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# On a certain discontinuous mapping as a continuous relation (summary)

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## 1 Introduction

In classical mathematics, the following mapping $F$ is a mapping of $\mathbb{R}$ into $\{0,1\}$, and is discontinuous at 0:

$$F(x) := \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

In Bishop's constructive mathematics (BISH) [2], we cannot prove that such a discontinuous mapping exists, since, in this framework, the existence implies a independent principle LPO (the least limited principle of Omniscience). Also, in Weihrauch's computable mathematics, $F$ is not computable, and therefore is not continuous (see [1] and [9]) . However, [1] gives some investigations of classical discontinuous mappings in the view point as relation, and does notions of continuous relations. But we have not known whcther the existence of a discontinuous mapping is justified by recognising it as a continuous relation in BISH. This paper gives an investigation of continuous relation in BISH.

Constructive mathematics is formalized in intuitionistic logic, and always requires a constructive proof, i.e. a proof which can be regarded as an algorithm. Bishop has given in [2] an informal framework of constructive mathematics, which is a proper subsystem of classical mathematics. Actually, it does not include an axiom that is refused in classical mathematics, and therefore its constructive proof is acceptable in classical mathematics. On the other hand, some classical theorems are independent on BISH. For example, the following property, called LPO (the least limited principle of Omniscience):

$$\forall\{a_n\} \in \{0,1\}^{\mathbb{N}} \left[\forall n(a_n = 0) \vee \exists n(a_n = 1)\right],$$

holds classically, but cannot be proved in BISH, since we cannot give an algorithm which decides whether $a_n = 0$ for all natural number $n$, or find a natural number $n$ with $a_n = 1$. But the negation of LPO does not hold; that is, LPO is independent on BISH. Moreover it is proved in BISH that LPO is equivalent to the following:

$$\forall x \in \mathbb{R}\,[x > 0 \vee x \leq 0]\,,$$

where $\mathbb{R}$ is the set of all Cauchy reals. We therefore cannot decide whether a given real number is either positive or non-positive. However the following *apartness property* holds in BISH (see [2, Corollary in p.24], [4, proposition (5.3)], etc.):

$$\forall a, b, c \in \mathbb{R}\,[a < b \rightarrow a < b \vee b < c]\,.$$

This is proved by comparison with two rational Cauchy sequence. This property has an important role, and also is used in the proof of Theorem 9.

Section 2 gives a constructive investigation of continuous properties given in [1]. In Section 3, we consider some continuity properties of a mapping for further study of continuous relations

# 2 Continuity properties of a relation

Let $\mathbb{R}$ denote a Euclid space of all Cauchy reals, $X$, $Y$ and $V$ subsets of $\mathbb{R}$, $R$ a relation on $X \times Y$ i.e. a subset of $X \times Y$, and $\varepsilon > 0$. Remark that "$A \neq \emptyset$" means that we can take an element of $A$. Let $x$ and $y$ be in $\mathbb{R}$. We now give some notations as follows:

$$[x]R := \{y \in Y : (x, y) \in R\}, \qquad R[y] := \{x \in X : (x, y) \in R\},$$

$$[V]R := \{y \in Y : R[y] \cap V \neq \emptyset\}, \qquad R[V] := \{x \in X : [x]R \cap V \neq \emptyset\},$$

$$\mathrm{dom}(R) := \{x \in X : \exists y \in Y((x, y) \in R)\},$$

$$\mathrm{range}(R) := \{y \in Y : \exists x \in X((x, y) \in R)\}.$$

$$N(x, \varepsilon) := \{y \in \mathbb{R} : |x - y| < \varepsilon\}.$$

We now consider notions of continuity of relation, defined in [1].

**Definition 1.**

1. Let $(x, y)$ a point in $X \times Y$. $R$ is *continuous at* $(x, y)$ if for each $k$ in $\mathbb{N}$, there exists $n$ in $\mathbb{N}$ such that $N_Y\left(y, 2^{-k}\right) \cap [x']R \neq \emptyset$ for all $x'$ in $N_X\left(x, 2^{-n}\right) \cap \mathrm{dom}(R)$.

2. *$R$ is continuous* if $R$ is continuous at all point $(x, y)$ in $R$.

3. *$R$ has a continuous restriction* if there exists a continuous relation $S \subset R$ such that that $\mathrm{dom}(S) = \mathrm{dom}(R)$.

4. *$R$ is weakly continuous* if for all $x$ in $\mathrm{dom}(R)$, there exists $y$ in $R[x]$ such that $R$ is continuous at $(x, y)$.

It is easy to show the following proposition.

