_n^2} \right) \right\|_2^2 < C$$

for some $C > 0$, hence by taking a subsequence, we have, as $n \to \infty$,

$$u_n(s) \rightharpoonup v \text{ weakly in } \in \dot{H}^1 \tag{26}$$

for some $v \in \dot{H}^1$.

Step 4. Properties of a limit function II. Statioinary solution.
Let $\varphi \in C_0^\infty(\mathbb{R}^N)$. By Proposition 3.1, we have

$$\begin{aligned}\int_{-1}^{\delta} ds \int_{\mathbb{R}^N} dy \partial_s u_n(s) \varphi &= \int_{-1}^{\delta} ds \int_{\mathbb{R}^N} dy \Delta u_n \varphi \\ &\quad + \int_{-1}^{\delta} ds \int_{\mathbb{R}^N} dy u_n |u_n|^{2^*-2} \varphi. \end{aligned} \tag{27}$$

By the convergence (22), we see

$$\begin{aligned}\left| \int_{-1}^{\delta} ds \int_{\mathbb{R}^N} dy \partial_s u_n(s) \varphi \right| &\leq \sqrt{(\delta+1)|\operatorname{supp}\varphi|} \left(\int_{-1}^{\delta} \|\partial_s u_n(s)\|_2^2 ds \right)^{\frac{1}{2}} \\ &= o(1) \end{aligned} \tag{28}$$

as $n \to \infty$ ($|\operatorname{supp}\varphi|$ denotes the Lebesgue measure of $\operatorname{supp}\varphi$). Moreover, from (25), the convergence (18) and (26), we have

$$\begin{aligned}\int_{-1}^{\delta} ds \int_{\mathbb{R}^N} dy \Delta u_n \varphi &= -\int_{-1}^{\delta} ds \int_{\mathbb{R}^N} dy \nabla u_n \nabla \varphi \\ &= -\int_{-1}^{\delta} ds \int_{\mathbb{R}^N} dy \nabla u \nabla \varphi + o(1) \\ &= (1+\delta) \int_{\mathbb{R}^N} dy \Delta u \varphi + o(1), \end{aligned}$$

$$\begin{aligned}\int_{-1}^{\delta} ds \int_{\mathbb{R}^N} dy u_n |u_n|^{2^*-2} \varphi &= \int_{-1}^{\delta} ds \int_{\mathbb{R}^N} dy u |u|^{2^*-2} \varphi + o(1) \\ &= (1+\delta) \int_{\mathbb{R}^N} dy u |u|^{2^*-2} \varphi + o(1), \end{aligned}$$

Thereofore combining these relation with (27), we have

$$\int_{\mathbb{R}^N} dy (\Delta u + u|u|^{2^*-2}) \varphi = 0, \quad \varphi \in C_0^\infty(\mathbb{R}^N). \tag{29}$$

This together with (18) and (25) yields $u = v$, thus

$$u \in \dot{H}^1.$$

Hence, by (25) and (29), we see that u is a nontrivial weak stationary solution of (P). By the standard classical elliptic regularity, u satisfies

$$-\Delta u = u|u|^{2^*-2}, \quad x \in \mathbb{R}^N.$$

Step 5. End of the proof.

By testing this equation with u, we have

$$\|\nabla u\|_2^2 = \|u\|_{2^*}^{2^*},$$

which together with the nontriviality of u and the Sobolev inequality yield

$$\|\nabla u\|_2^2 = \|u\|_{2^*}^{2^*} \geq S^{\frac{N}{2}}. \tag{30}$$

Moreover, by (18) and Proposition 3.1, we obtain

$$\begin{aligned} \|u\|_{2^*,B_R}^{2^*} &:= \int_{B_R} |u|^{2^*} = \int_{B_R} |u_n(s)|^{2^*} + o(1) \\ &\leq \int_{\mathbb{R}^N} |u_n(s)|^{2^*} + o(1) \\ &= \int_{\mathbb{R}^N} \left| u\left(t_n + \frac{s}{\lambda_n^2}\right) \right|^{2^*} + o(1), \end{aligned}$$

which together with (30) and the Sobolev inequality implies

$$S^{\frac{N}{2}} \leq \|u\|_{2^*}^{2^*} = \lim_{R\to\infty} \|u\|_{2^*,B_R}^{2^*} \leq \int_{\mathbb{R}^N} \left| u\left(t_n + \frac{s}{\lambda_n^2}\right) \right|^{2^*} + o(1) \leq \sup_{t\in[0,\infty)} \|u(t)\|_{2^*}^{2^*},$$

which contradicts with (8). ∎

3.3 A profile decomposition of Gérard-Jaffard

In order to analyze the asymptotic behavior of time-global solutions in the critical case, we rely on the following compactness device, see Gérard [16, THÉORÈME 1.1, REMARQUES 1.2.(b)], see also Jaffard [31, Theorem 1].

Proposition 3.4 (Profile decomposition)

Let $(u_n) \subset \dot{H}^1(\mathbb{R}^N)$ be a bounded sequence. Then there exist $(\lambda_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}_+$, $(x_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}^N$ $(j = 1, \cdots)$, $(\psi^j)_{j\in\mathbb{N}} \subset \dot{H}^1(\mathbb{R}^N)$ such that, for

$$\psi_n^j(x) := \left(\lambda_n^j\right)^{\frac{N-2}{2}} \psi^j(\lambda_n^j(x - x_n^j)),$$

there hold the following.
(a) *There holds*

$$\frac{\lambda_n^i}{\lambda_n^j} + \frac{\lambda_n^j}{\lambda_n^i} + \frac{|x_n^i - x_n^j|}{\lambda_n^i} \to \infty \text{ as } n \to \infty \text{ for } i \neq j.$$

(b) *For any $l \in \mathbb{N}$, there holds*

$$\lim_{l\to\infty} \lim_{n\to\infty} \|r_n^l\|_{2^*} = 0,$$

where $r_n^l := u_n - \sum_{j=1}^l \psi_n^j$.
(c) *There hold*

$$\begin{aligned} \|\nabla u_n\|_2^2 &= \sum_{j=1}^{l} \|\nabla\psi^j\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1), \\ \|u_n\|_{2^*}^{2^*} &= \sum_{j=1}^{\infty} \|\psi^j\|_{2^*}^{2^*} + o(1) \end{aligned}$$

as $n \to \infty$.

Remark 3.5 (The meaning of the profile decompisition)
As is mentioned in Proposition 3.1, norms of $\dot{H}^1$ and L^{2^*} have a scale and a translation invariance in the sence that $\|\nabla u_{\lambda,y}\|_2 = \|\nabla u\|_2$ and $\|u_{\lambda,y}\|_{2^*} = \|u\|_{2^*}$, where

$$u_{\lambda,y}(x) = \lambda^{\frac{N-2}{2}} u(\lambda(x-y)), \quad \lambda \in \mathbb{R}_+, \ y \in \mathbb{R}^N. \tag{31}$$

By using this invariance, it is easy to construct a bounded sequence $(u_n) \subset \dot{H}^1$ which is not strongly convergent in L^{2^*}. Indeed, let

$$u_n(x) := \lambda_n^{\frac{N-2}{2}} \varphi(\lambda_n x), \quad \lambda_n \to \infty,$$

where $\varphi \in C_0^\infty$. Then it is easy to see that (u_n) is bounded in $\dot{H}^1$ since $\|\nabla u_n\|_2 = \|\nabla\varphi\|_2$ by the scale invariance mentioned above and, $u_n(x) \to 0$ a.e. $x \in \mathbb{R}^N$ as $n \to \infty$. These together with the Sobolev embedding imply

$u_n \rightharpoonup 0$ in L^{2^*} but (u_n) cannot be strongly convergent to 0 in L^{2^*} since $\|u_n\|_{2^*} = \|\varphi\|_{2^*}$ again by the scale invariance.

The proposition above says that a kind of the converse is also true, i.e., the lack of the compactness of $\dot{H}^1 \hookrightarrow L^{2^*}$ only comes from the invariance above. Namely, for bounded sequence (u_n) in $\dot{H}^1$, if one substract l "profiles" which are the rescaling and a translation of $\varphi^j(\cdot)$, then the remainder term r_n^l tends to 0 strongly in L^{2^*} as $n \to \infty$ and $l \to \infty$. Moreover, by (a), the rescalings and translations are "mutually orthogonal" in $\dot{H}^1$. Namely, if one consider, for fixed $l \in \mathbb{N}$,

$$\begin{aligned} v_n^{j_0}(y) &:= \left(\frac{1}{\lambda_n^{j_0}}\right)^{\frac{N-2}{2}} u_n\left(x_n^{j_0} + \frac{y}{\lambda_n^{j_0}}\right) \\ &= \psi^{j_0}(y) + \sum_{i \neq j_0, 1 \leq i \leq l} \left(\frac{\lambda_n^i}{\lambda_n^{j_0}}\right)^{\frac{N-2}{2}} \psi^i\left(\frac{\lambda_n^i}{\lambda_n^{j_0}}\left[y + \frac{x_n^{j_0} - x_n^i}{\lambda_n^i}\right]\right) \\ &\quad + \left(\frac{1}{\lambda_n^{j_0}}\right)^{\frac{N-2}{2}} r_n^l\left(x_n^{j_0} + \frac{y}{\lambda_n^{j_0}}\right) \end{aligned}$$

which is a scale back of u_n focusing on the j_0-th "bubble", then $v_n^{j_0} \rightharpoonup \psi^{j_0}$ in $\dot{H}^1$ by virtue of (a), i.e., bubbles other than the j_0-th one "disappears" from the asymptotics of $v_n^{j_0}$. ∎

Remark 3.6 (Abstract aspect of the profile decomposition)

The invariance of $\dot{H}^1$-norm under (31) can be viewed as an invariance under the action of $\mathbb{R}^N \ltimes \mathbb{R}_+$, the semidirect product of $\mathbb{R}^N$ and $\mathbb{R}_+$, see §7. This structure plays an important role in generalizing Proposition 3.4 to the abstract setting, see Proposition 5.2.

4 Proofs of main results

4.1 On the potential-well structure

4.1.1 Proof of Theorem 1.1 (a)

In this subsubsection, we assume that

$$\text{there exists } t_0 \in [0, T_m) \text{ satisfying } u(t_0) \in W_{2^*} \cup \{0\}. \tag{1}$$

Proposition 4.1 (Invariance of the stable set)

There holds $u(t) \in W_{2^*} \cup \{0\}$ for any $t \in [t_0, T_m)$.

