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The fractional Liouville equation in dimension 1 Geometry, Compactness and quantization

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Abstract

We discuss some recent results about the fractional Liouville equation in dimension 1 and related questions. These results, resting on a geometric interpretation in terms of holomorphic maps, on the study of geodesics in a conformal metric and on a classical work of Blank about immersions of the disk into the plane, is a fractional counterpart of the celebrated works of Brézis-Merle and Li-Shafrir on the 2-dimensional Liouville equation, but providing sharp quantization estimates under weak assumptions which are not known in dimension 2.

1 Introduction

The purpose of this work is to study the fine compactness properties of the fractional Liouville equations $(-\Delta)^{\frac{1}{2}}u=Ke^{u}-1$ on S^{1} and $(-\Delta)^{\frac{1}{2}}u=Ke^{u}$ in $\mathbb{R}$ under very weak and natural geometric assumptions.

Let us recall that if (Σ, g_{0}) is a smooth, closed Riemann surface with Gauss curvature $K_{g_{0}}$, for any $u\in C^{\infty}$ the conformal metric $g_{u}:=e^{2u}g$ has Gaussian curvature K determined by the Gauss equation:

$$-\Delta_{g_{0}}u=Ke^{2u}-K_{g_{0}} \quad \text{on } \Sigma, \tag{1}$$

where $\Delta_{g_{0}}$ is the Laplace-Beltrami operator on (Σ, g_{0}). In particular when $\Sigma=\Omega\subset \mathbb{R}^{2}$ or $\Sigma=S^{2}$, the Gauss equation (1) reads respectively

$$-\Delta u=Ke^{2u} \quad \text{in } \Omega\subset \mathbb{R}^{2} \tag{2}$$

and

$$-\Delta_{S^{2}}u=Ke^{2u}-1, \quad \text{on } S^{2}. \tag{3}$$

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Both equations (2) and (3) have been largely studied in the literature. For what concerns e.g. the compactness properties of (2), H. Brézis and F. Merle [2] showed among other things the following blow-up behavior:

Theorem 1.1 (Brézis-Merle [2]) *Given an open subset Ω of $\mathbb{R}^2$, assume that $(u_k) \subset L^1_{\mathrm{loc}}(\Omega)$ is a sequence of weak solutions to (2) with $K = K_k \geq 0$ and such that*

$$\|K_k\|_{L^\infty} \leq \bar{\kappa}\,, \quad \|e^{2u_k}\|_{L^1} \leq \bar{A}\,.$$

Then up to subsequences either

1. *u_k is bounded in $L^\infty_{\mathrm{loc}}(\Omega)$, or*

2. *there is a finite (possibly empty) set $B = \{x_1, \ldots, x_N\} \subset \Omega$ (the blow-up set) such that $u_k(x) \to -\infty$ locally uniformly in $\Omega \setminus B$, and*

$$K_k e^{2u_k} \overset{*}{\rightharpoonup} \sum_{i=1}^{N} \alpha_i \delta_{x_i} \quad \textit{for some numbers } \alpha_i \geq 2\pi.$$

Notice that here $K_k \geq 0$. Theorem 1.1 implies that the amount of concentration of curvature α_i at each blow-up point is at least 2π, which is *half* of the total curvature of S^2. On the other hand, as shown by Y-Y. Li and I. Shafrir [12], if one assumes that $K_k \to K_\infty$ in $C^0(\Omega)$, then a stronger and deeper quantization result holds, namely α_i is an integer multiple of 4π. This result was then extended to higher even dimension $2m$ in the context of Q-curvature and GJMS-operators by several authors [7, 15, 18, 20, 22, 19], always under the strong assumption that the curvatures are continuous and converge in C^0 (sometimes even in C^1), but, at least in [18, 19] giving up the requirement that the curvatures are non-negative. The main ingredient here is that for uniformly continuous curvatures, blow-up leads to metrics of constant curvature, and such a constant is necessarily positive (by results of [17, 18]). Finally positive and constant curvature leads to spherical metrics, thanks to various classification results (e.g. [4, 14, 16]), and this is in turn responsible for the constant 4π (or $(n-1)!|S^n|$ in even dimension $n \geq 4$) in the quantization results, i.e. each blow-up point carries the total curvature of a sphere, or a multiple of it.