**Proposition 1.** *In Definition 1, (2) $\implies$ (3) $\implies$ (4) holds.*

We here define $F$ as a relation.

$$F := \{(x, y) \in \mathbb{R} \times \{0, 1\} : (x \leq 0 \to y = 0) \wedge (x > 0 \to y = 1)\}.$$

**Proposition 2.**

1. $F \subset \mathbb{R} \times \mathbb{R}$.
2. $\mathrm{dom}(F) = \{x \in \mathbb{R} : x \leq 0\} \cup \{x \in \mathbb{R} : x > 0\}$.
3. $\mathbb{R} = \mathrm{dom}(F)$ *if and only if LPO holds.*
4. $\mathrm{range}(F) = \{0, 1\}$.
5. $N_{\{0,1\}}\left(0, 2^{-k}\right) = \{0\}$ *for all* $k$.
6. $\left[2^{-k}\right] R = \{1\}$ *for all* $k$.

**Theorem 3.** *$F$ is not continuous at $(0, 0)$, and hence is not weak continuous.*

*Proof.* $N_{\{0,1\}}\left(0, 2^{-k}\right) \cap \left[2^{-(k+1)}\right] R = \emptyset$ for all $k$ in $\mathbb{N}$ by (5) and (6) of Proposition 2. That is, $F$ is not continuous at $(0, 1)$. Since $(0, 1) \notin R$, $F$ is not weakly continuous. □

We now consider an analogy of $F$. Let $\varepsilon > 0$, and set

$$F_\varepsilon := \{(x, y) : (x < \varepsilon \to y = 0) \wedge (x > 0 \to y = 1)\}.$$

It is clear that $F_\varepsilon$ is not a mapping but a relation and that $\mathrm{dom}(F_\varepsilon) = \mathbb{R}$ by the apartness property.

**Theorem 4.** *For any $\varepsilon > 0$, $F_\varepsilon$ is continuous.*

*Proof.* Fix any $\varepsilon > 0$. Let $(x, y)$ be an element of $F_\varepsilon$, and any $k$ in $\mathbb{N}$. Then $0 < x$ or $x < \varepsilon$. In the former case, we have $y = 1$. Take some $n$ in $\mathbb{N}$ with $2^{-n} < x$. Then, for any $x'$ in $N_{\mathbb{R}}\left(x, 2^{-n}\right)$,

$$N_{\{0,1\}}\left(1, 2^{-k}\right) \cap [x'] F_\varepsilon = \{1\}.$$

In the latter case, we have $y = 0$. Take some $n$ in $\mathbb{N}$ with $x + 2^n < \varepsilon$, Then, for $x'$ in $N_{\mathbb{R}}\left(x, 2^{-n}\right)$,

$$N_{\{0,1\}}\left(0, 2^{-k}\right) \cap [x'] F_\varepsilon = \{0\}.$$

□

We next consider weaker continuity of a relation.

**Definition 2.**

1. Let $(x, y)$ a point in $X \times Y$. $R$ is *left continuous at* $(x, y)$ if for each $k$ in $\mathbb{N}$, there exists $n$ in $\mathbb{N}$ such that $N_Y\left(y, 2^{-k}\right) \cap [x'] R \neq \emptyset$ for all $x'$ in $N_X\left(x, 2^{-n}\right) \cap \mathrm{dom}(R) \cap (-\infty, x)$.

2. *R is left continuous* if $R$ is left continuous at all point $(x, y)$ in $R$.

3. *R has a left continuous restriction* if there exists a left continuous relation $S \subset R$ such that $\mathrm{dom}(S) = \mathrm{dom}(R)$.

4. *R is weakly left continuous* if for all $x$ in $\mathrm{dom}(R)$, there exists $y$ in $R[x]$ such that $R$ is left continuous at $(x, y)$.

Left continuity for a mapping is given in [5, p.57], [8, Sect B, Ch. 508] and so on.

We clearly have the following proposition.

**Proposition 5.** *In Definition 2, (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) holds.*

In classical mathematics, $F$ is also a left continuous mapping on $\mathbb{R}$. We similarly obtain this matter in BISH as follows:

**Theorem 6.** *F is left continuous.*

*Proof.* Let $(x, y)$ be any point of $F$, and any $k$ in $\mathbb{N}$. Then $x$ in $\mathrm{dom}(F)$, and therefore $x \leq 0$ or $0 < x$. In the former case, we take any $n$ in $\mathbb{N}$, and have $N_{\{0,1\}}\left(0, 2^{-k}\right) \cap [x']F = \{0\}$ for all $x'$ in $N_{\mathbb{R}}\left(x, 2^{-n}\right) \cap \mathrm{dom}(F) \cap (-\infty, x)$. In the latter case, we can choose $n$ in $\mathbb{N}$ with $2^{-n} < x$, and then $N_{\{0,1\}}\left(0, 2^{-k}\right) \cap [x']F = \{1\}$ for all $x'$ in $N_{\mathbb{R}}\left(x, 2^{-n}\right) \cap \mathrm{dom}(F) \cap (-\infty, x)$. □

# 3 On continuity properties of a mapping

In this section, we consider some standard continuity properties of a mapping for more investigation of a continuous relation. Let $f$ be a mapping of a subset $X$ of $\mathbb{R}$ into $\mathbb{R}$ and $x$ in $X$. We say that *f is continuous at x* if for each $k$ in $\mathbb{N}$, there exists $N$ in $\mathbb{N}$ such that, given any $n \geq N$,

$$f\left(N_X\left(x, 2^{-n}\right)\right) \subset N_{\mathbb{R}}\left(f(x), 2^{-k}\right),$$

where $f(X)$ denotes the image by $f$. *f is continuous* if $f$ is continuous at all point $X$. It is easy to show that $f$ is a continuous mapping if and only if it is a continuous relation.

