Every strongly definable $C^r G$ vector bundle admits a unique strongly definable $C^\infty G$ vector bundle structure

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Abstract

Let $G$ be a compact subgroup of $GL_n(\mathbb{R})$. We prove that every strongly definable $C^r G$ vector bundle over an affine definable $C^\infty G$ manifold admits a unique strongly definable $C^\infty G$ vector bundle structure up to definable $C^\infty G$ vector bundle isomorphism ($0 \leq r < \infty$).

1 Introduction

By [12], if $s$ is a non-negative integer, then every $C^s$ Nash map between affine Nash manifolds is approximated in the definable $C^s$ topology by Nash maps. This definable $C^s$ topology is a new topology defined in [12].

In this paper, $G$ denotes a compact subgroup of $GL_n(\mathbb{R})$, every definable map is continuous and any manifold does not have boundary, unless otherwise stated. Under our assumption, $G$ is a compact algebraic subgroup of $GL_n(\mathbb{R})$ (e.g. 2.2 [10]). We consider an equivariant definable version of the above theorem in an o-minimal expansion $\mathcal{M} = (\mathbb{R}, +, \cdot, <, \ldots)$ of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. General references on o-minimal structures are [1], [3], see also [13]. Further properties and constructions of them are studied in [2], [4], [11].

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We consider strongly definable \( C^\infty G \) vector bundle structures of strongly definable \( C^r G \) vector bundles (\( 0 \leq r < \infty \)).

Everything is considered in \( M \) and the term “definable” is used throughout in the sense of “definable with parameters in \( M \)”, each definable map is assumed to be continuous.

2 Preliminaries

An ordered structure \((R, <)\) with a dense linear order \(<\) without endpoints is \( o\)-minimal (order minimal) if every definable set of \( R \) is a finite union of open intervals and points, where open interval means \((a, b)\), \( -\infty \leq a < b \leq \infty \).

If \((R, +, \cdot, <)\) is a real closed field, then it is \( o\)-minimal and the collection of definable sets coincides that of semialgebraic sets.

The topology of \( R \) is the interval topology and the topology of \( R^n \) is the product topology.

Let \( X \subset R^n \) and \( Y \subset R^m \) be definable sets. A continuous map \( f : X \to Y \) is definable if the graph of \( f \) \((\subset X \times Y \subset R^n \times R^m)\) is a definable set. A definable map \( f : X \to Y \) is a definable homeomorphism if there exists a definable map \( f' : Y \to X \) such that \( f \circ f' = id_Y \), \( f' \circ f = id_X \).

A group \( G \) is a definable group if \( G \) is a definable set and the group operations \( G \times G \to G \) and \( G \to G \) are definable.

Let \( G \) be a definable group. A pair \((X, \phi)\) consisting a definable set \( X \) and a \( G \) action \( \phi : G \times X \to X \) is a definable \( G \) set if \( \phi \) is definable. We simply write \( X \) instead of \((X, \phi)\) and \( gx \) instead of \( \phi(g, x) \).

A definable map \( f : X \to Y \) between definable \( G \) sets is a definable \( G \) map if for any \( x \in X, g \in G \), \( f(gx) = gf(x) \). A definable \( G \) map is a definable \( G \) homeomorphism if it is a homeomorphism.

Definition 1 A topological fiber bundle \( \eta = (E, p, X, F, K) \) is called a definable fiber bundle over \( X \) with fiber \( F \) and structure group \( K \) if the following two conditions are satisfied:

1. The total space \( E \) is a definable space, the base space \( X \) is a definable set, the structure group \( K \) is a definable group, the fiber \( F \) is a definable set with an effective definable \( K \) action, and the projection \( p : E \to X \) is a definable map.

2. There exists a finite family of local trivializations \( \{U_i, \phi_i : p^{-1}(U_i) \to U_i \times F\}_i \) of \( \eta \) such that each \( U_i \) is a definable open subset of \( X \), \( \{U_i\}_i \) is a finite
open covering of $X$. For any $x \in U_i$, let $\phi_{i,x} : p^{-1}(x) \to F, \phi_{i,z}(z) = \pi_i \circ \phi_i(z)$, where $\pi_i$ stands for the projection $U_i \times F \to F$. For any $i$ and $j$ with $U_i \cap U_j \neq \emptyset$, the transition function $\theta_{ij} := \phi_{j,x} \circ \phi_{i,x}^{-1} : U_i \cap U_j \to K$ is a definable map. We call these trivializations definable. Definable fiber bundles with compatible definable local trivializations are identified.

