Brauer indecomposability of Scott modules and local subgroups

Hiroki Ishioka
Department of Mathematics,
Tokyo University of Science

1. INTRODUCTION

Let $p$ be a prime number and $k$ an algebraically closed field of characteristic $p$. For a $p$-subgroup $Q$ of a finite group $G$ and a $kG$-module $M$, the Brauer quotient $M(Q)$ of $M$ with respect to $Q$ is naturally a $kN_G(Q)$-module. A $kG$-module $M$ is said to be Brauer indecomposable if $M(Q)$ is indecomposable or zero as a $kC_G(Q)$-module for any $p$-subgroup $Q$ of $G$ ([6]). Brauer indecomposability of $p$-permutation modules is important for constructing stable equivalences of Morita type between blocks of finite groups (see [2]).

There is a connection between Brauer indecomposability of $p$-permutation $kG$-modules and fusion systems, as shown in [6]. The main result in [6] is the following.

**Theorem 1 ([6, Theorem 1.1]).** Let $P$ be a $p$-subgroup of $G$ and $M$ an indecomposable $p$-permutation $kG$-module with vertex $P$. If $M$ is Brauer indecomposable, then $\mathcal{F}_P(G)$ is a saturated fusion system.

In the case that $P$ is abelian and $M$ is the Scott $kG$-module $S(G,P)$, it is known that the converse of the above theorem holds.

**Theorem 2 ([6, Theorem 1.2]).** Let $P$ be an abelian $p$-subgroup of $G$. If $\mathcal{F}_P(G)$ is saturated, then $S(G,P)$ is Brauer indecomposable.

In general, the above theorem does not hold in the case that $P$ is non-abelian. However, there are some cases in which the Scott $kG$-module $S(G,P)$ is Brauer indecomposable for non-abelian $P$ (see [5, 7]). Moreover, it was shown that there are some relationships between Brauer indecomposability of Scott modules and fusion systems ([3, 5]). In particular, we proved the following theorem in [3].

**Theorem 3 ([3, Theorem 1.3]).** Let $G$ be a finite group and $P$ a $p$-subgroup of $G$. Suppose that $M = S(G,P)$ and that $\mathcal{F}_P(G)$ is saturated. Then the following are equivalent.

(i) $M$ is Brauer indecomposable.

(ii) $\text{Res}^{N_G(Q)}_{Q}(S(N_G(Q),N_P(Q)))$ is indecomposable for each fully normalized subgroup $Q$ of $P$.

If these conditions are satisfied, then $M(Q) \cong S(N_G(Q),N_P(Q))$ for each fully normalized subgroup $Q \leq P$.

The above theorem gives a criterion to determine whether the Scott module $S(G,P)$ is Brauer indecomposable.
We investigate the possibility of providing applications of the above theorem. In this paper, we will prove the following result.

**Theorem 4.** Let $G$ be a finite group and $P$ a $p$-subgroup of $G$. Suppose that $\mathcal{F} := \mathcal{F}_P(G)$ is a saturated fusion system. Consider the following two conditions:

(i) $S(N_G(Q), N_P(Q))$ is Brauer indecomposable for each fully $\mathcal{F}$-normalized subgroup $Q \leq P$.

(ii) $S(G, P)$ is Brauer indecomposable.

Then (i) implies (ii), and the converse holds if $\mathcal{F} = \mathcal{F}_P(N_G(P))$.

The above theorem shows that there exists some relationship between $G$ and its local subgroups in terms of the Brauer indecomposability of Scott modules, and will be a useful tool for the study of the Brauer indecomposability of Scott modules.

2. **Preliminaries**

2.1. **Scott modules.** First, We recall the definition of Scott modules and some of its properties:

**Definition 5.** For a subgroup $H$ of $G$, the Scott $kG$-module $S(G, H)$ with respect to $H$ is the unique indecomposable summand $M$ of $\text{Ind}_H^G k_H$ such that $k_G | M$.

If $P$ is a Sylow $p$-subgroup of $H$, then $S(G, H)$ is isomorphic to $S(G, P)$. By definition, the Scott $kG$-module $S(G, P)$ is a $p$-permutation $kG$-module.

By Green's indecomposability criterion, the following result holds.

**Lemma 6.** Let $H$ be a subgroup of $G$ such that $|G : H| = p^a$ (for some $a \geq 0$). Then $\text{Ind}_H^G k_H$ is indecomposable. In particular, we have that $S(G, H) \cong \text{Ind}_H^G$.