A subset $A$ of $\mathbb{R}$ is *open* in $\mathbb{R}$ if for each $x$ in $A$, there exists some $\delta > 0$ such that $N_{\mathbb{R}}(x, \delta) \subset A$

We can easily show the following theorem.

**Theorem 7.** *Let $f$ be a mapping of $\mathbb{R}$ into $\mathbb{R}$. Then $f$ is continuous if and only if, whenever $O$ is an open subset of $\mathbb{R}$, then the inverse $f^{-1}(O)$ is open in $\mathbb{R}$.*

We next consider sequential continuity of a mapping. In BISH, a continuous mapping of a metric space into a metrics is sequentially continuous, but the converse cannot be proved (see [7] and so on). We therefore need to investigate sequential continuity of a mapping as another continuity property, for considering sequentially continuous relations.

Let $A$ be a subset of $\mathbb{R}$. $\overline{A}$ means the closure of $A$ i.e.

$$\overline{A} := \{x \in \mathbb{R} : \forall \varepsilon > 0\,[N_{\mathbb{R}}(x,\varepsilon) \cap A \neq \emptyset]\}$$

A subset of $\mathbb{R}$ is *closure* in $\mathbb{R}$ if $\overline{A} = A$.

The following lemma is proved in the same way as *Ishihara's Trick* ([3, Lemma 3.2.1] and [6, Lemma 1]).

**Lemma 8.** *Let $f$ be a mapping from $\mathbb{R}$ into $\mathbb{R}$, $\{x_n\}$ a sequence in $\mathbb{R}$ and $x$ a real number. Assume that $f$ satisfies $f(\overline{A}) \subset \overline{f(A)}$ for all subset $A$ of $\mathbb{R}$. Then, for positive real numbers $a$ and $b$ with $a < b$, $|f(x_n) - f(x)| > a$ for some $n$, or $|f(x_n) - f(x)| < b$ for all $n$.*

We finally show the equivalence between sequential continuity and another continuity properties.

**Theorem 9.** *Let $f$ be a mapping from $\mathbb{R}$ into $\mathbb{R}$. Then the followings are equivalent.*

1. *$f$ is sequentially continuous i.e. $f$ satisfies that for all $x$ in $\mathbb{R}$ and sequence $\{x_n\}$ in $\mathbb{R}$, if $\{x_n\}$ converges to $x$ in $\mathbb{R}$, then the sequence $\{f(x_n)\}$ converges to $f(x)$ in $\mathbb{R}$.*

2. *For all $x$ in $\mathbb{R}$ and sequence $\{x_n\}$ in $\mathbb{R}$, if $\{x_n\}$ converges to $x$ in $\mathbb{R}$, then there exists a subsequence $\{f(x_{n_k})\}$ of $\{f(x_n)\}$ converging to $f(x)$ in $\mathbb{R}$.*

3. *For all subset $C$ of $\mathbb{R}$, if $C$ is closed in $\mathbb{R}$, then the subset $f^{-1}(C)$ is closed in $\mathbb{R}$.*

4. *For all subset $A$ of $\mathbb{R}$, $f(\overline{A}) \subset \overline{f(A)}$.*

We here note the most important part of the proof of (4) $\Longrightarrow$ (1). Let $x$ be a real number, and $\{x_n\}$ a sequence converging to $x$ in $\mathbb{R}$. Take a strictly increasing sequence $\{N_k\}$ in $\mathbb{N}$ such that $|x_n - x| < 2^{-k}$ for any $n \geq N_k$. Let $\varepsilon$ be any positive number. By Lemma 8, construct an increasing binary sequence $\{\lambda_k\}$ such that

$$\begin{aligned} \lambda_k = 0 \quad &\Longrightarrow \quad \forall i \leq k \left[\exists n \geq N_i\, |f(x_n) - f(x)| > \frac{\varepsilon}{2}\right] \\ \lambda_k = 1 \quad &\Longrightarrow \quad \exists i \leq k \left[\forall n \geq N_i\, |f(x_n) - f(x)| < \varepsilon\right]. \end{aligned}$$

Then the assumption (4) implies $\lambda_k = 1$ for some $k$.

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