Proof of Proposition 4.1.

Suppose that the conlusion is false, namely, suppose that there exists $t_1 \in (t_0, T_m)$ such that

$$u(t_1) \notin W_{2^*} \cup \{0\}.$$

By the monotonicity of $t \mapsto J_{2^*}(u(t))$ and by the definition of W_{2^*}, we obtain

$$J_{2^*}(u(t_1)) \leq J_{2^*}(u(t_0)) < \left(\frac{1}{2} - \frac{1}{p}\right) S^{\frac{2^*}{2^*-2}},$$

which yields

$$-\|\nabla u(t_1)\|_2^2 + \|u(t_1)\|_{2^*}^{2^*} \geq 0 \tag{2}$$

since $u(t_1) \notin W_{2^*} \cup \{0\}$. By the Sobolev's inequality and (2), we have

$$S\|u(t_1)\|_{2^*}^2 \leq \|\nabla u(t_1)\|_2^2 \leq \|u(t_1)\|_{2^*}^{2^*}, \tag{3}$$

which together with $u(t_1) \neq 0$ yields

$$S^{\frac{2^*}{2^*-2}} \leq \|u(t_1)\|_{2^*}^{2^*}. \tag{4}$$

On the other hand, by $u(t_0) \in W_{2^*}$, we see

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{2^*}\right) \|u(t_0)\|_{2^*}^{2^*} &< \frac{1}{2}\|\nabla u(t_0)\|_2^2 - \frac{1}{2^*}\|u(t_0)\|_{2^*}^{2^*} = J(u(t_0)) \\ &< \left(\frac{1}{2} - \frac{1}{2^*}\right) S^{\frac{2^*}{2^*-2}}\end{aligned}$$

whence

$$\|u(t_0)\|_{2^*}^{2^*} < S^{\frac{2^*}{2^*-2}}. \tag{5}$$

Now the continuity of $t \mapsto \|u(t)\|_{2^*}$ together with (4) and (5) yields the existence of $t_2 \in (t_0, t_1]$ satisfying

$$\|u(t_2)\|_{2^*}^{2^*} = S^{\frac{2^*}{2^*-2}}. \tag{6}$$

By noting $t_0 < t_2$, one see

$$\begin{aligned}J(u(t_2)) &= \frac{1}{2}\|\nabla u(t_2)\|_2^2 - \frac{1}{2^*}\|u(t_2)\|_{2^*}^{2^*} \leq J(u(t_0)) \\ &< \left(\frac{1}{2} - \frac{1}{2^*}\right) S^{\frac{2^*}{2^*-2}}. \end{aligned} \tag{7}$$

Relations (6) and (7) imply that

$$\begin{aligned}\|\nabla u(t_2)\|_2^2 &< 2\left(\left(\frac{1}{2}-\frac{1}{2^*}\right)S^{\frac{2^*}{2^*-2}}+\frac{1}{2^*}\|u(t_2)\|_{2^*}^{2^*}\right)\\ &= 2\left(\left(\frac{1}{2}-\frac{1}{2^*}\right)S^{\frac{2^*}{2^*-2}}+\frac{1}{2^*}S^{\frac{2^*}{2^*-2}}\right)\\ &= S^{\frac{2^*}{2^*-2}}.\end{aligned} \tag{8}$$

Hence by (6) and (8), we have

$$\frac{\|\nabla u(t_2)\|_2^2}{\|u(t_2)\|_{2^*}^2}<\frac{S^{\frac{2^*}{2^*-2}}}{S^{\frac{2}{2^*-2}}}=S:=\inf_{u\in\dot{H}^1\setminus\{0\}}\frac{\|\nabla u(t)\|_2^2}{\|u(t)\|_{2^*}^2}, \tag{9}$$

a contradiction. ∎

Note that if $u \in W_{2^*}$, then

$$\left(\frac{1}{2}-\frac{1}{2^*}\right)\|u\|_{2^*}^{2^*}<\frac{1}{2}\|\nabla u\|_2^2-\frac{1}{2^*}\|u\|_{2^*}^{2^*}=J_{2^*}(u)<\left(\frac{1}{2}-\frac{1}{2^*}\right)S^{\frac{2^*}{2^*-2}},$$

thus

$$\|u\|_{2^*}^{2^*}<S^{\frac{2^*}{2^*-2}}. \tag{10}$$

This relation and Proposition 4.1 imply that

$$\sup_{t<T_m}\|u(t)\|_{2^*}^{2^*}<S^{\frac{2^*}{2^*-2}}(=S^{\frac{N}{2}}),$$

whence follows

$$T_m=\infty$$

from the ε-regularity, i.e., Proposition 3.3.

Also (10) yields, for any $u \in W_{2^*}$,

$$\begin{aligned}\|\nabla u\|_2^2 &= 2J_{2^*}(u)+\frac{2}{2^*}\|u\|_{2^*}^{2^*}<2\left(\frac{1}{2}-\frac{1}{2^*}\right)S^{\frac{2^*}{2^*-2}}+\frac{2}{2^*}S^{\frac{2^*}{2^*-2}}\\ &= S^{\frac{2^*}{2^*-2}}.\end{aligned} \tag{11}$$

Then this relation and the decreasing property of $t \mapsto J_{2^*}(u(t))$ imply the existence of $c := \lim_{t\uparrow T_m} J_{2^*}(u(t)) > -\infty$. Note that the existence of the energy limit c comes from the boundedness of W_{2^*} in $\dot{H}^1$ together with the invariance of W_{2^*} under the flow associated with u and does not use (8). Hence we do not need any additional assumption on u such as a nonnegativity to assure the existence of the energy limit c.

Proposition 4.2 (Asymptotic behavior of the orbit which intersects with the stable set)
There holds $\lim_{t\to\infty}\|\nabla u(t)\|_2 = 0$. *Consequently,* $\lim_{t\to\infty} J_{2^*}(u(t)) = 0$ *and* $\lim_{t\to\infty}\|u(t)\|_{2^*} = 0$.

Proof of Proposition 4.2.
By Proposition 3.2, there exists $t_n \to \infty$ such that

$$\|\nabla u(t_n)\|_2^2 - \|u(t_n)\|_{2^*}^{2^*} = o(1) \tag{12}$$

as $n \to \infty$. This together with (8) yields

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{2^*}\right)\|\nabla u(t_n)\|_2^2 &= \frac{1}{2}\|\nabla u(t_n)\|_2^2 - \frac{1}{2^*}\|u(t_n)\|_{2^*}^{2^*} + o(1) \\ &= J_{2^*}(u(t_n)) + o(1) = c + o(1). \qquad (13)\end{aligned}$$

From this relation and (12), we see that

$$\|u(t_n)\|_{2^*}^{2^*} + o(1) = \|\nabla u(t_n)\|_2^2 + o(1) = \frac{c}{\frac{1}{2} - \frac{1}{2^*}} =: A. \tag{14}$$

Assume that $c > 0$. Then, by letting $n \to \infty$ in the Sobolev inequality

$$S\|u(t_n)\|_{2^*}^2 \leq \|\nabla u(t_n)\|_2^2 = \|u(t_n)\|_{2^*}^{2^*} + o(1)$$

and by (14), we have

$$S^{\frac{2^*}{2^*-2}} \leq A. \tag{15}$$

On the other hand, by (10), Proposition 4.1 and the assumption (1), we see that, for large n,

$$\|u(t_n)\|_{2^*}^{2^*} \leq \frac{J_{2^*}(u(t_0))}{\frac{1}{2} - \frac{1}{2^*}} < S^{\frac{2^*}{2^*-2}},$$

hence $A < S^{\frac{2^*}{2^*-2}}$, which contradicts (15) and we conclude that $c = 0$. The other results follow form this relation and (14). ∎

4.1.2 Proof of Theorem 1.1 (b)

We start with the claim which says that V_{2^*} is also an invariant set under the flow associated with (P). The proof is similar to the one for Proposition 4.1.

Proposition 4.3 (Invariance of the unstable set)

If there exists $t_0 \in [0, T_m)$ satisfying $u(t_0) \in V_{2^*}$, then $u(t) \in V_{2^*}$ for any $t \in [t_0, T_m)$.

Proof of Proposition 4.3.

Suppose that the conlusion is false, namely, suppose that there exists $t_1 \in (t_0, T_m)$ such that

$$u(t_1) \notin V_{2^*}.$$

By the monotonicity of $t \mapsto J_{2^*}(u(t))$ and by the definition of V_{2^*}, we obtain

$$J_{2^*}(u(t_1)) \leq J_{2^*}(u(t_0)) < \left(\frac{1}{2} - \frac{1}{2^*}\right) S^{\frac{2^*}{2^*-2}},$$

which yields

$$-\|\nabla u(t_1)\|_2^2 + \|u(t_1)\|_{2^*}^{2^*} \leq 0$$

since $u(t_1) \notin V_{2^*} \cup \{0\}$. This relation togehter with the definition of V_{2^*} and the decreasing property of $t \mapsto J_{2^*}(u(t))$ assures

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{2^*}\right) \|u(t_1)\|_{2^*}^{2^*} &\leq \frac{1}{2}\|\nabla u(t_1)\|_2^2 - \frac{1}{2^*}\|u(t_1)\|_{2^*}^{2^*} = J_{2^*}(u(t_1)) \leq J_{2^*}(u(t_0)) \\ &< \left(\frac{1}{2} - \frac{1}{2^*}\right) S^{\frac{2^*}{2^*-2}},\end{aligned}$$

whence

$$\|u(t_1)\|_{2^*}^{2^*} < S^{\frac{2^*}{2^*-2}}. \tag{16}$$

On the other hand, by the Sobolev's inequality and the definition of V_{2^*}, we have

$$S\|u(t_0)\|_{2^*}^2 \leq \|\nabla u(t_0)\|_2^2 < \|u(t_0)\|_{2^*}^{2^*},$$

which together with $u(t_0) \neq 0$ (note that $0 \notin V_{2^*}$) yields

$$S^{\frac{2^*}{2^*-2}} < \|u(t_0)\|_{2^*}^{2^*}. \tag{17}$$

Now the continuity of $t \mapsto \|u(t)\|_{2^*}$ together with (16) and (17) yields the existence of $t_2 \in (t_0, t_1]$ satisfying

$$\|u(t_2)\|_{2^*}^{2^*} = S^{\frac{2^*}{2^*-2}}. \tag{18}$$

By noting $t_0 < t_2$ and the decreasing property of $J_{2^*}(u(t))$, one see

$$\begin{aligned} J_{2^*}(u(t_2)) &= \frac{1}{2}\|\nabla u(t_2)\|_2^2 - \frac{1}{2^*}\|u(t_2)\|_{2^*}^{2^*} \le J_{2^*}(u(t_0)) \\ &< \left(\frac{1}{2} - \frac{1}{2^*}\right) S^{\frac{2^*}{2^*-2}}. \end{aligned} \tag{19}$$