(3) A definable fiber bundle is a definable vector bundle if $F = \mathbb{R}^n$ and $K = GL(n, \mathbb{R})$.

Definition 2 (1) Let $0 \leq r \leq \infty$. A Hausdorff space $X$ is an $n$-dimensional definable $C^r$ manifold if there exist a finite open cover $\{U_i\}_{i=1}^k$ of $X$, finite open sets $\{V_i\}_{i=1}^k$ of $\mathbb{R}^n$, and a finite collection of homeomorphisms $\{\phi_i : U_i \to V_i\}_{i=1}^k$ such that for any $i$, $j$ with $U_i \cap U_j \neq \emptyset$, $\phi_i(U_i \cap U_j)$ is definable and $\phi_j \circ \phi_i^{-1} : \phi(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable $C^r$ diffeomorphism. This pair $\{(U_i)_{i=1}^k, \{\phi_i : U_i \to V_i\}_{i=1}^k\}$ of sets and homeomorphisms is called a definable $C^r$ coordinate system.

(2) A definable $C^r$ manifold $G$ is a definable $C^r$ group if $G$ is a group and the group operations $G \times G \to G, G \to G$ are definable $C^r$ maps.

(3) Let $G$ be a definable group. A pair $(X, \phi)$ consisting a definable $C^r$ manifold $X$ and a $G$ action $\phi : G \times X \to X$ is a definable $C^r G$ manifold if $\phi$ is a definable $C^r$ map. We simply write $X$ instead of $(X, \phi)$ and $gx$ instead of $\phi(g, x)$.

Definition 3 ([6]) Let $G$ be a definable $C^r$ group and $0 \leq r \leq \infty$.

(1) A definable $C^r G$ vector bundle is a definable $C^r$ vector bundle $\eta = (E, p, X)$ satisfying the following three conditions.

(a) The total space $E$ and the base space $X$ are definable $C^r G$ manifolds.

(b) The projection $p : E \to X$ is a definable $C^r G$ map.

(c) For any $x \in X$ and $g \in G$, the map $p^{-1}(x) \to p^{-1}(gx)$ is linear.

(2) Let $\eta$ and $\zeta$ be definable $C^r G$ vector bundles over $X$. A definable $C^r$ vector bundle morphism $\eta \to \zeta$ is called a definable $C^r G$ vector bundle morphism if it is a $G$ map. A definable $C^r G$ vector bundle morphism $f : \eta \to \zeta$ is said to be a definable $C^r G$ vector bundle isomorphism if there exists a definable $C^r G$ vector bundle morphism $h : \zeta \to \eta$ such that $f \circ h = id$ and $h \circ f = id$. If $r = 0$, then a definable $C^0 G$ vector bundle (resp. a definable $C^0 G$ vector bundle morphism, a definable $C^0 G$ vector bundle isomorphism) is simply called a definable $G$ vector bundle (resp. a definable $G$ vector bundle morphism, a definable $G$ vector bundle isomorphism).

(3) A definable $C^r$ section of a definable $C^r G$ vector bundle is a definable $C^r G$ section if it is a $G$ map.
Definition 4 ([8], [6]) Let $0 \leq r \leq \infty$.

(1) A group homomorphism (resp. A group isomorphism) from $G$ to $O_n(\mathbb{R})$ is a *definable group homomorphism* (resp. *a definable group isomorphism*) if it is a definable map (resp. a definable homeomorphism).

Note that a definable group homomorphism (resp. a definable group isomorphism) between $G$ and $O_n(\mathbb{R})$ is a definable $C^\infty$ map (resp. a definable $C^\infty$ diffeomorphism) because $G$ and $O_n(\mathbb{R})$ are Lie groups.

(2) An $n$-dimensional representation of $G$ means $\mathbb{R}^n$ with the linear action induced by a definable group homomorphism from $G$ to $O_n(\mathbb{R})$. In this paper, we assume that every representation of $G$ is orthogonal.

