数理解析研究所講究録 第2084巻 2018年 39-47

On Rieffel's theorem

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Abstract

Let $\theta \in \mathbb{R}\backslash\mathbb{Q}$. We defined a notion of quantum 2-torus T_θ in [1] and studied its model theoretic properties. In the subsequent paper [2], we introduced the notion of *geometric equivalence* and also of *Morita equivalence* between such quantum 2-tori.

We showed that this notion is closely connected with the fundamental notion of *Morita equivalence* of non-commutative geometry. Namely, we proved that the quantum 2-tori T_{θ_1} and T_{θ_2} are Morita equivalent if and only if $\theta_2 = \dfrac{a\theta_1 + b}{c\theta_1 + d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$. This is our version of Rieffel's Theorem [4] which characterizes Morita equivalence of quantum tori in the same terms.

In this note we reconsider the relation between the original version of Rieffel's theorem and our model theoretic version.

1 Quantum 2-torus T_q

In this section we give a quick review of the construction of a quantum 2-torus T_θ described in [1].

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ and put $q = \exp(2\pi i\theta)$. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Consider a $\mathbb{C}^*$-algebra $\mathcal{A}_q$ generated by operators U, U^{-1}, V, V^{-1} satisfying

$$VU = qUV, \quad UU^{-1} = U^{-1}U = VV^{-1} = V^{-1}V = I.$$

Let $\Gamma_\theta = q^{\mathbb{Z}} = \{q^n : n \in \mathbb{Z}\}$ be a cyclic multiplicative subgroup of $\mathbb{C}^*$. From now on in this note we work in an uncountable $\mathbb{C}$-module $\mathcal{M}$ such that $\dim \mathcal{M} \geq |\mathbb{C}|$.

*joint work with Boris Zilber, Oxford University

1.1 Quantum line bundles

For each pair $(u, v) \in \mathbb{C}^* \times \mathbb{C}^*$, we will construct two $\mathcal{A}_q$-modules $M_{|u,v\rangle}$ and $M_{\langle v,u|}$ so that both $M_{|u,v\rangle}$ and $M_{\langle v,u|}$ are sub-modules of $\mathcal{M}$.

The module $M_{|u,v\rangle}$ is generated by linearly independent elements labeled $\{\mathbf{u}(\gamma u, v) \in \mathcal{M} : \gamma \in \Gamma_\theta\}$ satisfying

$$\begin{array}{lcl} U & : & \mathbf{u}(\gamma u, v) \mapsto \gamma u \mathbf{u}(\gamma u, v), \\ V & : & \mathbf{u}(\gamma u, v) \mapsto v \mathbf{u}(q^{-1}\gamma u, v). \end{array} \tag{1}$$

Next let $\phi : \mathbb{C}^*/\Gamma_\theta \to \mathbb{C}^*$ such that $\phi(x\Gamma_\theta) \in x\Gamma_\theta$ for each $x\Gamma_\theta \in \mathbb{C}^*/\Gamma_\theta$. Put $\Phi = \mathrm{ran}(\phi)$. We call ϕ a *choice function* and Φ the system of representatives.

Set for $\langle u, v\rangle \in \Phi^2$

$$\begin{array}{lcl} \Gamma \cdot \mathbf{u}(u, v) & := & \{\gamma \mathbf{u}(u, v) : \gamma \in \Gamma_\theta\}, \\ \mathbf{U}_{\langle u,v\rangle} & := & \bigcup_{\gamma \in \Gamma_\theta} \Gamma_\theta \cdot \mathbf{u}(\gamma u, v) = \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \gamma_1, \gamma_2 \in \Gamma_\theta\}. \end{array} \tag{2}$$

And set

$$\begin{array}{lcl} \mathbf{U}_\phi & := & \bigcup_{\langle u,v\rangle \in \Phi^2} \mathbf{U}_{\langle u,v\rangle} \\ & = & \{\gamma_1 \cdot \mathbf{u}(\gamma_2 u, v) : \langle u, v\rangle \in \Phi^2, \gamma_1 . \gamma_2 \in \Gamma_\theta\}, \\ \mathbb{F}^*\mathbf{U}_{\phi_1} & := & \{x \cdot \mathbf{u}(\gamma u, v) : \langle u, v\rangle \in \Phi^2, x \in \mathbb{F}^*, \gamma \in \Gamma_\theta\}. \end{array} \tag{3}$$