We now ask what happens if we remove *both* the positivity and uniform continuity assumptions on the curvature, only relying on an L^∞ bound. We will address this question in dimension 1, where the analogue of (3) is

$$(-\Delta)^{\frac{1}{2}}\lambda = Ke^{\lambda} - 1, \quad \text{on } S^1,$$

whose geometric interpretation in terms of conformal maps plays a crucial role in having the following precise understanding of the blow-up behaviour.

Theorem 1.2 (Da Lio-Martinazzi-Rivière [6]) *Let $(\lambda_k) \subset L^1(S^1, \mathbb{R})$ be a sequence-satisfying*

$$(-\Delta)^{\frac{1}{2}}\lambda_k = \kappa_k e^{\lambda_k} - 1 \quad \textit{in } S^1, \tag{4}$$

under the bounds

$$\|e^{\lambda_k}\|_{L^1(S^1)} \le \bar{L}, \quad \|\kappa_k\|_{L^\infty(S^1)} \le \bar{\kappa}. \tag{5}$$

Then up to subsequence we have $\kappa_k e^{\lambda_k} \rightharpoonup \mu$ *weakly in* $W^{1,p}_{\mathrm{loc}}(S^1 \setminus B)$ *for every* $p < \infty$, *where* μ *is a Radon measure,* $B := \{a_1, \dots, a_N\}$ *is a (possibly empty) subset of* S^1 *and* $\kappa_k \overset{*}{\rightharpoonup} \kappa_\infty$ *in* $L^\infty(S^1)$. *Set* $\bar{\lambda}_k := \frac{1}{2\pi}\int_{S^1} \lambda_k d\theta$. *Then one of the following alternatives holds:*
i) $\bar{\lambda}_k \to -\infty$ *as* $k \to \infty$, $N = 1$ *and* $\mu = 2\pi\delta_{a_1}$. *In this case*

$$v_k := \lambda_k - \bar{\lambda}_k \rightharpoonup v_\infty \quad \text{in } W^{1,p}_{\mathrm{loc}}(S^1 \setminus \{a_1\}) \text{ for every } p < \infty,$$

where $v_\infty(e^{\imath\theta}) = -\log(2(1-\cos(\theta-\theta_1)))$ *for* $a_1 = e^{\imath\theta_1}$, *solving*

$$(-\Delta)^{\frac{1}{2}} v_\infty = -1 + 2\pi\delta_{a_1} \quad \text{in } S^1. \tag{6}$$

ii) $\bar{\lambda}_k \to -\infty$ *as* $k \to \infty$, $N = 2$ *and* $\mu = \pi(\delta_{a_1} + \delta_{a_2})$. *In this case*

$$v_k := \lambda_k - \bar{\lambda}_k \rightharpoonup v_\infty \quad \text{in } W^{1,p}_{\mathrm{loc}}(S^1 \setminus \{a_1, a_2\}) \text{ for every } p < \infty,$$

where

$$v_\infty(e^{\imath\theta}) = -\frac{1}{2}\log(2(1-\cos(\theta-\theta_1))) - \frac{1}{2}\log(2(1-\cos(\theta-\theta_2))), \quad a_1 = e^{\imath\theta_1},\ a_2 = e^{\imath\theta_2}$$

solves

$$(-\Delta)^{\frac{1}{2}} v_\infty = -1 + \pi\delta_{a_1} + \pi\delta_{a_2} \quad \text{in } S^1. \tag{7}$$

iii) $|\bar{\lambda}_k| \le C$ *and* $\mu = \kappa_\infty e^{\lambda_\infty} + \pi(\delta_{a_1} + \dots + \delta_{a_N})$ *for some* $\lambda_\infty \in W^{1,p}_{\mathrm{loc}}(S^1 \setminus B)$, *with* $\lambda_\infty, e^{\lambda_\infty} \in L^1(S^1)$ *and*

$$(-\Delta)^{\frac{1}{2}} \lambda_\infty = \kappa_\infty e^{\lambda_\infty} - 1 + \sum_{i=1}^N \pi\delta_{a_i} \quad \text{in } S^1. \tag{8}$$

For a discussion of the above result we refer to [6]. Here instead we want to devote our attention to the case of the real line. The analogue of (2) is

$$(-\Delta)^{\frac{1}{2}} u = Ke^u, \quad \text{in } \mathbb{R}.$$

Here the additional difficulty is that we cannot a priori control the behaviour at infinity of the solutions. Nonetheless we obtain the following result.