Relations (18) and (19) imply that

$$\begin{aligned} \|\nabla u(t_2)\|_2^2 &< 2\left(\left(\frac{1}{2} - \frac{1}{2^*}\right) S^{\frac{2^*}{2^*-2}} + \frac{1}{2^*}\|u(t_2)\|_{2^*}^{2^*}\right) \\ &= 2\left(\left(\frac{1}{2} - \frac{1}{2^*}\right) S^{\frac{2^*}{2^*-2}} + \frac{1}{2^*} S^{\frac{2^*}{2^*-2}}\right) \\ &= S^{\frac{2^*}{2^*-2}}. \end{aligned} \tag{20}$$

Hence by (18) and (20), we have

$$\frac{\|\nabla u(t_2)\|_2^2}{\|u(t_2)\|_{2^*}^2} < \frac{S^{\frac{2^*}{2^*-2}}}{S^{\frac{2}{2^*-2}}} = S := \inf_{u \in \dot{H}^1 \setminus \{0\}} \frac{\|\nabla u(t)\|_2^2}{\|u(t)\|_{2^*}^2}, \tag{21}$$

a contradiction. ∎

End of the proof of Theorem 1.1 (b) Recall that since we assume the nonnegativity of u, we have (7). Hence the existence of $t_1 \in [0, T_m)$ satisfying $J_{2^*}(u(t_1)) < 0$ immediately yields $T_m < \infty$. Therefore hereafter we assume that

$$J_{2^*}(u(t)) \ge 0, \quad t \in [0, T_m). \tag{22}$$

Suppose on the contrary, assume that $T_m = \infty$. Then Proposition 3.2 assures the existence of (t_n) satsifying $t_n \to \infty$ and

$$\left(\frac{1}{2} - \frac{1}{2^*}\right) \|u(t_n)\|_{2^*}^{2^*} + o(1) = J_{2^*}(u(t_n)). \tag{23}$$

Note that Proposition 4.3 implies $u(t_n) \in V_{2^*}$ for large n. Hence by the definition of V_{2^*} and the decreasing property of $J_{2^*}(u(t))$ in (7), we see that there exist $\gamma' > 0$ satsifying

$$J_{2^*}(u(t_n)) < \left(\frac{1}{2} - \frac{1}{2^*}\right) S^{\frac{2^*}{2^*-2}} - \gamma' \tag{24}$$

for large n. Then we have

$$\|u(t_n)\|_{2^*}^{2^*} < S^{\frac{2^*}{2^*-2}} - \gamma \tag{25}$$

for some $\gamma > 0$ by (23) and (24). Thereofore, by the Sobolev inequality,

$$\begin{aligned} & -\|\nabla u(t_n)\|_2^2 + \|u(t_n)\|_{2^*}^{2^*} \le -S\|u(t_n)\|_{2^*}^2 + \|u(t_n)\|_{2^*}^{2^*} \\ = \; & (-S + \|u(t_n)\|_{2^*}^{2^*-2})\|u(t_n)\|_{2^*}^2 \le -\gamma'\|u(t_n)\|_{2^*}^2 < 0 \end{aligned}$$

for some $\gamma' > 0$, which together with (24) says $u(t_n) \in W_{2^*} \cap V_{2^*}$. On the other hand, it is easy to see that $W_{2^*} \cap V_{2^*} = \emptyset$ from the definition, hence we have a contradiction and $T_m < \infty$ follows.

4.1.3 Proof of Theorem 1.2

Classification of the modulus of initial data Let us take a nonnegative function $\varphi \in L^\infty \cap H^1$ and let us denote the solution of (P) with initial data $\lambda\varphi$ by u_λ, where $\lambda > 0$. Let

$$\begin{aligned} \Lambda_0 &:= \{\lambda \in \mathbb{R}_+;\ u_\lambda(t_0) \in W_{2^*} \text{ for some } t_0 \in [0, T_m)\}, \\ \Lambda_b &:= \{\lambda \in \mathbb{R}_+;\ u_\lambda(t_0) \in V_{2^*} \text{ for some } t_0 \in [0, T_m)\}, \\ \Lambda_c &:= \{\lambda \in \mathbb{R}_+;\ u_\lambda(t) \notin W_{2^*} \cup V_{2^*} \text{ for all } t \in [0, T_m)\}. \end{aligned}$$

It is easy to see that

$$\mathbb{R}_+ = \Lambda_0 \cup \Lambda_b \cup \Lambda_c$$

and, by the comparison principle,

$$\Lambda_0 \text{ and } \Lambda_b \text{ are ordred sets in } \mathbb{R},$$

where the comparison principle of (P) assures that if $u_0 \le v_0$ a.e. in $\mathbb{R}^N$, then solutions u, v of (P) with initinal u_0, v_0 satisfy $u \le v$ a.e. in $\mathbb{R}^N \times [0, T_m(v_0))$, where $T_m(v_0)$ denotes the maximal existence time of v.

Also, by the openness of W_{2^*} and V_{2^*} in $\dot{H}^1$ and the continuity of

$$\lambda \in \mathbb{R}_+ \mapsto \lambda\varphi \in \dot{H}^1 \mapsto u_\lambda(t) \in \dot{H}^1$$

for fixed $t > 0$ (the continuous dependence of a solution of (P) with respect to the initial data in $\dot{H}^1$), we obtain

$$\Lambda_0 \text{ and } \Lambda_b \text{ are open sets in } \mathbb{R}.$$

Finally, it is easy to see that

$$\Lambda_0 \text{ and } \Lambda_b \text{ are nonempty sets in } \mathbb{R}.$$

Combining thse results, we have the existence of $0 < \underline{\lambda} \le \overline{\lambda} < \infty$ satisfying

$$\Lambda_0 = (0, \underline{\lambda}), \quad \Lambda_c = [\underline{\lambda}, \overline{\lambda}], \quad \Lambda_0 = (\overline{\lambda}, \infty).$$

4.2 On global bounds for time-global solutions

In this subsection, we always assume that u is a time-global solution of (P) with $p = 2^*$ and $\Omega = \mathbb{R}^N$.

Let (t_n) be any time sequence with

$$\text{(A)} \quad t_n \to \infty \text{ as } n \to \infty \text{ and } \sup_{n\in\mathbb{N}} \|u(t_n)\|_{2^*} < \infty.$$

By (A) and (8), we have

$$\sup_{n\in\mathbb{N}} \|\nabla u(t_n)\|_2 < \infty,$$

hence $u_n := u(t_n)$ satisfies the assumption of Proposition 3.4. The key claim to have main results is the following:

Proposition 4.4 (Profiles are stationary solutions)
ψ^j appeared in Proposition 3.4 for $(u(t_n))$ is a stationary solution of (P).

A sketch of the proof of Proposition 4.4 will be given in §4.2.2.

For a while, we assume Proposition 4.4 is correct and prove Theorem 1.3 and Theorem 1.4.

4.2.1 Proof of Theorem 1.3

We start with a substitute of Corollary 2.1 in the subcritical case:

Proposition 4.5 (Liminf is finite in the critical case)
There holds

$$\liminf_{t\to\infty} \|u(t)\|_{2^*}^{2^*} \le \frac{d}{\frac{1}{2} - \frac{1}{2^*}},$$

where $d = \lim_{t\to\infty} J_{2^*}(u(t)) (> -\infty)$.

Proof of Proposition 4.5.
By Proposition 3.2, we have the existence of (t_n) satisfying $t_n \to \infty$ and

$$\|\nabla u(t_n)\|_2^2 = \|u(t_n)\|_{2^*}^{2^*} + o(1)$$

as $n \to \infty$. Combining this with (8), the decreasing property of the energy with finite limit d, we see that

$$\begin{aligned} d &= J_{2^*}(u(t_n)) + o(1) = \frac{1}{2}\|\nabla u(t_n)\|_2^2 - \frac{1}{2^*}\|u(t_n)\|_{2^*}^{2^*} \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right) \|u(t_n)\|_{2^*}^{2^*} + o(1) \end{aligned}$$

as $n \to \infty$, hence the conclusion follows. ∎

Next we prove a substitute of Lemma 2.1 in the subcritical case:

Proposition 4.6 (Non-oscillation theorem for $\|u(t)\|_p$ in the critical case)

Let (t_n) be a time sequence satisfying the assumption (A). *Then there holds*

$$\|u(t_n)\|_{2^*}^{2^*} \le \frac{d}{\frac{1}{2} - \frac{1}{2^*}} + o(1)$$

as $n \to \infty$, where $d = \lim_{t\to\infty} J_{2^*}(u(t))(> -\infty)$.

Remark 4.1 (How to exclude the oscillation of $\|u(t)\|_p$ in the critical case)

Proposition 4.6 (for the critical case) plays a role of Lemma 2.1 (for the subcritical case), i.e., exclusion of an oscillation of $\|u(t)\|_p$. In the subcritical case, this exclusion is obtained by using a decay estimate of the heat kernel $e^{t\Delta}$ as is observed in the proof of Lemma 2.1. In the critical case, the decay estimate cannot be applied directly as is stated in §2.2. Here, instead of the quantative information such as the decay estimate of $e^{t\Delta}$, we use the qualitative information such as a profile decomposition to prove the non-oscillation of $\|u(t)\|_p$. ∎

Proof of Proposition 4.6.

By the assumption and (8), we see $\sup_{n\in\mathbb{N}} \|\nabla u(t_n)\|_2 < \infty$. This together with Proposition 3.4 yields the existence of $(\lambda_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}_+$, $(x_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}^N$ $(j = 1, \cdots)$ and $(\psi^j)_{j\in\mathbb{N}} \subset \dot{H}^1(\mathbb{R}^N)$ such that the conclusion of Proposition 3.4 holds. Now take $\varepsilon > 0$. Proposition 3.4 (b) yields $\sum_{j=1}^{\infty} \|\psi^j\|_{2^*}^{2^*} < \infty$, whence follows the existence of $l \in \mathbb{N}$ satsfying

$$\sum_{j=l+1}^{\infty} \|\psi^j\|_{2^*}^{2^*} < \varepsilon. \tag{26}$$

For this $l \in \mathbb{N}$, again by Proposition 3.4 (b), we have

$$\|u(t_n)\|_{2^*}^{2^*} = \sum_{j=1}^{\infty} \|\psi^j\|_{2^*}^{2^*} + o(1), \tag{27}$$

$$\|\nabla u(t_n)\|_2^2 = \sum_{j=1}^{l} \|\nabla \psi^j\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1) \tag{28}$$

as $n \to \infty$.