We call $\Gamma \cdot \mathbf{u}(u, v)$ a Γ-**set** over the pair (u, v), $\mathbf{U}_\phi$ a Γ-bundle over $\mathbb{C}^* \times \mathbb{C}^*/\Gamma$, and $\mathbb{C}^*\mathbf{U}_\phi$ a **line-bundle** over $\mathbb{C}^*$. Notice that $\mathbf{U}_\phi$ can also be seen as a bundle inside $\bigcup_{\langle u,v\rangle} M_{|u,v\rangle}$. Notice also that the line bundle $\mathbb{C}^*\mathbf{U}_\phi$ is closed under the action of the operators U and V satisfying the relations (1).

We define the module $M_{\langle v,u|}$ generated by linearly independent elements labeled $\{\mathbf{v}(\gamma v, u) \in \mathcal{M} : \gamma \in \Gamma\}$ satisfying

$$\begin{array}{lcl} U & : & \mathbf{v}(\gamma v, u) \mapsto u \mathbf{v}(q\gamma v, u), \\ V & : & \mathbf{v}(\gamma v, u) \mapsto \gamma v \mathbf{v}(\gamma v, u), \end{array} \tag{4}$$

and also

$$\begin{array}{lcl} U^{-1} & : & \mathbf{u}(\gamma u, v) \mapsto \gamma^{-1} u^{-1} \mathbf{u}(\gamma u, v), \\ V^{-1} & : & \mathbf{u}(\gamma u, v) \mapsto v^{-1} \mathbf{u}(q\gamma u, v). \end{array} \tag{5}$$

Similarly a Γ-set $\Gamma \cdot \mathbf{v}(v, u)$ over the pair (v, u), a Γ-bundle $\mathbf{V}_\phi$ over $\mathbb{C}^*/\Gamma \times \mathbb{C}^*$, and $\mathbb{C}^*\mathbf{V}_\phi$ a **line-bundle** over $\mathbb{C}^*$ are defined.

To define the line bundles $\mathbb{C}^*\mathbf{U}_\phi$ and $\mathbb{C}^*\mathbf{V}_\phi$, we do not need any particular properties of the element $q = \exp(2\pi i\theta)$ or the choice function ϕ. Therefore we have:

Proposition 1 (Proposition 2 [1]) *Let $\mathbb{F}$, $\mathbb{F}'$ be fields and $q \in \mathbb{F}$, $q' \in \mathbb{F}'$ such that there is an field isomorphism i from $\mathbb{F}$ to $\mathbb{F}'$ sending q to q'. Then i can be extended to an isomorphism from the Γ-bundle $\mathbf{U}_\phi$ to the Γ'-bundle $\mathbf{U}_{\phi'}$ and also from the line-bundle $\mathbb{F}^*\mathbf{U}_\phi$ to the line-bundle $(\mathbb{F}^*)'\mathbf{U}_{\phi'}$. The same is true for the line-bundles $\mathbb{F}^*\mathbf{V}_\phi$ and $(\mathbb{F}')^*\mathbf{V}_{\phi'}$.*

In particular the isomorphism type of Γ-bundles and line-bundles does not depend on the. choice function.

Proof: Let i be an isomorphism from $\mathbb{F}$ to $\mathbb{F}'$ sending q to q'. Set $i(x \cdot \mathbf{u}(\gamma u, v)) = i(x) \cdot \mathbf{u}(i(\gamma u), i(v))$. Then this defines an isomorphism from $\mathbb{F}^*\mathbf{U}_\phi$ to $(\mathbb{F}')^*\mathbf{U}_{\phi'}$. ■