Theorem 1.3 (Da Lio-Martinazzi [5]) *Let* $(u_k) \subset L_{\frac{1}{2}}(\mathbb{R})$ *be a sequence of solutions to*

$$(-\Delta)^{\frac{1}{2}} u_k = K_k e^{u_k} \quad \text{in } \mathbb{R} \tag{9}$$

and assume that for some $\bar{\kappa}, \bar{L} > 0$ *and for every* k *it holds*

$$\|e^{u_k}\|_{L^1} \le \bar{L}, \quad \|K_k\|_{L^\infty} \le \bar{\kappa}. \tag{10}$$

Up to a subsequence assume that $K_k \overset{*}{\rightharpoonup} K_\infty$ *in* $L^\infty(\mathbb{R})$, *and that* $K_k e^{u_k} \rightharpoonup \mu$ *as Radon measures. Then there exists a finite (possibly empty) set* $B := \{x_1, \dots, x_N\} \subset \mathbb{R}$ *such that, up to extracting a further subsequence, one of the following alternatives holds:*

1. $u_k \to u_\infty$ *in* $W^{1,p}_{\mathrm{loc}}(\mathbb{R} \setminus B)$ *for* $p < \infty$, *where*

$$(-\Delta)^{\frac{1}{2}} u_\infty = \mu = K_\infty e^{u_\infty} + \sum_{i=1}^{N} \pi \delta_{x_i} \quad \text{in } \mathbb{R} \tag{11}$$

(compare to Fig. 1).

2. $u_k \to -\infty$ *locally uniformly in* $\mathbb{R} \setminus B$ *and*

$$\mu = \sum_{j=1}^{N} \alpha_j \delta_{x_j}$$

for some $\alpha_j \geq \pi$, $1 \leq j \leq N$ *(compare to Fig. 3).*

Let us compare the above theorem with the result of Brézis and Merle. The cost to pay for allowing K_k to change sign is that even in case 1, in which u_k has a non-trivial weak-limit, there can be blow-up, and in this case a *half-quantization* appears: the constant π in (11) is half of the total-curvature of S^1. In case 2, instead we are able to recover the analogue of case 2 in the Brézis-Merle theorem. On the other hand, the proof is now much more involved, as near a blow-up point regions of negative and positive curvatures can (and in general do) accumulate, and one needs a way to take into account the various cancelations. A direct blow-up approach does not seem to work because there can be infinitely many scales at which non-trivial contributions of curvature appear. In general it would be easy to prove that

$$|K_k| e^{u_k} \overset{*}{\rightharpoonup} \sum_{j=1}^{N} \alpha_j \delta_{x_j}$$

for some $\alpha_j \geq \pi$, but removing the absolute values we need to prove that there is "more" positive than negative curvature concentrating at each blow-up point. This is turn will be reduced to a theorem of differential topology about the degree of closed curves in the plan, inspired by a classical work of S. J. Blank [1], and to the blow-up analysis provided in [6], which will allow us to choose a suitable blow-up scale and estimate the curvature left in the other scales.

Things simplify and the above theorem can be sharpened if we assume that $K_k \geq 0$, hence falling back into a statement of Brézis-Merle type.

Theorem 1.4 (Da Lio-Martinazzi [5]) *Let* (u_k) *and* (K_k) *be as in Theorem 1.3 and additionally assume that* $K_k \geq 0$. *Then, up to a subsequence, in case 1 of Theorem 1.3 we have* $N = 0$ *and in case 2 we have* $\alpha_j > \pi$ *for* $1 \leq j \leq N$.