Proposition 4.4 says that ψ^j is a stationary solution of (P) for each $j \in \mathbb{N}$. Hence we see that

$$-\Delta\psi^j = \psi^j|\psi^j|^{2^*-2} \text{ in } \mathbb{R}^N.$$

Multiplying ψ^j to both sides and integrating over $\mathbb{R}^N$, we obtain

$$\|\nabla\psi^j\|_2^2 = \|\psi^j\|_{2^*}^{2^*}.$$

Then we have

$$\begin{aligned}
d + o(1) &= J_{2^*}(u(t_n)) = \frac{1}{2}\|\nabla u(t_n)\|_2^2 - \frac{1}{2^*}\|u(t_n)\|_{2^*}^{2^*} \\
&= \frac{1}{2}\left(\sum_{j=1}^{l}\|\nabla\psi^j\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1)\right) \\
&\quad -\frac{1}{2^*}\left(\sum_{j=1}^{l}\|\psi^j\|_{2^*}^{2^*} + \sum_{j=l+1}^{\infty}\|\psi^j\|_{2^*}^{2^*} + o(1)\right) \\
&= \left(\frac{1}{2} - \frac{1}{2^*}\right)\sum_{j=1}^{l}\|\psi^j\|_{2^*}^{2^*} + \frac{1}{2}\|\nabla r_n^l\|_2^2 - \frac{1}{2^*}\sum_{j=l+1}^{\infty}\|\psi^j\|_{2^*}^{2^*} + o(1) \\
&\geq \left(\frac{1}{2} - \frac{1}{2^*}\right)\sum_{j=1}^{l}\|\psi^j\|_{2^*}^{2^*} - \frac{1}{2^*}\varepsilon + o(1)
\end{aligned}$$

as $n \to \infty$, where we used (26), (27) and (28). Hence there holds

$$\frac{d + \frac{1}{p}\varepsilon}{\frac{1}{2} - \frac{1}{2^*}} \geq \sum_{j=1}^{l}\|\psi^j\|_{2^*}^{2^*}.$$

Since $\varepsilon > 0$ is an arbitrary, we have

$$\frac{d}{\frac{1}{2} - \frac{1}{2^*}} \geq \sum_{j=1}^{l}\|\psi^j\|_{2^*}^{2^*}$$

and by letting $l \to \infty$ and using (27), we have the conclusion. ∎

End of the proof of Theorem 1.3 Now assume that $\limsup_{t\to\infty} \|u(t)\|_{2^*}^{2^*} = \infty$. Then this assumption and Proposition 4.5 yield the existence of (t_n) satisfying $t_n \to \infty$ and

$$\|u(t_n)\|_{2^*}^{2^*} = 2\frac{d}{\frac{1}{2}-\frac{1}{2^*}} \text{ for any } n \tag{29}$$

as $n \to \infty$. Then since (t_n) satisfies the assumption (A), Proposition 4.6 implies

$$\|u(t_n)\|_{2^*}^{2^*} \le \frac{d}{\frac{1}{2}-\frac{1}{2^*}},$$

which contradicts (29). This completes the proof.

4.2.2 Identification of "profiles". Sketch of the proof for Proposition 4.4

Now we prove Proposition 4.4. Since the proof is rather technical, we proceed in a sketchy way in this section. For the detail of the argument, see [28].

The proof of Proposition 4.4 consists of 3 steps. Let (t_n) be a time sequence which satisfies

$$\sup_{n\in\mathbb{N}} \|u(t_n)\|_{2^*} < \infty. \tag{30}$$

Step 1. Introduction of a rescaled sequence. The energy decreasing property (8) together with (30) implies that

$$\sup_{n\in\mathbb{N}} \|\nabla u(t_n)\|_2 < \infty. \tag{31}$$

This relation and Proposition 3.4 yield the existence of $(\mu_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}_+$, $(x_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}^N$, $(\psi^j)_{j\in\mathbb{N}} \subset \dot{H}^1(\mathbb{R}^N)$ which satisfy the following: for

$$\psi_n^j(x) := \left(\frac{1}{\mu_n^j}\right)^{\frac{N-2}{2}} \psi^j\left(\frac{x-x_n^j}{\mu_n^j}\right),$$

passing to a subsequence if necessary, we have

$$\frac{\mu_n^j}{\mu_n^i} + \frac{\mu_n^i}{\mu_n^j} + \mu_n^i|x_n^i - x_n^j| \to \infty \text{ as } n \to \infty \text{ for } i \neq j \tag{32}$$

and

$$\begin{aligned}
\lim_{l\to\infty}\lim_{n\to\infty}\|r_n^l\|_{2^*} &= 0,\\
\|\nabla u(t_n)\|_2^2 &= \sum_{j=1}^{l}\|\nabla\psi^j\|_2^2+\|\nabla r_n^l\|_2^2+o(1),\\
\|u(t_n)\|_{2^*}^{2^*} &= \sum_{j=1}^{\infty}\|\psi^j\|_{2^*}^{2^*}+o(1)
\end{aligned}$$

as $n\to\infty$ for each $l\in\mathbb{N}$, where $r_n^l := u(t_n)-\sum_{j=1}^{l}\psi_n^j$.

We would like to show that ψ^{j_0} is a stationary solution of (P) for each $j_0\in\mathbb{N}$. To this end, we introduce a rescaled sequence $(v_n^{j_0})$ which is the scale back of $(u(t_n))$ focusing on the j_0-th "bubble":

$$v_n^{j_0}(y,s) \;:=\; \left(\mu_n^{j_0}\right)^{\frac{N-2}{2}} u_n\left(x_n^{j_0}+\mu_n^{j_0}y, t_n+(\mu_n^{j_0})^2 s\right),$$

where $(y,s)\in\mathbb{R}^N\times[0,1]$ (note that we denote λ_n^j in Proposition 3.4 by $\frac{1}{\mu_n^j}$ for the notational simplicity). Note that

$$\begin{aligned}
v_n^{j_0}(y,0_s) &:= \left(\mu_n^{j_0}\right)^{\frac{N-2}{2}} u_n\left(x_n^{j_0}+\mu_n^{j_0}y,t_n\right)\\
&= \psi^{j_0}(y)+\sum_{i\neq j_0,1\leq i\leq l}\left(\frac{\mu_n^{j_0}}{\mu_n^i}\right)^{\frac{N-2}{2}}\psi^i\left(\frac{\mu_n^{j_0}}{\mu_n^i}\left[y+\mu_n^i(x_n^{j_0}-x_n^i)\right]\right)\\
&\quad+\left(\mu_n^{j_0}\right)^{\frac{N-2}{2}} r_n\left(x_n^{j_0}+\mu_n^{j_0}y\right).
\end{aligned}$$

As is already mentioned in Remark 3.5, it is easy to see that

$$v_n^{j_0}(0_s)\to\psi^{j_0} \text{ weakly in } \dot{H}^1 \text{ and strongly in } L^2_{\mathrm{loc}} \tag{33}$$

as $n\to\infty$ by virtue of (32), i.e., the orthogonality of $(\mu_n^i,y_n^i)_n$ and $(\mu_n^j,y_n^j)_n$ for $i\neq j$.

By using Proposition 3.1, the scale invariance of (P), we also see that $v_n^{j_0}$ satisfies (P):

$$\partial_s v_n^{j_0}=\Delta v_n^{j_0}+v_n^{j_0}|v_n^{j_0}|^{2^*-2} \text{ in } \mathbb{R}^N\times[0,1]. \tag{34}$$

By the energy equality (6) and the boundedness (31), we have $d \in \mathbb{R}$ satisfying (8). Then again by the energy inequality (6), we have

$$\begin{aligned}
\int_0^1 ds \|\partial_s v_n^{j_0}(s)\|_2^2 &= J_{2^*}(v_n^{j_0}(0)) - J_{2^*}(v_n^{j_0}(1)) \\
&= J_{2^*}(u(t_n)) - J_{2^*}(u(t_n + (\mu_n^{j_0})^2)) \\
&= d - d + o(1) = o(1), \qquad (35)
\end{aligned}$$

where we used the scale invariance of the energy functional Proposition 3.1. Moreover, by multiplying $\phi \in C_0^\infty(\mathbb{R}^N)$ to (34) and integrating it over $\mathbb{R}^N \times [0,1]$, we see that

$$\begin{aligned}
\int_0^1 ds \int_{\mathbb{R}^N} dy \partial_s v_n^{j_0} \phi &= \int_0^1 ds \int_{\mathbb{R}^N} dy \Delta v_n^{j_0} \phi \\
&\quad + \int_0^1 ds \int_{\mathbb{R}^N} dy v_n^{j_0} |v_n^{j_0}|^{2^*-2} \phi. \qquad (36)
\end{aligned}$$

We will show in the following that

$$0 = \int_{\mathbb{R}^N} dy \Delta \psi^{j_0} \phi + \int_{\mathbb{R}^N} dy \psi^{j_0} |\psi^{j_0}|^{2^*-2} \phi, \qquad (37)$$

i.e., ψ^{j_0} is a weak stationary solution of (P). Then by the classical elliptic regularity and $\psi^{j_0} \in \dot{H}^1$, we have the conclusion.

Step 2. Bounds for large norms of rescaled sequence. From now on, we will show (37). Let $K := \operatorname{supp} \phi$ and let $\eta \in C_0^\infty(\mathbb{R}^N)$ be a cut-off function to K satisfying $0 \leq \phi \leq 1$ in $\mathbb{R}^N$ and $\eta = 1$ in K.

First we claim that by multiplying (34) by $v_n^{j_0}\eta$ and by taking $\int_0^1 ds \int_{\mathbb{R}^N} dy$, we have

$$\left| -\int_0^1 ds \int_{\mathbb{R}^N} dy |\nabla v_n^{j_0}|^2 \eta + \int_0^1 ds \int_{\mathbb{R}^N} dy |v_n^{j_0}|^{2^*} \eta \right| < C_1 \qquad (38)$$

for some $C_1 > 0$. The derivation of this relation needs (35) and we omit it to avoid a technical complexity.