1.2 Pairing function

Recall next the notion of *pairing function* $\langle \cdot \mid \cdot \rangle$ which plays the rôle of an *inner product* of two Γ-bundles $\mathbf{U}_\phi$ and $\mathbf{V}_\phi$:

$$\langle \cdot \mid \cdot \rangle : \Big(\mathbf{V}_\phi \times \mathbf{U}_\phi\Big) \cup \Big(\mathbf{U}_\phi \times \mathbf{V}_\phi\Big) \to \Gamma. \tag{6}$$

having the following properties:

1. $\langle \mathbf{u}(u, v) | \mathbf{v}(v, u) \rangle = 1$,
2. for each $r, s \in \mathbb{Z}$, $\langle U^r V^s \mathbf{u}(u, v) | U^r V^s \mathbf{v}(v, u) \rangle = 1$,
3. for $\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma$,

$$\langle \gamma_1 \mathbf{u}(\gamma_2 u, v) | \gamma_3 \mathbf{v}(\gamma_4 v, u) \rangle = \langle \gamma_3 \mathbf{v}(\gamma_4 v, u) | \gamma_1 \mathbf{u}(\gamma_2 u, v) \rangle,$$

4. $\langle \gamma_1 \mathbf{u}(\gamma_2 u, v) | \gamma_3 \mathbf{v}(\gamma_4 v, u) \rangle = \gamma_1^{-1} \gamma_3 \langle \mathbf{u}(\gamma_2 u, v) | \mathbf{v}(\gamma_4 v, u) \rangle$, and
5. for $v' \notin \Gamma \cdot v$ or $u' \notin \Gamma \cdot u$, $\langle q^s \mathbf{v}(v', u) | q^r \mathbf{u}(u', v) \rangle$ is not defined.

Proposition 2 (Proposition 3 [1]) *The* **pairing** *function (6) defined above satisfies the following: for any $m, k, r, s \in \mathbb{N}$ we have*

$$\langle q^s \mathbf{v}(q^m v, u) | q^r \mathbf{u}(q^k u, v) \rangle = q^{r-s-km} \tag{7}$$

and

$$\langle q^r \mathbf{u}(q^k u, v) | q^s \mathbf{v}(q^m v, u) \rangle = q^{km+s-r} = \langle q^s \mathbf{v}(q^m v, u) | q^r \mathbf{u}(q^k u, v) \rangle^{-1}. \tag{8}$$

1.3 Quantum 2-torus

We call the three sorted structure $\langle \mathbf{U}_\phi, \mathbf{V}_\phi, \langle \cdot \mid \cdot \rangle \rangle$ a quantum 2-torus and denoted by T_θ.

From Proposition 1 we know that the structure of the line-bundles does not depend on the choice function. The next proposition tells us that the structure of the quantum 2-torus $T_q^2(\mathbb{C})$ depends only on $\mathbb{C}$, q and not on the choice function.

Proposition 3 (cf. Proposition 4.4, [6]) *Given $q \in \mathbb{F}^*$ not a root of unity, any two structures of the form $T_q^2(\mathbb{F})$ are isomorphic over $\mathbb{F}$. In other words, the isomorphism type of $T_q^2(\mathbb{F})$ does not depend on the system of representatives Φ.*

Remark 4 *In our construction of quantum line bundles and quantum 2-tori, the $\mathbb{C}^*$-algebra $\mathcal{A}_q$ does not play a major role, but a minor one.*

2 Geometrically equivalent quantum 2-tori

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Note that this θ has no relation with the one used in the previous section.

From now on we work in the structure $\mathbb{C}^\theta = (\mathbb{C}, +, \cdot, 1, x^\theta)$ (raising to real power θ in the complex numbers).

We define

$$x^\theta = \exp(\theta \cdot (\ln x + 2\pi i \mathbb{Z})) = \{\exp(\theta \cdot (\ln x + 2\pi i k)) : k \in \mathbb{Z}\}.$$

as a multi-valued function and by $y = x^\theta$ we mean the relation $\exists z\, (x = \exp(z) \wedge y = \exp(z\theta))$.