The proofs of Theorems 1.3 and 1.4 are strongly based on the following geometric interpretation of Equation (9) (compare also [11]).

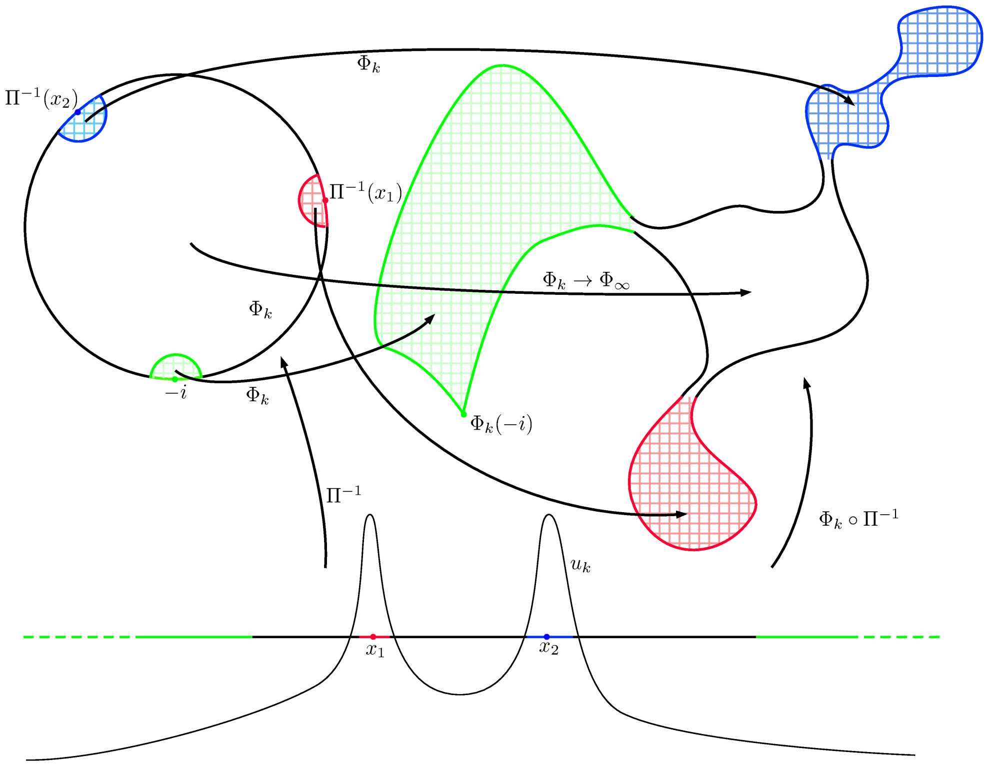

Figure 1: The case 1 of Theorem 1.3 with $N = 2$, in the interpretation given by Theorem 1.5. From the function u_k blowing up at a_1 and a_2 (and possibly at infinity, in the sense that some curvature vanishes at infinity) we construct $\Phi_k : \bar{D}^2 \to \mathbb{C}$ blowing up at $a_1 = \Pi^{-1}(x_1)$, $a_2 = \Pi^{-1}(x_2)$ and possibly $-i$, but converging to an immersion Φ_∞ away from $\{a_1, a_2, -i\}$.

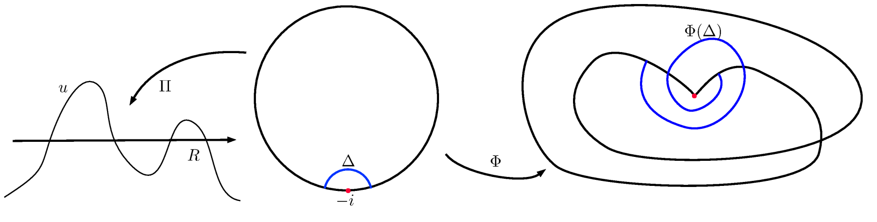

Figure 2: The map Φ given by Theorem 1.5 is in general singular at $-i$. Because of this the curve $\Phi|_{S^1}$ can have rotation index greater than 1 (it is 2 in the above example). The image of the curve Δ should facilitate the intuition of the geometry of Φ near $-i$.