Secondly, by multiplying (34) by $v_n^{j_0}\eta$ and by taking $\int_0^1 d\sigma \int_0^\sigma ds \int_K dy$, we have, for sufficiently small positive ζ,

$$\left| -\frac{1-\zeta}{2} \int_0^1 ds \int_{\mathbb{R}^N} dy |\nabla v_n^{j_0}|^2 \eta + \frac{1}{2^*} \int_0^1 ds \int_{\mathbb{R}^N} dy |v_n^{j_0}|^{2^*} \eta \right| < C_{2,\zeta}, \qquad (39)$$

where $C_{2,\zeta}$ is a positive number depending on ζ. To derive this relation, we have to choose a cut-off η more wisely and to use (35), but again we omit it to avoid a technical complexity.

Note that (38) and (39) yield the existence of $C_3 > 0$ satisfying

$$\int_0^1 ds \int_{\mathbb{R}^N} dy |\nabla v_n^{j_0}|^2 \eta < C_3, \tag{40}$$

$$\int_0^1 ds \int_{\mathbb{R}^N} dy |v_n^{j_0}|^{2^*} \eta < C_3 \tag{41}$$

for $n \in \mathbb{N}$. Relations (35) and (40) imply that

$$(v_n^{j_0}) \text{ is a bounded sequence in } H^1(Q_K), \tag{42}$$

where $Q_K := K \times [0,1]$ and $H^1(Q_K)$ denotes the Sobolev space of functions with $(N+1)$-varilable (y,s).

Step 3. Convergence of rescaled sequence. Now it is easy to see from (42) that

$$\begin{aligned} v_n^{j_0} \to v^{j_0} \qquad & \text{weakly in } H^1(Q_K), && (43)\\ & \text{weakly in } L^{2^*}(Q_K), && (44)\\ & \text{strongly in } L^2(Q_K). && (45) \end{aligned}$$

The last convergence together with the stabilizing-in-time result (35) yields

$$v^{j_0} \text{ is a time-independent function.} \tag{46}$$

Also, we can prove from (43) and (44) that

$$\begin{aligned} \int_0^1 ds \int_{\mathbb{R}^N} dy \nabla v_n^{j_0} \nabla \phi &= \int_0^1 ds \int_{\mathbb{R}^N} dy \nabla v^{j_0} \nabla \phi + o(1), \\ \int_0^1 ds \int_{\mathbb{R}^N} dy v_n^{j_0} |v_n^{j_0}|^{2^*-2} \phi &= \int_0^1 ds \int_{\mathbb{R}^N} dy v^{j_0} |v^{j_0}|^{2^*-2} \phi + o(1) \end{aligned}$$

as $n \to \infty$. By plugging these reation and (35) into (36) and taking $n \to \infty$, we have

$$0 = -\int_0^1 ds \int_{\mathbb{R}^N} dy \nabla v^{j_0} \nabla \phi + \int_0^1 ds \int_{\mathbb{R}^N} dy v^{j_0} |v^{j_0}|^{2^*-2} \phi.$$

Then we have from this relation

$$0 = -\int_{\mathbb{R}^N} dy \nabla v^{j_0} \nabla \phi + \int_{\mathbb{R}^N} dy v^{j_0} |v^{j_0}|^{2^*-2} \phi \tag{47}$$

by noting that v^{j_0} is time-independent by (46) and ϕ is also time-independent. Finally, by (33), (35) and (45), we can derive

$$v^{j_0} = \psi^{j_0}.$$

This relation together with (47) implies (37), i.e., ψ^{j_0} is a weak stationary solution of (P). This completes the sketch of the proof.

4.3 Proof of Theorem 1.4

Let us assume, on the contrary, the conclusion does not hold. Then there exists a time sequence (t_n) and $\varepsilon > 0$ satisfying $t_n \to \infty$ and

$$\mathrm{dist}_{L^{2^*}}(u(t_n), E_\infty(u_0)) \geq \varepsilon. \tag{48}$$

By the assumption $\sup_{t>0} \|\nabla u(t)\|_2 < \infty$ and by the Sobolev inequality, we know

$$\sup_n \|u(t_n)\|_{2^*} < \infty.$$

Hence Proposition 3.4 and Proposition 4.4 yield the existence of $(\lambda_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}_+$, $(x_n^j)_{j\in\mathbb{N}} \subset \mathbb{R}^N$ $(j = 1, \cdots)$, a family of stationary solutions $(\psi^j)_{j\in\mathbb{N}} \subset \dot{H}^1(\mathbb{R}^N)$ of (P) which satisfy (a) – (c) of Proposition 3.4. Now take $\eta > 0$. Proposition 3.4 (b) yields $\sum_{j=1}^{\infty} \|\psi^j\|_{2^*}^{2^*} < \infty$, whence follows the existence of $L_1 \in \mathbb{N}$ satsfying

$$\sum_{j=l+1}^{\infty} \|\psi^j\|_{2^*}^{2^*} < \eta \text{ for any } l > L_1. \tag{49}$$

Moreover, by Proposition 4.4 (c), there exists $L_2 \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \|r_n^l\|_{2^*} < \frac{\varepsilon}{2} \text{ for any } l > L_2, \tag{50}$$

where r_n^l satsifies

$$u(t_n) = \sum_{j=1}^{l} (\lambda_n^j)^{\frac{N-2}{2}} \psi^j(\lambda_n^j(\cdot - x_n^j)) + r_n^l. \tag{51}$$

Now take any $l \in \mathbb{N}$ satsifying $l > \max(L_1, L_2)$ and let

$$w_n^l := \sum_{j=1}^{l} (\lambda_n^j)^{\frac{N-2}{2}} \psi^j(\lambda_n^j(\cdot - x_n^j)).$$

Then by (8) and Proposition 3.4 (b) and (c), passing to subsequence if necessary, we have

$$\begin{aligned}
J_{2^*}(u_0) &\geq \frac{1}{2}\|\nabla u(t_n)\|_2^2 - \frac{1}{2^*}\|u(t_n)\|_{2^*}^{2^*} \\
&= \frac{1}{2}\left(\sum_{j=1}^{l} \|\nabla \psi^j\|_2^2 + \|\nabla r_n^l\|_2^2 + o(1)\right) \\
&\quad -\frac{1}{2^*}\left(\sum_{j=1}^{l} \|\psi^j\|_{2^*}^{2^*} + \sum_{j=l+1}^{\infty} \|\psi^j\|_{2^*}^{2^*} + o(1)\right) \\
&\geq \frac{1}{2}\sum_{j=1}^{l} \|\nabla \psi^j\|_2^2 - \frac{1}{2^*}\sum_{j=1}^{l} \|\psi^j\|_{2^*}^{2^*} - \frac{1}{2^*}\eta \\
&= \sum_{j=1}^{l} J_{2^*}(\psi^j) - \frac{1}{2^*}\eta
\end{aligned}$$

as $n \to \infty$. Since $\eta > 0$ is an arbitrary, from this relation, we have

$$J_{2^*}(u_0) \geq \sum_{j=1}^{l} J_{2^*}(\psi^j). \tag{52}$$

This relation shows that $w_n^l \in E_\infty(u_0)$, which together with (51) and (50) implies

$$\mathrm{dist}_{L^{2^*}}(u(t_n), E_\infty(u_0)) \leq \|u(t_n) - w_n^l\|_{2^*} = \|r_n^l\|_{2^*} < \frac{\varepsilon}{2},$$

which contradicts to (48). This completes the proof of Theorem 1.4.

Remark 4.2

In the proof above, we do not use any information on the number of nonzero profiles and we can easily see that the number is finite. Indeed, by (52) and the fact that $J_{2^*}(\psi^j) \geq \frac{S^{\frac{N}{2}}}{N}$ for a stationary solution ψ^j of (P), where $S := \inf_{u \in \dot{H}^1 \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2}$ is the best Sobolev constant, we see that the number of j for which $\psi^j \neq 0$ is at most $\frac{Nd}{S^{\frac{N}{2}}}$. ∎

5 Discussions

5.1 On global bounds for time-global solutions: toward an abstract theory for dynamical systems with noncompact orbit

In this subsection, we try to understand Theorem 1.4 from an abstract point of view. We start by reviewing such a framework for the subcritical problem.

5.1.1 A compact case: the LaSalle principle

Let us recall Proposition 2.1 which treats a subcritical and bounded domain case. Proposition 2.1 (a) says that every global-in-time solution induces a bounded orbit in $\dot{H}^1$ and, (b) indicates that every (global-in-time) orbit is absorbed to a set of equilibrium. The abstract version of the latter asymptotics is called the LaSalle principle, see e.g. Cazenave-Haraux [4, §9]. In this subsection, we review this. Let (Z, d) be a complete metric space.

- **(Abstract dynamical system)** A dynamical system on Z is a family $(S_t)_{t\geq 0}$ of mappings on Z such that

  (a) $S_t \in C(Z; Z)$ for any $t \geq 0$,

  (b) $S_0 = I$,

  (c) $S_{t+s} = S_t \circ S_s$ for any $t, s \geq 0$,

  (d) The function $t \mapsto S_t z$ is in $C([0, \infty); Z)$ for all $z \in Z$.

  The set $O(z) := \{S_t z;\ t \geq 0\} \subset Z$ is called an orbit of z.

- **(ω-limit set)** Let $z \in Z$. The set

$$\omega(z) := \{y \in Z;\ \text{there exists } (t_n) \text{ which satisfies } t_n \to \infty \text{ and } S_{t_n} z \to y \text{ as } n \to \infty\}$$

  is called an omega-limit set of z.

- **(Equilibrium)** $z \in Z$ is called an equilibrium if $S_t z = z$ for any $t \geq 0$. A set consists of all equilibrium points is denoted by E.

- **(Lyapunov functional)** Let J be a continuous functional on Z.

  (a) J is said to be a Lyapunov functional for $(S_t)_{t\geq 0}$ if $J(S_t z) \leq J(z)$ for any $z \in Z$ and $t \geq 0$.

(b) A Lyapunov functional J is said to have a strict Lyapunov property if for any $z \in Z$ satisfying $J(S_t z) = J(z)$ for all $t \geq 0$, then there holds $z \in E$.