The following theorem gives us information of restrictions of Scott modules.

**Theorem 7 ([4, Theorem 1.7]).** Let $P$ be a $p$-subgroup of $G$. Let $Q$ be a maximal element of $P \cap G = \{^gP \cap H \mid g \in G\}$. Then $S(H, Q)$ is a direct summand of $\text{Res}_H^G S(G, P)$.

2.2. **Brauer quotients.** Let $M$ be a $kG$-module and $H$ a subgroup of $G$. We denote by $M^H$ the set of $H$-fixed elements in $M$. For subgroups $L$ of $H$, we denote by $\text{Tr}_L^H$ the trace map $\text{Tr}_L^H : M^L \to M^H$. Brauer quotients are defined as follows.

**Definition 8.** Let $M$ be a $kG$-module. For a $p$-subgroup $Q$ of $G$, the Brauer quotient of $M$ with respect to $Q$ is the $k$-vector space $M(Q) := M^Q / (\sum_{R<Q} \text{Tr}_R^Q(M^R))$.

This $k$-vector space has a natural structure of $kN_G(Q)$-module.

Brauer quotients have the following well-known properties.

**Proposition 9.** Let $P$ be a $p$-subgroup of $G$ and $M = S(G, P)$. Then $M(P) \cong S(N_G(P), P)$.

**Proposition 10.** Let $M$ be an indecomposable $p$-permutation $kG$-module with vertex $P$. Let $Q$ be a $p$-subgroup of $G$. Then $Q \leq_G P$ if and only if $M(Q) \neq 0$. 

2.3. Fusion systems. For subgroups $Q$, $R$ of $G$, we denote by $\text{Hom}_G(Q,R)$ the set of all group homomorphisms from $Q$ to $R$ which are induced by conjugation in $G$. For a $p$-subgroup $P$ of $G$, the fusion system $\mathcal{F}_P(G)$ of $G$ over $P$ is the category whose objects are the subgroups of $P$ and whose morphism set from $Q$ to $R$ is $\text{Hom}_G(Q,R)$. We refer the reader to [1] for background involving fusion systems.

**Definition 11.** Let $P$ be a $p$-subgroup of $G$

(i) A subgroup $Q$ of $P$ is said to be fully normalized in $\mathcal{F}_P(G)$ if $|N_P(\gamma Q)| \leq |N_P(Q)|$ for all $x \in G$ such that $xQ \leq P$.

(ii) A subgroup $Q$ of $P$ is said to be fully automated in $\mathcal{F}_P(G)$ if $p \nmid |N_G(Q) : N_P(Q)C_G(Q)|$.

(iii) A subgroup $Q$ of $P$ is said to be receptive in $\mathcal{F}_P(G)$ if it has the following property: for each $R \leq P$ and $\varphi \in \text{Iso}_{\mathcal{F}_P(G)}(R, Q)$, if we set

$$N_\varphi := \{g \in N_P(Q) | \exists h \in N_P(R), c_g \circ \varphi = \varphi \circ c_h\},$$

then there is $\overline{\varphi} \in \text{Hom}_{\mathcal{F}_P(G)}(N_\varphi, P)$ such that $\overline{\varphi}|_R = \varphi$.

Saturated fusion systems are defined as follows.

**Definition 12.** Let $P$ be a $p$-subgroup of $G$. The fusion system $\mathcal{F}_P(G)$ is saturated if the following two conditions are satisfied:

(i) $P$ is fully normalized in $\mathcal{F}_P(G)$.

(ii) For each subgroup $Q$ of $P$, if $Q$ is fully normalized in $\mathcal{F}_P(G)$, then $Q$ is receptive in $\mathcal{F}_P(G)$.

For example, if $P$ is a Sylow $p$-subgroup of $G$, then $\mathcal{F}_P(G)$ is saturated.

3. PROOF OF THEOREM 4

In this section, we give a proof of Theorem 4.

For a saturated fusion system $\mathcal{F}$ over $p$-group $P$ and a subgroup $Q$ of $P$, the normalizer fusion system $N_\mathcal{F}(Q)$ of $Q$ is defined and is a fusion system over $N_P(Q)$ (see [1, II, §2]). We note that if $\mathcal{F} = \mathcal{F}_P(G)$, then $N_\mathcal{F}(Q) = N_\mathcal{F}_P(N_G(Q))$.