(3) A definable $C^r$ submanifold of a definable $C^r$ manifold $X$ is called a *definable $C^r$ submanifold* of $X$ if it is $G$ invariant.

(4) A definable $C^r$ manifold is called *affine* if it is definably $C^r$ diffeomorphic (definably $G$ homeomorphic if $r = 0$) to a definable $C^r$ submanifold of some representation of $G$.

(5) A *definable $C^r$ manifold with boundary* is defined similarly.

If $0 \leq r < \infty$, then every definable $C^r$ manifold is affine ([8], [7]) and if $\mathcal{M}$ is exponential, then each compact definable $C^\infty$ manifold is affine [8].

Recall universal $G$ vector bundles (e.g. [6]) and existence of a Nash $G$ tubular neighborhood of a Nash $G$ submanifold of a representation of $G$ ([9]).

Let $\Omega$ be an $n$-dimensional representation of $G$ induced by a definable group homomorphism $B : G \to O_n(\mathbb{R})$. Suppose that $M(\Omega)$ denotes the vector space of $n \times n$ matrices with the action $(g, A) \in G \times M(\Omega) \mapsto B(g)AB(g)^{-1} \in M(\Omega)$. For any positive integer $k$, we define the vector bundle $\gamma(\Omega, k) = (E(\Omega, k), u, G(\Omega, k))$ as follows:

\[
G(\Omega, k) = \{ A \in M(\Omega) | A^2 = A,^tA = A, TrA = k \},
\]

\[
E(\Omega, k) = \{ (A, v) \in G(\Omega) \times \Omega | Av = v \},
\]

\[
u : E(\Omega, k) \to G(\Omega, k), u((A; v)) = A,
\]

where $^tA$ denotes the transposed matrix of $A$ and $TrA$ stands for the trace of $A$. Then $\gamma(\Omega, k)$ is an algebraic vector bundle. Since the action on $\gamma(\Omega, k)$
is algebraic, it is an algebraic $G$ vector bundle. We call it the universal $G$ vector bundle associated with $\Omega$ and $k$. Remark that $G(\Omega, k) \subset M(\Omega)$ and $E(\Omega, k) \subset M(\Omega) \times \Omega$ are nonsingular algebraic $G$ sets. In particular, they are Nash $G$ submanifolds of $M(\Omega)$ and $M(\Omega) \times \Omega$, respectively.

**Theorem 5 ([9])** Every Nash $G$ submanifold $X$ of a representation $\Omega$ of $G$ has a Nash $G$ tubular neighborhood $(U, \theta)$ of $X$ in $\Omega$.

**Definition 6 ([6])** (1) Let $G$ be a definable group. A definable $G$ vector bundle $\eta = (E, p, X)$ over a definable $G$ set $X$ is called strongly definable if there exist a representation $\Omega$ of $G$ and a definable $G$ map $f : X \to G(\Omega, k)$ such that $\eta$ is definably $G$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where $k$ denotes the rank of $\eta$.

(2) Let $G$ be a definable $C^r$ group and $0 \leq r \leq \infty$. A definable $C^rG$ vector bundle $\eta = (E, p, X)$ over an affine definable $C^rG$ manifold $X$ is called strongly definable if there exist a representation $\Omega$ of $G$ and a definable $C^rG$ map $f : X \to G(\Omega, k)$ such that $\eta$ is definably $C^rG$ vector bundle isomorphic to $f^*(\gamma(\Omega, k))$, where $k$ denotes the rank of $\eta$.

### 3 Our results

**Theorem 7 ([5])** If $0 \leq s < \infty$ and $M$ admits $C^\infty$ cell decomposition and exponential, then every definable $C^sG$ map between affine definable $C^\inftyG$ manifolds is approximated in the definable $C^s$ topology by definable $C^\inftyG$ maps.

Our main result is the following.

**Theorem 8 ([5])** Let $X$ be an affine definable $C^\inftyG$ manifold and $M$ admits $C^\infty$ cell decomposition and exponential. If $0 \leq r < \infty$, then every strongly definable $C^rG$ vector bundle over $X$ admits a unique strongly definable $C^\inftyG$ vector bundle structure up to definable $C^\inftyG$ vector bundle isomorphism.
References