Example 5.1

Let $u_0 \in H^1 \cap L^\infty$ be an initial data which gives a global-in-time solution of (P) with subcritical p and bounded Ω and let $S_t u_0 := u(t)$. Then

$$\begin{aligned}
&(S_t)_{t\geq 0} \text{ is a dynamical system on } L^p,\\
&J_p(u) = \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{1}{p}\|u\|_p^p \text{ has a strict Lyapunov property,}\\
&E := \{\varphi;\ \varphi \text{ is a stationary solution of (P)}\}
\end{aligned}$$

hold. ∎

The novelty of this setting is that the information above (rather not so much!) gives an asymptotics of the orbit as in Proposition 2.1 (b):

Proposition 5.1 (The LaSalle principle)

Let J be a Lyapunov functional for $(S_t)_{t\geq 0}$ with a strict Lyapunov property, and let $z \in Z$ be such that

$$O(z) \text{ is relatively compact in } Z. \tag{1}$$

Then there holds $d(S_t z, E) \to 0$ as $t \to \infty$, i.e., $\omega(z) \subset E$.

On the applicability of the LaSalle principle to (P) Note that the proposition above immediately yields Proposition 2.1 (b). Indeed, in the subcritical and the bounded domain case, it is easy to extend the proof of Lemma 2.2 to assure

$$\text{the orbit } O(u_0) \text{ is relatively compact in } L^p. \tag{2}$$

In fact, it is proved in [25] that the Palais-Smale condition along the orbit is equivalent to the relative compactness of the orbit in $\dot{H}^1$. Since we are in the compact situation, it is easy to see that the Palais-Smale condition along the orbit holds and the result in [25] implies (2). Then, Proposition 5.1 yields Proposition 2.1 (b).

5.1.2 Suitable topology in the critical case: D-convergence of Tintarev

On the other hand, (P) with critical p defined on ball has a lack of compactness of the orbit. Indeed, a solution given in (16) concentrates at the origin with nonzero L^{2^*}-norm as is explained in Remark 1.10 and this suggests that we cannot rely on Proposition 5.1 in general to prove Theorem 1.4 in the critical case. Thus, it is natural to consider *the extension of the LaSalle principle above which is also valid for the critical case.* Since the typical noncompactness phenomena in the critical case is concentration as is observed in (16), this extension may need to introduce a generalized topology which allows to include concentration phenomena. In the proof of Theorem 1.3 and Theorem 1.4, the key machinary is Proposition 3.4, the profile decomposition. There exists an abstract version of it. For more detail concerning the fact below, see e.g. Tintarev-Fieseler [51, §3] and references therein. Also, articles in the blog of Terrence Tao [49, 50] give a good introduction to this topic.

Let H be a separable infinite-dimensional Hilbert space and let $(\cdot, \cdot)$ be its inner product.

- **(D-convergence)** Let D be a topological group of isometry acting on H. We say "u_n converges to u D-weakly" if

  $$\lim_{n\to\infty} \sup_{g\in D}(u_n - u, g\varphi) = 0$$

  for all $\varphi \in H$.

  Observe that this convergence is stronger than the weak convergence and weaker than the strong convergence.

- **(Dislocation space)** (H, D) is said to be a dislocation space if for any $(g_n) \subset D$ with $g_n \not\rightharpoonup 0$ in D and for any $(u_n) \subset H$ with $u_n \rightharpoonup 0$ in H, there exists a subsequence of (g_n) and (u_n) (denoted by the same symbol) such that $g_n u_n \rightharpoonup 0$ in H.

Under these abstract setting, we can formulate an abstract version of the profile decomposition which can also be seen as a refinement of Banach-Alaoglu theorem in H:

Proposition 5.2 (Abstract profile decomposition)

Let (H, D) be a dislocation space and let $(u_n) \subset H$ be a bounded sequence. Then there exist $J \subset \mathbb{N}$, $(\psi^j)_{j\in J} \subset H$, $(g_n^j)_{j\in J} \subset D$ with $g_n(1)$ =identity such that, for renumbered subsequence, there holds

$$\begin{aligned}
&r_n := u_n - \sum_{j\in N} g_n^j \psi^j \overset{D}{\rightharpoonup} 0,\\
&(g_n^j)^{-1} u_n \rightharpoonup \psi^j,\\
&(g_n^i)^{-1} g_n^j \rightharpoonup 0 \text{ for } i \neq j,\\
&\|u_n\|^2 - \sum_{j\in J} \|\psi^j\|^2 - \|r_n\|^2 \to 0
\end{aligned}$$

as $n \to \infty$.

For the proof, see Tintarev-Fieseler [51, §3] and references therein. Observe that $(\dot{H}^1, \mathbb{R}^N \ltimes \mathbb{R}_+)$ is a dislocation space, where $\ltimes$ denotes the semidirect product, see §7 for $\ltimes$ and see e.g. [51, Lemma 5.2] for the proof of this fact. Somewhat remarkably, D-convergence in this case coincides with the convergence in L^{2^*}, hence Proposition 5.2 is applicable to $(\dot{H}^1, \mathbb{R}^N \ltimes \mathbb{R}_+)$ and this gives Proposition 3.4, a profile decompostion in $\dot{H}^1$.

5.1.3 Toward an "extended" LaSalle principle

The orbit O of a solution of a differential equation defined in a function space X falls into one of the following three categories:

(a) O is unbounded in X,

(b) O is bounded but not compact in X,

(c) O is relatively compact in X.

The case (a) is rather a different behavior, since it corresponds to a blow-up in infinite time (or grow-up), and our problem (P) does not possesses such solutions for $X := L^p$ with $p < 2^*$ and bounded Ω by Proposition 2.1 or nonnegative solutions for (P) with $p = 2^*$ and $\Omega = \mathbb{R}^N$ by Theorem 1.3. Note that this category is also important and (16) says that this category of behaviour actually occurs for $X := L^\infty$ in the critical case.

The usual LaSalle principle Proposition 5.1 is applicable to the case (c), hence we have an abstract theory in this case. The remaining case is the case (b).

Note that a (system of) ordinary differential equation defines a dymanical system with a finite dimensional phase space Z. In this case, (b) cannot occur, since every bounded set in a finite dimensional topological vector space is always relatively compact by the Bolzano-Weierstrass theorem. Hence the case (b) only appears for the infinite-dimensional dynamical system. Our Theorem 1.4 indicates that for (P) with critical exponent (which defines an infinite-dimensional dynamical system in, say, L^{2^*}), there indeed exists an essentially different phenomena from the one which is described by Proposition 5.1, the LaSalle principle. Hence it is natural to consider the extention of the LaSalle principle to the case (b) where the orbit is bounded but not compact. Such an extension may possible if one combines the framework of the LaSalle principle together with the D-convergence. This issue will be discussed in the forthcoming paper [29].

5.2 Open problems

We collect in this section some basic open problems concerning (P) with critical exponent. Though these problems seem very simple and rather elemental, it still remain open.

On the asymptotic behavior of time-global solution

Open problem 5.1 (The validity of the existence of the energy limit in the changing-sign case)
One of the basis of the proof of Theorem 1.3 and Theorem 1.4 is (7), the finiteness of the energy limit $d \geq 0$ which implies a "stabilization" in time of time-global solutions as $t \to \infty$. As is mentioned in Remark 1.1, the validity of (7) for (P) with $\Omega = \mathbb{R}^N$ and $p = 2^*$ (even for $p < 2^*$) is only proved so far for nonnegative solutions and we cannot exclude the possibility of the existence of a sign-changing time-global solution u of (P) satisfying (12), i.e.,

$$\lim_{t\to\infty} \|\nabla u(t)\|_2 = \infty \text{ and } \lim_{t\to\infty} J_{2^*}(u(t)) = -\infty.$$

In this case, there also holds

$$\|u(t)\|_2^2 \geq Ct, \text{ thus } \lim_{t\to\infty} \|u(t)\|_2 = \infty \tag{3}$$

for some $C > 0$, see Corollary 6.1. The existence of a sign-changing solution satisifying above seems an open problem and (3) shows that the validity of (7) is heavily related with the exclusion of the possibility of solutions which grow-up in L^2-sense, see also Open problem 5.3. ∎

Open problem 5.2 (Extension of the topology in Theorem 1.4)
Let u be a time-global solution u of (P) in $\mathbb{R}^N$ with $p = 2^*$ satisfying $\sup_{t>0} \|\nabla u(t)\|_2 < \infty$. Then Theorem 1.4 gives an asymptotic behavior for which the topology of the convergence is taken in L^{2^*}. The extention of the topology from L^{2^*} to $\dot{H}^1$ is an open problem.

In [28], it is proved that if u is a *nonnegative* time-global solution of (P), then

$$\mathrm{dist}_{\dot{H}^1}(u(t), E_\infty(u_0)) \to 0 \tag{4}$$

as $t \to \infty$. The proof needs the "quantization of $d(= \lim_{t\to\infty} J(u(t)))$" which we do not know in the sign-changing case due to the lack of the knowledge of quantizations of norms of sign-changing stationary solutions of (P). If the extension (4) for the sign-changing case is possible, then we know *a posteriori* that $(u(t_n))$ is a Palais-Smale sequence of J for every sequence (t_n) with $t_n \to \infty$ as $n \to \infty$, see Remark 3.3. ∎

Open problem 5.3 (On bounds for L^2-norm)
For a time-global solution u of (P) on a bounded domain Ω, we can derive

$$\sup_{t>0} \|u(t)\|_2 < \infty \tag{5}$$

directly, see e.g. Ôtani [43], Cazenave-Lions [5] and Cazenave-Haraux [4, §8]. The method of proof is also applicable to the critical case of (P) on a bounded domain. On the other hand, the validity of (5) for (P) with critical p on the entire domain is an open problem. This is due to the lack of the knowledge of an effect of the unboundedness of the domain. Indeed, in the asymptotics

$$u(\cdot, t_n) - \sum_j (\lambda_n^j)^{\frac{N-2}{2}} \varphi^j(\lambda_n^j(\cdot - y_n^j)) = o(1) \text{ in } L^{2^*}$$

in $\mathbb{R}^N$ which is given in (16), if there exists $j \in \mathbb{N}$ such that $\lambda_n^j \to 0$ as $n \to \infty$ (which only occurs for the unbounded domain case), then we have $\|u(t_n)\|_2 \to \infty$ as $n \to \infty$ heuristically. Hence this question is closely related with the asymptotics of λ_n^j as $n \to \infty$. ∎

Open problem 5.4 (The finite-dimensional reduction of the dynamics)

