数理解析研究所講究録 第2084巻 2018年 27-28

A Proof of Hales-Jewett Theorem using a Non-Standard Method

Akito Tsuboi
University of Tsukuba

Shelah's proof of Hales-Jewett theorem was explained in [1]. Following the ideas explained there, I give a proof using a nonstandard method.

Theorem 1. *For all $n, n_c \in \omega$, we can find $N = N_{n,n_c} \in \omega$ with the following property: For a given coloring $c : n^N \to n_c$ there is a word $w(x) \in (n \cup \{x\})^N \smallsetminus n^N$ such that $\{w(i) : i \in n\}$ is c-monochromatic.*

Proof. Throughout n_c is fixed. We prove the theorem by induction on n. So we assume that the theorem is true for n. Let $N = N_{n,n_c}$. Now we work in a (sufficiently saturated) nonstandard model $M \succ (\omega, 0, 1, +, \cdot, <, \dots)$. Let $a \in M \smallsetminus \omega$ and let

$$c^* : (n+1)^a \to n_c$$

be given. We can assume, by coding method, c^* lives in M. We want to find a c^*-monochromatic line in $(n+1)^a$. First fix an indiscernible sequence

$$d_0 < d_1 < \cdots < d_i < \cdots < a$$

over c^*. For numbers $s \le t \le u \le v$ and x, let

$$C(s,t,u,v;x) = (n)_{t-s}\,\hat{}\,(x)_{u-t}\,\hat{}\,(0)_{v-u} = \langle \underbrace{n \dots n}_{t-s}, \underbrace{x \dots x}_{u-t}, \underbrace{0 \dots 0}_{v-u} \rangle.$$

Then, for $\bar{e} = e_0, \dots, e_{N-1} \in (n+1)^N$, $f(\bar{e})$ is the sequence

$$(n)_{d_0}\,\hat{}\,C(d_{0,1,2,3}; e_0)\hat{}\,C(d_{3,4,5,6}; e_1)\,\hat{}\cdots\hat{}\,C(d_{3N-3,3N-2,3N-1,3N}; e_{N-1})\,\hat{}\,(0)_{a-d_{3N}},$$

of length a, where $d_{i,j,k,l}$ denotes the sequence d_i, d_j, d_k, d_l. For $\bar{e} \in (n+1)^N$, let $\hat{e}$ be the sequence obtained from $\bar{e}$ by replacing every term $e_\imath = n$ with 0. For example, if $\bar{e} = \langle n, n-1, n, 1, \dots \rangle$, then $\hat{e} = \langle 0, n-1, 0, 1, \dots \rangle$. $\hat{e}$ belongs to n^N.

Claim A. $c^*(f(\bar{e})) = c^*(f(\hat{e}))$.

We assume $e_k = n$. Then the following equations are true.

$$\begin{aligned} c^*(f(\bar{e})) &= c^*(\ldots\,\hat{}\,C(d_{3k,3k+1,3k+2,3k+3}; n)\,\hat{}\,\ldots)) \\ &= c^*(\ldots\,\hat{}\,C(d_{3k,3k+2,3k+2,3k+3}; n)\,\hat{}\,\ldots)) \quad (1) \\ &= c^*(\ldots\,\hat{}\,C(d_{3k,3k+1,3k+1,3k+3}; n)\,\hat{}\,\ldots)) \quad (2) \\ &= c^*(\ldots\,\hat{}\,C(d_{3k,3k+1,3k+2,3k+3}; 0)\,\hat{}\,\ldots)) \quad (3) \\ &= c^*(f(,\bar{e}')). \end{aligned}$$

The equality (1) holds because the two cells $C(d_{3k,3k+1,3k+2,3k+3}; n)$ and $C(d_{3k,3k+2,3k+2,3k+3}; n)$ are the same. The equality (2) holds because of the indiscernibility of $\bar{d}$ over c^*. The equlality (3) holds because the two cells $C(d_{3k,3k+1,3k+1,3k+3}; n)$ and $C(d_{3k,3k+1,3k+2,3k+3}; 0)$ are the same. (End of Proof of Claim)

Now we consider the coloring $c' : (n+1)^N \to n_c$ defined by $c'(\bar{e}) = c^*(f(\bar{e}))$. By our choice of N, if c' is restricted to the domain n^N, there is a word $w(x) \in (n \cup \{x\})^N \smallsetminus n^N$ such that $\{w(i) : i \in n\}$ is c'-monochromatic. By Claim A and the choice of $w(x)$, $\{w(i) : i \in n+1\}$ is also monochromatic. Let $w^*(x)$ denote the word $f(w(x))$. It is a sequence in $((n+1) \cup \{x\})^{N^*} \smallsetminus (n+1)^{N^*}$ and the following claim clearly holds.

Claim B. $\{w^*(i) : i \in n+1\}$ *is* c^**-monochromatic.*

Now we have shown that the following statement holds in M:

$\exists a,\ \forall c^* : (n+1)^a \to n_c,\ \exists w^*(x)$ s.t. $\{w^*(i) : i \in n+1\}$ is a singleton.

Since ω is an elementary substructure of M, the same statement holds in ω. This provides the induction step of our proof. □

References

[1] Pierre Matet, Shelah's proof of the Hales-Jewett thorem revisited, European Journal of Combinatorics 28(2007) 1742-1745.

[2] S. Shelah, Primitive recursive bounds for van der Waerden numbers, Journal of the American Mathematical Society 1 (1988) 683-697.

[3] Nikolaos Kragiannis, A combinatorial proof of an infinite version of the Hales-Jewett theorem.