Notation 5 *$C_\theta(x, y)$ denotes the binary relation $y = x^\theta$ as defined above.*

Let $\theta_1, \theta_2 \in \mathbb{R} \setminus \mathbb{Q}$. Set $q_1 = \exp(2\pi i \theta_1)$ and $q_2 = \exp(2\pi i \theta_2)$. Put $\Gamma_{\theta_1} = \langle q_1 \rangle$ and $\Gamma_{\theta_2} = \langle q_2 \rangle$.

Definition 6 *Let $a, b \in \mathbb{C}^*$.*

(1) We say that C_θ sends the coset $a \cdot \Gamma_{q_1}$ of Γ_{θ_1} to the coset $b \cdot \Gamma_{\theta_2}$ of Γ_{θ_2} if

$$\forall x' \in a \cdot \Gamma_{\theta_1}\ \forall y' \in \mathbb{C}^* \Big(y' \in b \cdot \Gamma_{\theta_2} \iff C_\theta(x', y') \Big).$$

(2) *We say that C_θ sends the cosets of Γ_{θ_1} to the cosets of Γ_{θ_2} if C_θ gives rise to a one-to-one correspondence from the cosets of Γ_{θ_1} to the cosets of Γ_{θ_2}.*

Let Φ_1 be the system of representatives for a choice function $\phi_1 : \mathbb{C}^*/\Gamma_{\theta_1} \to \mathbb{C}^*$. Let T_{θ_2} be quantum 2-tori constructed as explained in the previous section.

Suppose $(u,v) \in (\Phi_1)^2$. We identify the modules $M_{|u,v\rangle}$ constitutes the quantum 2-torus T_{θ_1} with its canonical basis denoted by $E_{|u,v\rangle}$. Put

$$E_{|u,v\rangle} = \{q^{nl}\mathbf{u}(q^n u, v) : l, n \in \mathbb{Z}\}.$$

We see the Γ_{q_1}-bundle $\mathbf{U}_{\phi_1}$ as a bundle inside $\bigcup_{(u,v)\in(\Phi_1)^2} M_{|u,v\rangle}$. Thus knowing the set of bases of $\mathbf{U}_{\phi_1}$ that is the set $\bigcup_{(u,v)\in(\Phi_1)^2} E_{|u,v\rangle}$, we can determine the quantum 2-torus T_{θ_1}.

Let Φ_2 be the system of representatives for a choice function $\phi_2 : \mathbb{C}^*/\Gamma_{\theta_2} \to \mathbb{C}^*$. Let T_{θ_2} be quantum 2-tori constructed as explained in the previous section.

We define a similar set $E_{|u',v'\rangle}$ which is a canonical basis for $M_{|u',v'\rangle}$ where $(u',v') \in (\Phi_2)^2$ and the set $\bigcup_{(u',v')\in(\Phi_2)^2} E_{|u',v'\rangle}$ determines the quantum 2-torus T_{θ_2}.

We now introduce the notion called *geometric equivalence* between quantum 2-tori.

Definition 7 (Geometric equivalence) *We say that the quantum 2-torus T_{θ_1} is geometrically equivalent to T_{θ_2}, written $T_{\theta_1} \simeq_\theta T_{\theta_2}$, if*

(1) *C_θ sends the cosets of Γ_{q_1} to the cosets of Γ_{q_2}, and*

(2) *there is a one-to-one correspondence L_θ from $\bigcup_{\langle u,v\rangle} E_{|u,v\rangle}$ to $\bigcup_{\langle u',v'\rangle} E_{|u',v'\rangle}$ such that for each $(u,v) \in (\Phi_1)^2$ and $(u',v') \in (\Phi_2)^2$ satisfying $C_\theta(u,u')$ and $C_\theta(v,v')$ we have*

$$L_\theta(q_1^{nl}\mathbf{u}(q_1^n u, v)) = q_2^{nl}\mathbf{u}(q_2^n u', v')).$$