As is seen in Theorem 1.4, long-time asymptotics of time-global solutions is reduced to the finite-dimensional dynamics. Indeed, Theorem 1.4 says that the asymtotics of u is decomposed as $u(t) = w(t) + \varepsilon(t)$, where $\varepsilon(t)$ is an error part such that

$$\|\varepsilon(t)\|_{L^{2^*}} = \mathrm{dist}_{L^{2^*}}(u(t), E_\infty(u_0)) \to 0 \text{ as } t \to \infty$$

and $w(t)$ is a principal part of u satisfying

$$\mathrm{dist}_{L^{2^*}}(u(t), E_\infty(u_0)) = \mathrm{dist}_{L^{2^*}}(u(t), w(t))$$

with $w(t) \in E_\infty(u_0)$, thus is described by

$$w(t) = \sum_{\jmath} (\lambda^\jmath(t))^{\frac{N-2}{2}} \psi^\jmath(\lambda^\jmath(t)(x - x^\jmath(t))).$$

Hence the long-time asymptotics of u is governed by that of $(\lambda^\jmath(t), x^\jmath(t))$. It is an open problem to derive an effective equation of motion for $(\lambda^\jmath(t), x^\jmath(t))$ and to give a precise asymptotics of $(\lambda^\jmath(t), x^\jmath(t))$. For a formal result based on the matched asymptotic expansion in the radially symmetric setting, see e.g. Fila-King [13]. For the construction of such solutions in the nonradial setting, see del Pino [8]. For the similar finite-dimensional reduction of dynamics of gradient system, see e.g. Ei [9], Ei-Ishimoto [10] for for a pulse solution in reaction diffusion equations and Bahri-Coron [1] for a gradient flow for Yamabe functional. ∎

On the blow-up phenomena In this note, we only consider asymptotics of time-global solutions of (P) with critical exponent. The blow-up problem can be seen as a "dual" problem for it and there exist several fundamental open problems. We introduce some of them.

As is already mentioned in (4), since a solution u of (P) in the class (3) is a classical solution, it satisfies the blow-up alternative in L^∞-sense:

$$\text{if } T_m < \infty, \text{ then } \lim_{t \to T_m} \|u(t)\|_\infty = \infty.$$

Since $\dot{H}^1$ and L^p-norms also play a fundamental role for the analysis of (P) from the viewpoint of the energy structure, it is natural to ask whether

$$T_m < \infty \text{ implies } \|\nabla u(t)\|_2,\ \|u(t)\|_p \to \infty \text{ as } t \to T_m \tag{6}$$

or not. In the subcritical case, we can prove (6) by using Lemma 2.1:

Lemma 5.1

There holds (6) *for* (P) *with* $p < 2^*$.

Proof of Lemma 5.1.

Let $T_m < \infty$ and suppose on the contrary there holds $\liminf_{t \in [0,T_m)} \|u(t)\|_p < \infty$. Then we have the existence of (t_n) satisfying

$$t_n \uparrow T_m, \quad \sup_n \|u(t_n)\|_p =: L < \infty.$$

Now for t_n with $T_m - T(L) < t_n (< T_m)$, where $T(L)$ denotes a local existence time in (3), the solution $u(t)$ can be extended to $(T_m <) t_n + T(L)$ as a L^p-solution. Since in the subcritical case, a solution in L^p-sense is a classical solution (see e.g. Brezis-Cazenave [3] and Ruf-Terraneo [45]), this contradicts to the maximality of T_m in the L^∞-sense. Hence we have

$$\lim_{t \to T_m} \|u(t)\|_p = \infty \text{ if } T_m < \infty. \tag{7}$$

This togehter with the Sobolev inequality, we have (6) in the subcritical case. ∎

Note that the proof of Lemma 5.1 is based on Lemma 2.1, which needs the convergence of s-integral in (5) and the subcriticality of p is needed for this convergence. By virtue of this convergence, we can obtain a function $T(\cdot)$ in (3) which implies that the local existence time of solutions of (P) can be taken uniformly in L^p-norm of initial data. In the critical case, δ in (7) satisfies $\delta = 0$ and the integral in (5) diverges. Because of this, the local existence time cannot be taken uniformly in L^{2^*}-norm of initial data and the proof of Lemma 2.1 does not work for the critical case. Only facts known so far in the critical case for $T(\cdot)$ is that the local existence time can be taken uniformly for a compact set of initial data (not a bounded set of intial data as in the subcritical case), see Brezis-Cazenave [3] and Ruf-Terraneo [45].

These considerarions show that whether (6) happnes or not in the critical case is closely related with the behavior of an orbit which is bounded but noncompact in the Sobolev space and blows up in finite time in the classical sense. Such a phenomena is called a "bubbling in finite time" in the field of the analysis of geometric heat flow such as the Yamabe flow or the harmonic heat flow, see e.g. Ye [55] and Topping [52].

Recently, the existence of a finite time blow-up solution satisifying (6) is assured by Schweyer in [46]:

Proposition 5.3

Let $N = 4$. Then there exists a radially symmetric initial data $u_0^* \in \dot{H}^1(\mathbb{R}^4)$ which satsifes the following: the solution u^* of (P) *in $\mathbb{R}^N$ with $p = 4$ (critical Sobolev exponent in $N = 4$) starting from u_0^* blows up in finite time in the classical sense (i.e., $T_m < \infty$) and there exists $v \in \dot{H}^1(\mathbb{R}^4)$ with $\Delta v \in L^2(\mathbb{R}^4)$ such that*

$$u^*(t) - \frac{1}{\lambda(t)} U\left(\frac{x}{\lambda(t)}\right) \to v \text{ strongly in } \dot{H}^1(\mathbb{R}^4) \tag{8}$$

as $t \uparrow T_m$, where U denotes the unique positive stationary solution of (P) *(Talenti function). The function λ satsifies the following type II profile (see Open problem 5.7 for the word "type II"):*

$$\lambda(t) = c(u_0)(1 + o(1)) \frac{T_m - t}{|\log(T_m - t)|^2} \tag{9}$$

as $t \uparrow T_m$, where $c(u_0) > 0$.

Note that from (8), this solution u^* satisfies $\sup_{t \in [0, T_m)} \|\nabla u^*(t)\|_2 < \infty$, hence

$$\lim_{t \to T_m} J_{2^*}(u(t)) > -\infty \text{ and } T_m < \infty. \tag{10}$$

Hence in the critical case, (6) does not hold in general.

Open problem 5.5 (Blow-up alternative for the energy norm in the critical case)

Note that the proof in Schweyer in [46] relies on the explicit construction of the initial data by using the radial symmetry and the general situation is unclear. It is an important open problem to clarify what kind of situation (6) is valid in the critical case. This issue will be discussed in [30]. ∎

Open problem 5.6 (Blow-up of the energy)

In Open problem 5.1, we asked the finiteness of the energy limit d for time-global solutions. For the finite time blow-up solutions, it is natural to ask whether the blow-up of the energy holds:

$$\text{(BJ)} \quad \lim_{t \to T_m} J_p(u(t)) = -\infty \text{ if } T_m < \infty.$$

(BJ) is true for the subcritical problem $p < 2^*$, see Giga [17] and Baras-Cohen [2]. In the critical case, if we consider radially symmetric, nonnegative solutions u of (P), then (BJ) holds, see e.g. Galaktionov-Vazquez [15] and references therein. Also, Proposition 5.3 and (10) indicates that (BJ) is not true in general. It is an important open problem to clarify what kind of situation (BJ) is valid in the critical case. ∎

Open problem 5.7 (Existence of type II blow-up)

As is observed in Proposition 5.3, there exists a solution of (P) satisfying $T_m < \infty$ and $\lim_{t\to T_m} J_p(u(t)) > -\infty$ in the critical case. The existence of such a solution u may lead to the existence of "type II blow-up" which means that the blow-up rate of $\|u(t)\|_\infty$ is faster than the "type I blow-up" rate defined as the rate of a solution of $\dot{u} = u|u|^{p-2}$. In the subcritical case, it is known that every finite time blow-up solution blows up in type I rate, see Giga-Kohn [19, 20, 21]. In the critical case, for a radially symmetric nonnegative function, it is known that there exist no type II blow-up solution, see Matano-Merle [38]. On the other hand, Proposition 5.3 indicates that there exists a type II blow-up solution since (9) is a type II blow-up rate (and, in particular, u_0^* in Proposition 5.3 is sign-changing by Matano-Merle [38]).

In the framework of solutions without radial symmetry and nonnegativity, what kind of condition (on the initial data) gives the type II blow-up for (P) with critical p is an open problem. ∎

Application of the method to other critical heat flow The method proposed in this note is quite flexible and may be successfully applied to have a global bounds for time-global solutions such as a heat flow associated with a noncompact variational functional such as a Yang-Mills-Higgs functional. These problems will be discussed elsewhere.

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6 Appendix. The concavity argument of Payne-Sattinger-Levine

Let us prove (7) for a solution u of (P) with bounded Ω. In this section, we always assume that

$$\Omega \text{ is a bounded domain}$$

and we only assume $p > 2$ unless stated.

First we introduce a relevant equality. By multiplying u to (P) and integrating over Ω, we have

$$\frac{d}{dt}\frac{1}{2}\|u(t)\|_2^2 = -\|\nabla u(t)\|_2^2 + \|u(t)\|_p^p.$$

From the definition of the energy functional, we see that

$$-\|\nabla u(t)\|_2^2 = -2J_p(u(t)) - \frac{2}{p}\|u(t)\|_p^p$$

and these relations yield

$$\frac{d}{dt}\frac{1}{2}\|u(t)\|_2^2 = -2J_p(u(t)) + \left(1 - \frac{2}{p}\right)\|u(t)\|_p^p. \tag{1}$$

Now we show

Lemma 6.1
$T_m < \infty$ follows if $J_p(u(t_0)) < 0$ for some $t_0 \in [0, T_m)$.

Proof of Lemma 6.1.
Assume on the contrary, we have $T_m = \infty$ in spite of the existence of $t_0 \in [0, T_m)$ satisfying $J_p(u(t_0)) < 0$.