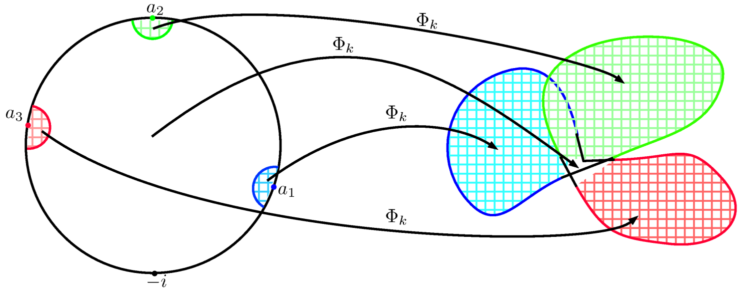

Figure 3: An example of multiple pinching.

Theorem 1.5 (Da Lio-Martinazzi-Rivière [5, 6]) *Let $u \in L_{\frac{1}{2}}(\mathbb{R})$ with $e^u \in L^1(\mathbb{R})$ satisfy*

$$(-\Delta)^{\frac{1}{2}}u = Ke^u \quad \text{in } \mathbb{R} \tag{12}$$

for some function $K \in L^\infty(\mathbb{R})$. Then there exists $\Phi \in C^0(\bar{D}^2, \mathbb{C})$ with $\Phi|_{S^1} \in W^{2,p}_{\mathrm{loc}}(S^1 \setminus \{-i\}, \mathbb{C})$ for every $p < \infty$ such that Φ is a holomorphic immersion of $\bar{D}^2 \setminus \{-i\}$ into $\mathbb{C}$,

$$|\Phi'(z)| = \frac{2}{1+\Pi(z)^2}e^{u(\Pi(z))}, \quad \text{for } z \in S^1 \setminus \{-i\}, \tag{13}$$

and the curvature of the curve $\Phi|_{S^1\setminus\{-i\}}$ is $\kappa := K \circ \Pi$, where $\Pi : S^1 \setminus \{-i\} \to \mathbb{R}$ is the stereographic projection given by $\Pi(z) = \frac{\Re z}{1+\Im z}$.

Another ingredient in the proof of Theorem 1.3 comes from differential geometry and roughly speaking says that if a closed positively oriented curve $\gamma : S^1 \to \mathbb{C}$ of class C^1 except at finitely many points can be extended to a function $F \in C^0(D^2, \mathbb{C})$ which is a C^1-immersion except at finitely many boundary points, then the rotation index of γ is at least 1. This is obvious if $F \in C^1(\bar{D}^2, \mathbb{C})$ is an immersion everywhere (no corners on the boundary), and in fact the rotation index of $\gamma = F|_{S^1}$ is 1 in this case, but in the general case the rotation index can be arbitrarily high and the proof that it must be strictly positive rests on ideas introduced by Blank to study which regular closed curves can be extended to an immersion of the disk into the plane.

Another consequence of Theorem 1.5 is a new and geometric proof, not relying on a Pohozaev-type identity, nor on the moving plane technique (moving point in this case), of the classification of the solutions to the non-local equation

$$(-\Delta)^{\frac{1}{2}}u = e^u \quad \text{in } \mathbb{R}, \tag{14}$$

under the integrability condition

$$L := \int_{\mathbb{R}} e^u dx < \infty. \tag{15}$$

Theorem 1.6 *Every function $u \in L_{\frac{1}{2}}(\mathbb{R})$ solving (14)-(15) is of the form*

$$u_{\mu,x_0}(x) := \log\left(\frac{2\mu}{1+\mu^2|x-x_0|^2}\right), \quad x \in \mathbb{R}^n, \tag{16}$$

for some $\mu > 0$ and $x_0 \in \mathbb{R}$.

Previous proofs can be found e.g. in [3, 8, 11, 13, 21, 23, 24]. Similar higher-dimensional results, also in the fractional case have appeared in [3, 9, 10, 14, 16, 23].

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