We call L_θ a *geometric transformation* from $\bigcup_{\langle u,v\rangle} E_{|u,v\rangle}$ to $\bigcup_{\langle u',v'\rangle} E_{|u',v'\rangle}$ and we simply write as

$$L_\theta : E_{|u,v\rangle} \mapsto E_{|u',v'\rangle}.$$

For a geometric transformation L_θ, we have the following diagrams, for each $(u,v) \in (\Phi_1)^2$ and $(u',v') \in (\Phi_2)^2$:

$$\begin{array}{ccc}
\mathbf{u}((q_1)^n u, v) & \xrightarrow{L_\theta} & \mathbf{u}((q_2)^n u', v') \\
\Big\downarrow U & \circlearrowleft & U \Big\downarrow \\
(q_1)^n u \mathbf{u}((q_1)^n u, v) & \xrightarrow{L_\theta} & (q_2)^n u' \mathbf{u}((q_2)^n u', v')
\end{array}$$

and

$$\begin{array}{ccc}
\mathbf{u}((q_1)^n u, v) & \xrightarrow{L_\theta} & \mathbf{u}((q_2)^n u', v') \\
\Big\downarrow V & \circlearrowleft & V \Big\downarrow \\
v\mathbf{u}((q_1)^{-1}(q_1)^n u, v) & \xrightarrow{L_\theta} & v'\mathbf{u}((q_2)^{-1}(q_2)^n u', v')
\end{array}$$

Conversely, the existence of such diagrams is sufficient for L_θ to be a geometric transformation.

3 Rieffel's theorem

Recall

Definition 8 *Two algebras A and B are said to be Morita equivalent if the categories A-mod and B-mod of modules are equivalent.*

For quantum tori this notion was studied by M.Rieffel and in the particular case of 2-tori we have the following

Theorem 9 (Rieffel) *Let A_{θ_1} and A_{θ_2} be (the coordinate algebras of) quantum 2-tori. Then A_{θ_1} and A_{θ_2} are Morita equivalent if and only if there exist integers a, b, c, d such that $ad - bc = \pm 1$ and $\theta_2 = \dfrac{a\theta_1 + b}{c\theta_1 + d}$.*

For quantum tori T_{θ_1} and T_{θ_2} constructed as in the previous section, we say that T_{θ_1} and T_{θ_2} are Morita equivalent if their coordinate algebras $\mathcal{A}_{\theta_1}$ and $\mathcal{A}_{\theta_2}$ are Morita equivalent. We shall prove a theorem stating that: T_{θ_1} and T_{θ_2} are Morita equivalent if and only if T_{θ_1} and T_{θ_2} are geometrically equivalent.

Of course, in light of Rieffel's theorem it is enough to prove that the geometric equivalence of T_{θ_1} and T_{θ_2} amounts to the condition $\theta_2 = \dfrac{a\theta_1 + b}{c\theta_1 + d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$.

3.1 Relations giving rise to geometric transformations

Proposition 10 *For each* $\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$*, the binary relation*

$$C_\Theta(x,y), \quad \Theta = \frac{m_{11}\theta + m_{12}}{m_{21}\theta + m_{22}}$$

corresponding to

$$y = x^{\frac{m_{11}\theta+m_{12}}{m_{21}\theta+m_{22}}}$$

is positive quantifier-free definable in the structure $\mathbb{C}_\theta$*.*

Proof: Observe the following immediate equivalences:

- $y = x^{m\theta} \equiv C_\theta(x^m, y)$
- $y = x^{m\theta+n} \equiv C_\theta(x^m, yx^{-n})$
- $y = x^{\frac{1}{\theta}} \equiv C_\theta(y, x)$
- $y = x^{\frac{1}{m\theta+n}} \equiv x = y^{m\theta+n} \equiv C_\theta(y^m, xy^{-n})$

It follows

$$\begin{aligned} y = x^{\frac{m_{11}\theta+m_{12}}{m_{21}\theta+m_{22}}} &\equiv y^{m_{21}\theta+m_{22}} = x^{m_{11}\theta+m_{12}} \\ &\equiv (y^{m_{21}}x^{-m_{11}})^\theta = x^{m_{12}}y^{-m_{22}} \\ &\equiv C_\theta(y^{m_{21}}x^{-m_{11}}, x^{m_{12}}y^{-m_{22}}) \end{aligned}$$

■

Lemma 11 *Suppose that* C_θ *sends the cosets of* Γ_{q_1} *to the cosets of* Γ_{q_2}*. Then there is a geometric transformation from* T_{θ_1} *to* T_{θ_2}*, hence we have* $T_{\theta_1} \simeq_\theta T_{\theta_2}$*.*

Proof: Once we know the correspondence between the cosets of Γ_{q_1} and the cosets of Γ_{q_2}, it is easy to define a geometric transformation L_θ from T_{θ_1} to T_{θ_2}, and we have $T_{\theta_1} \simeq_\theta T_{\theta_2}$ ■

3.1.1 Main theorem

We now show the main theorem.