By noting the decreasing property of the energy (6) and the assumption, we see that $J_p(u(t)) < 0$ for $t \geq t_0$. Moreover, by the boundedess of the domain, we obtain, by using the Hölder inequality,

$$\|u(t)\|_2^p \leq C(\Omega)\|u(t)\|_p^p. \tag{2}$$

By plugging these relations to (1), we have

$$\frac{d}{dt}\frac{1}{2}\|u(t)\|_2^2 \geq C\|u(t)\|_2^p \tag{3}$$

for some $C > 0$ depending on the measure of Ω. It is easy to solve (3) and we have

$$\|u(t)\|_2^2 \geq \frac{1}{\|u(0)\|_2^{p-2} - \frac{C(p-2)}{2}t},$$

which means $\|u(t)\|_2 \to \infty$ as $t \uparrow \frac{2\|u(0)\|_2^{p-2}}{C(p-2)}$, contradicting the assumption $T_m = \infty$ (note that u satisfies (3), particularly $u \in C([0, T_m); L^2)$). This completes the proof. ∎

Remark 6.1
Note that Lemma 6.1 holds for $p > 2$ including even the supercritical case. On the other hand, it seems difficult to extend the proof above directly to unbounded domains. ∎

For (P) with unbounded domains, so far we have the following:

Lemma 6.2

Let u be a solution of (P) *in an unbounded domain with $p > 2$ and assume that there exists $t_0 < T_m$ satisfying $J_p(u(t_0)) < 0$. Then one of the following holds:*
(a) $T_m < \infty$, *or*
(b) $T_m = \infty$ *and* $\|u(t)\|_2 \geq Ct$ *for some* $C > 0$. *Moreover, if* $p = 2^*$ *and* $\Omega = \mathbb{R}^N$, *we have, in addition,* $\lim_{t\to\infty} J_{2^*}(u(t)) = -\infty$ *and* $\lim_{t\to\infty} \|\nabla u(t)\|_2 = \infty$.

Proof of Lemma 6.2.

Assume (a) does not hold, thus assume $T_m = \infty$. Then by the decreasing property of the energy (6), we have

$$J_p(u(t)) \leq J_p(u(t_0))(< 0) \text{ for any } t \geq t_0.$$

Then this relation and (1) imply

$$\frac{d}{dt}\frac{1}{2}\|u(t)\|_2^2 = -2J_p(u(t)) + \left(1 - \frac{2}{p}\right)\|u(t)\|_p^p \geq -2J(u(t_0))(> 0),$$

hence by putting $C := -4J_p(u(t_0))(> 0)$, we have the first assertion in (b). Now we assume $p = 2^*$ and $\Omega = \mathbb{R}^N$. Then by the decreasing property (6) of $J_{2^*}(u(t))$, we have $\lim_{t\to\infty} J_{2^*}(u(t)) = d \in [-\infty, J_{2^*}(u(t_0))]$. Suppose $d > -\infty$. Then by following the argument in the proof of Proposition 3.2, we have the existence of $t_n \to \infty$ satisfying $\|\nabla u(t_n)\|_2^2 = \|u(t_n)\|_{2^*}^{2^*} + o(1)$ as $n \to \infty$, which yields

$$\lim_{t\to\infty} J_{2^*}(u(t)) \geq \lim_{n\to\infty} J_{2^*}(u(t_n)) = \left(\frac{1}{2} - \frac{1}{2^*}\right)\lim_{n\to\infty} \|\nabla u(t_n)\|_2^2 \geq 0,$$

contradicting the assumption $J_{2^*}(u(t_0)) < 0$ and (6). Hence we have $\lim_{t\to\infty} J_{2^*}(u(t)) = -\infty$. This immediately yields $\lim_{t\to\infty} \|\nabla u(t)\|_2 = \infty$, since otherwise there exists (t_n) such that $\lim_{n\to\infty} \|\nabla u(t_n)\|_2 < \infty$ and

$$|J_{2^*}(u(t_n))| \leq C_1\|\nabla u(t_n)\|_2 < C_2,$$

whence follows $\lim_{t\to\infty} J_{2^*}(u(t)) > -\infty$ by (6) and this contradicts the second assertion. ∎

The same proof immediately yields the following:

Corollary 6.1

Let u be a time-global solution of (P) *in* $\mathbb{R}^N$ *with* $p = 2^*$. *Then one of the following holds:*
(a) $\lim_{t\to\infty} J_{2^*}(u(t)) > -\infty$, *or*
(b) $\lim_{t\to\infty} J_{2^*}(u(t)) = -\infty$, $\lim_{t\to\infty} \|\nabla u(t)\|_2 = \infty$ *and* $\|u(t)\|_2 \geq Ct$ *for some* $C > 0$.

7 Appendix. An isometircal action of the semi-direct product $\mathbb{R}^N \ltimes \mathbb{R}_+$ to $\dot{H}^1$

The invariance of $\dot{H}^1$-norm under (31) indicates that a transformation group obtained from $\mathbb{R}^N$ and $\mathbb{R}_+$, a semi-direct product of them, acts isometrically on $\dot{H}^1$. We introduce this structure.

General facts Let G_1, G_2 be a group and $\rho : G_2 \to \operatorname{Aut} G_1$ be a homomorphism. Then $G_1 \times G_2$ becomes a group by a product

$$[x', y'] * [x, y] := [y\rho_{y'}(x), y'y] \tag{1}$$

for $[x, y]$, $[x', y'] \in G_1 \times G_2$. $(G_1 \times G_2, *)$ is called a semi-direct product of G_1 and G_2 by ρ denoted by $G_1 \ltimes_\rho G_2$, or $G_1 \ltimes G_2$.

The action of $\mathbb{R}^N$ and $\mathbb{R}_+$ on $\dot{H}^1$ Let $a \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}_+$. We first define the action of $\mathbb{R}^N$ and $\mathbb{R}_+$ on $\mathbb{R}^N$ by

$$T_a x := x + a, \quad D_\lambda x := \lambda x, \qquad x \in \mathbb{R}^N, \tag{2}$$

and denote $H_1 := \{T_a;\, a \in \mathbb{R}^N\}$ and $H_2 := \{D_\lambda;\, \lambda \in \mathbb{R}_+\}$. We lift the action of H_1 and H_2 on $\mathbb{R}^N$ to that on $\dot{H}^1$ by

$$\widehat{T_a}u(x) := u(T_a x) = u(x+a), \quad \widehat{D_\lambda}u(x) := \lambda^{\frac{N-2}{2}}u(D_\lambda x) = \lambda^{\frac{N-2}{2}}u(\lambda x) \tag{3}$$

for $u \in \dot{H}^1$ and denote $G_1 := \{\widehat{T_a};\, a \in \mathbb{R}^N\}$ and $G_2 := \{\widehat{D_\lambda};\, \lambda \in \mathbb{R}_+\}$.

It is easy to see that the transformation of u given in (31) coincides with

$$u \mapsto \widehat{T_{-\lambda y}}\widehat{D_\lambda}u.$$

We show that this can be interpreted as an action of $G_1 \ltimes G_2$.

It is easy to see that

$$
\begin{aligned}
\widehat{T_{a'}}\widehat{D_{\lambda'}}\widehat{T_a}\widehat{D_\lambda}u(x) &= \widehat{T_{a'}}\widehat{D_{\lambda'}}\widehat{T_a}\lambda^{\frac{N-2}{2}}u(\lambda x) = \widehat{T_{a'}}\widehat{D_{\lambda'}}\lambda^{\frac{N-2}{2}}u(\lambda x + a) \\
&= \widehat{T_{a'}}(\lambda'\lambda)^{\frac{N-2}{2}}u(\lambda'\lambda x + \lambda' a) \\
&= (\lambda'\lambda)^{\frac{N-2}{2}}u(\lambda\lambda' x + \lambda' a + a') \\
&= \widehat{T_{\lambda' a + a'}}\widehat{D_{\lambda\lambda'}}u(x), \qquad (4)
\end{aligned}
$$

hence if we denote

$$\widehat{T_a}\widehat{D_\lambda} \text{ by } [a, \lambda], \text{ where } (\widehat{T_a}, \widehat{D_\lambda}) \in G_1 \times G_2 \qquad (5)$$

for the brevity, then we see

$$[a', \lambda'] \circ [a, \lambda] = [\lambda' a + a', \lambda'\lambda] \qquad (6)$$

from (4).

Let us introduce

$$\rho : G_2 \ni \widehat{D_\lambda} \mapsto \rho_{\widehat{D_\lambda}} \in X := \{\psi : G_1 \to G_1\}$$

by

$$\rho_{\widehat{D_\lambda}} : G_1 \ni \widehat{T_a} \mapsto \widehat{T_{\lambda a}} \in G_1.$$

Proposition 7.1

$(G_1 \times G_2, \circ)$ *coincides with* $G_1 \ltimes_\rho G_2$.

We prove Proposition 7.1. First we show that

Lemma 7.1

(a) *Foe* $\lambda \in \mathbb{R}_+$, $\rho_{\widehat{D_\lambda}}$ *is an isomorphism on* G_1. *Hence* $\rho : G_2 \to \operatorname{Aut} G_1$.

(b) ρ *is a homomorphism.*

Proof of Lemma 7.1.

Let us denote $\rho_{\widehat{D_\lambda}}$ by ρ_λ.

$$\rho_\lambda(\widehat{T_{a'}}\widehat{T_a}) = \rho_\lambda\widehat{T_{a'+a}} = \widehat{T_{\lambda a' + \lambda a}} = \widehat{T_{\lambda a'}}\widehat{T_{\lambda a}} = \rho_\lambda\widehat{T_{a'}}\rho_\lambda\widehat{T_a},$$

hence ρ_λ is a homomorphism on G_1. Take any $\widehat{T_a} \in G_1$. Then for $a' := \frac{a}{\lambda}$, it is obvious that $\rho_\lambda\widehat{T_{a'}} = \widehat{T_a}$. Moreover, if $\rho_\lambda\widehat{T_{a'}} = \rho_\lambda\widehat{T_a}$, then $a' = a$ holds.

Thus $\widehat{T_{a'}} = \widehat{T_a}$. These relations say that $\rho_\lambda \in \operatorname{Aut} G_1$., i.e., (a). We also have

$$\rho_{\lambda'\lambda}\widehat{T_a} = \widehat{T_{\lambda'\lambda a}} = \rho_{\lambda'}\widehat{T_{\lambda a}} = \rho_\lambda\rho_{\lambda'}\widehat{T_a}, \qquad \widehat{T_a} \in G_1,$$

hence $\rho.$ is a homomorphism, i.e., (b) holds. ∎

Proof of Proposition 7.1.

Note that Lemma 7.1 together with (1) implies that $G_1 \times G_2$ is a group under the product $*$ defined in (1). By using the notation introduced by (5), $*$ has a form

$$[a', \lambda'] * [a, \lambda] := [a' + \rho_{\lambda'}(a), \lambda'\lambda] = [a' + \lambda' a, \lambda'\lambda].$$

This relation coincides with the rule in (6) and we see that $(G_1 \times G_2, \circ)$ coincides with $G_1 \ltimes_\rho G_2$. ∎