**Proof of Theorem 4.** Suppose that (i) holds. Let $Q$ be a fully $\mathcal{F}$-normalized subgroup of $P$. Then $S(N_G(Q), N_P(Q))(Q)$ is indecomposable, and we have that

$$S(N_G(Q), N_P(Q))(Q) \cong S(N_G(Q), N_P(Q))(Q).$$

Therefore, $S(G, P)$ is Brauer indecomposable by Theorem 3.

Next, suppose that (ii) and $\mathcal{F} = \mathcal{F}_P(N_G(P))$ hold. Then any subgroup $Q$ of $P$ is fully $\mathcal{F}$-normalized. Let $Q$ be any subgroup of $P$. Then $\mathcal{F}_{N_p(Q)}(N_G(Q)) = N_\mathcal{F}(Q)$ is saturated by [1, II, Theorem 2.1]. Let $R$ be a fully $\mathcal{F}_P(N_G(Q))$-normalized subgroup of $N_P(Q)$. It is sufficient to show that $S(N_{N_G(Q)}(R), N_{N_P(Q)}(R))$ is indecomposable as $kC_{N_G(Q)}(R)$-module by Theorem 3.

Since $QR$ is fully $\mathcal{F}$-normalized, $S(N_G(QR), N_P(QR))$ is indecomposable as $kC_{N_G(QR)}$-module, and hence is also indecomposable as $kC_{N_G(Q)}(R)$-module. Therefore, it is sufficient to show that

$$\text{Res}_{N_{N_G(Q)}(R)}^{N_G(QR)} S(N_G(QR), N_P(QR)) \cong S(N_{N_G(Q)}(R), N_{N_P(Q)}(R)),$$
and if we show that $N_{N_{P}(Q)}(R)$ is a maximal element of $N_{P}(QR) \cap_{N_{G}(QR)} N_{N_{G}(Q)}(R)$, then the isomorphism holds by Theorem 7 and the indecomposability of $S(N_{G}(QR), N_{P}(QR))$ as a $N_{N_{G}(Q)}(R)$-module.

Let $g$ be an element of $N_{G}(QR)$ such that $N_{N_{P}(Q)}(R) \leq N_{P}(QR) \cap_{N_{G}(QR)} N_{N_{G}(Q)}(R)$. Then we have $Q^{g} \leq (QR)^{g} = QR \leq P$ and hence there is $h \in N_{G}(P)$ such that $gh^{-1} \in C_{G}(Q)$ since $\mathcal{F} = \mathcal{F}_{P}(N_{G}(P))$. We have that

\[
|N_{N_{P}(Q)}(R)| \leq |N_{P}(QR) \cap N_{N_{G}(Q)}(R)|
= |N_{P} \cap N_{G}(QR) \cap N_{G}(Q) \cap N_{G}(R)|
= |P \cap N_{G}(Q) \cap N_{G}(R)|
= |N_{N_{P}(Q)}(R)|
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= |N_{N_{P}(QR)}(R)|.
\]

On the other hand, since

\[
R^{gh^{-1}} \leq N_{N_{P}(Q)}(R)^{gh^{-1}}
\leq (N_{P}(QR) \cap N_{N_{G}(Q)}(R))^{gh^{-1}}
\leq (P \cap N_{G}(Q))^{gh^{-1}}
= P^{h^{-1}} \cap N_{G}(Q^{gh^{-1}})
= P \cap N_{G}(Q)
= N_{P}(Q)
\]

and $gh^{-1} \in C_{G}(Q) \leq N_{G}(Q)$, the conjugation map $(\cdot)^{gh^{-1}} : R \to R^{gh^{-1}}$ is an isomorphism in $N_{F}(Q)$. Since $R$ is fully $N_{F}(Q)$-normalized, we have that $|N_{N_{P}(Q)}(R)^{gh^{-1}}| \leq |N_{N_{P}(Q)}(R)|$. Therefore, $N_{N_{P}(Q)}(R) = N_{P}(QR) \cap N_{N_{G}(Q)}(R)$, and $N_{N_{P}(Q)}(R)$ is maximal in $N_{P}(QR) \cap_{N_{G}(QR)} N_{N_{G}(Q)}(R)$, as desired. \(\square\)

REFERENCES


DEPARTMENT OF MATHEMATICS
TOKYO UNIVERSITY OF SCIENCE
1-3 KAGURAZAKA, SHINJUKU-KU, TOKYO 162-8601
JAPAN

E-mail address: 114701@ed.tus.ac.jp