Theorem 12 *Let $\theta_1, \theta_2 \in \mathbb{R} \setminus \mathbb{Q}$. Then $T_{\theta_1} \simeq_\theta T_{\theta_2}$ if and only if $\theta_2 = \dfrac{a\theta_1 + b}{c\theta_1 + d}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$.*

Proof: By Lemma 11 $T_{\theta_1} \simeq_\theta T_{\theta_2}$ if and only if C_θ sends cosets of Γ_{θ_1} to Γ_{θ_2}. In particular, C_θ induces a group isomorphism $\Gamma_{\theta_1} = \langle q_1 \rangle$ to $\Gamma_{\theta_2} = \langle q_2 \rangle$:

$$\exp(2\pi i(\mathbb{Z}\theta_1 + \mathbb{Z})) \xmapsto{\theta} \exp(2\pi i((\mathbb{Z}\theta_1 + \mathbb{Z})\theta)) = \exp(2\pi i(\mathbb{Z}\theta_2 + \mathbb{Z})).$$

The isomorphism is completely determined by the images of $q_1 = \exp(2\pi i\theta_1)$ and 1 both in $\Gamma_{[theta_1}$. Thus it suffices to know the images of θ_1 and 1 by this isomorphism i.e., multiplication by θ. Hence we have

$$\left\{ \begin{array}{lcrcl} \theta_1 & \xmapsto{\theta} & \theta_1\theta & = & a\theta_2 + b \\ 1 & \xmapsto{\theta} & \theta & = & c\theta_2 + d \end{array} \right. \quad \text{where}\, a, b, c, d \in \mathbb{Z} \text{ and } |ad - bc| = 1.$$

It follows that

$$\theta = \frac{a\theta_2 + b}{\theta_1} = c\theta_2 + d. \tag{9}$$

Solving for θ_2 we get

$$\theta_2 = \frac{d\theta_1 - b}{-c\theta_1 + a}. \tag{10}$$

Since $|ad - bc| = 1$ we have

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \in \mathrm{GL}_2(\mathbb{Z}).$$

And this completes the proof. ■

3.2 Relation between modularity and Morita equivalence

Let $\mathcal{A}_{\theta_1}$ and $\mathcal{A}_{\theta_2}$ be (the coordinate algebras of) quantum 2-tori T_{θ_1} and T_{θ_2}.

Combining Rieffel's Theorem and Theorem 12, we see that the following three properties are equivalent:

(1) $\mathbb{C}^*$-algebras $\mathcal{A}_{\theta_1}$ and $\mathcal{A}_{\theta_2}$ are Morita equivalent,

(2) quantum-tori T_{θ_1} and T_{θ_2} are geometrically equivalent,

(3) there exist integers a, b, c, d such that $ad - bc = \pm 1$ and

$$\theta_2 = \frac{a\theta_1 + b}{c\theta_1 + d}.$$

Keeping this relation in mind, we introduce an equivalence relation $E_\theta(\theta_1, \theta_2)$ over $\mathbb{R} \setminus \mathbb{Q}$ defined as follows; we work in the structure $\mathbb{C}^\theta = (\mathbb{C}, +, \cdot, 1, x^\theta)$ (raising to real power θ in the complex numbers), and take $\theta_1, \theta_2 \in \mathbb{R} \setminus \mathbb{Q}$. Put

$$E_\theta(\theta_1, \theta_2) \Longleftrightarrow T_{\theta_1} \simeq_\theta T_{\theta_2}.$$

Our next objective is to investigate the structure $(\mathbb{R} \setminus \mathbb{Q})/E_\theta$.

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