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Characters of finite permutation groups and Krein parameters

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1 Introduction

A finite permutation group is called *multiplicity-free* if the permutation character is a sum of distinct irreducible characters. A transitive permutation group is called regular if every non-identity element is fixed point-free. For a transitive permutation group G on a finite set Ω, let $\Lambda_0, \Lambda_1, \ldots, \Lambda_d$ be the orbits of G on $\Omega \times \Omega$. Then $(\Omega, \{\Lambda_i\}_{i=0}^{d})$ is an association scheme and the adjacency algebra $\mathcal{A} = \langle A_0, A_1, \ldots, A_d\rangle_{\mathbb{C}}$ is isomorphic to $\mathrm{End}_G(\mathbb{C}[\Omega])$, where $\mathbb{C}[\Omega]$ is the permutation module of G on Ω.

If G is multiplicity-free, then $\mathcal{A}$ is commutative and the permutation module $\mathbb{C}[\Omega]$ decomposes into $d+1$ non-isomorphic irreducible G-modules: $\mathbb{C}[\Omega] = V_0 \oplus V_1 \oplus \cdots \oplus V_d$. For $i = 0, 1, \ldots, d$, let E_i be the orthogonal projection from $\mathbb{C}[\Omega]$ onto V_i. Then E_i are primitive idempotents of $\mathcal{A}$ and $\{E_0, E_1, \ldots, E_d\}$ is a basis of $\mathcal{A}$. Since $\mathcal{A}$ is closed under the Hadamard product $\circ$, there exist scalars $q_{i,j}^{k}$ such that

$$E_i \circ E_j = \frac{1}{|\Omega|} \sum_{k=0}^{d} q_{i,j}^{k} E_k.$$

These scalars are called *Krein parameters*. Let χ_i be the irreducible character corresponding to the irreducible G-module V_i. Scott's Theorem reveals the following relations between Krein parameters and inner products of characters: if $q_{i,j}^{k} \neq 0$, then $(\chi_i\chi_j, \chi_k) \neq 0$ (see Theorem 3.3). The converse of this implication is not always true. As a counterexample, we have the action of the symmetric group S_{2d} on all d-element subsets of a $2d$-element set (see [2, Section 2.8]).

In this paper, we prove the converse of the implication for transitive permutation groups of semidirect product type whose regular normal subgroup is abelian. If a permutation group G has a regular abelian normal subgroup, then G is multiplicity-free. Under this condition, we can describe all irreducible characters which appear in the permutation character. We find that the inner products of characters and Krein parameters are similar in terms of their components (see Theorem 5.4). In particular, the inner products and Krein parameters are equal for some transitive permutation groups.

2 Commutative association schemes

Let X be a set with $|X| = n$ and $R_i \subset X \times X$ $(i = 0, 1, \dots, d)$. For R_i $(i = 0, 1, \dots, d)$, the i^{th}-*adjacency matrix* A_i *of* $\mathcal{X}$ is the square matrix indexed by X such that $(A_i)_{xy} = 1$ if $(x, y) \in R_i$, $(A_i)_{xy} = 0$ otherwise.

Definition 2.1. The pair $(X, \{R_i\}_{i=0}^d)$ is called a *commutative association scheme* if following hold:

(i) $A_0 = I_n$.

(ii) $J_n = \sum_{i=0}^d A_i$.

(iii) For any $i \in \{0, \dots, d\}$, There exists $i' \in \{0, 1, \dots, d\}$ such that $A_i^T = A_{i'}$.

(iv) There exist $p_{i,j}^k$ such that $A_i A_j = \sum_{k=0}^d p_{i,j}^k A_k$ for all $i, j \in \{0, \dots, d\}$.

(v) $A_i A_j = A_j A_i$ for all $i, j \in \{0, \dots, d\}$.

Let $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ be a commutative association scheme with $|X| = n$ and A_i $(i = 0, \dots, d)$ be the adjacency matrices of $\mathcal{X}$. $k_i = p_{i,i'}^0$ is called the *valency* of R_i. The subalgebra $\mathcal{A} = \langle A_0, A_1, \dots, A_d \rangle$ of the matrix algebra $M_n(\mathbb{C})$ is called the *adjacency algebra of* $\mathcal{X}$. The adjacency algebra $\mathcal{A}$ has $d + 1$ primitive idempotents $E_0, E_1, \dots, E_d$. The integers $m_i = \text{rank}(E_i)$ are called the *multiplicities of* $\mathcal{X}$.

Since both of $\{A_0, A_1, \dots, A_d\}$ and $\{E_0, E_1, \dots, E_d\}$ are bases of $\mathcal{A}$, let

$$A_i = \sum_{j=0}^{r} p_i(j) E_j,$$

$$E_i = \frac{1}{n} \sum_{j=0}^{d} q_i(j) A_j$$

for $i = 0, 1, \dots, d$.

Since $\mathcal{A}$ is closed under the Hadamard product, we may define Krein parameters.

Definition 2.2. Let $E_0, E_1, \dots, E_d$ be the primitive idempotents of the adjacency algebra $\mathcal{A}$ of $\mathcal{X}$. Set

$$E_i \circ E_j = \frac{1}{n} \sum_{k=0}^{d} q_{i,j}^k E_k$$

for some $q_{i,j}^k \in \mathbb{C}$, where $\circ$ denotes the Hadamard product. The coefficients $q_{i,j}^k$ are called the *Krein parameters of* $\mathcal{X}$.

Lemma 2.3 ([2, Theorem 3.6 in Chapter II]). *For any* $i, j, k \in \{0, 1, \dots, d\}$,

$$q_{i,j}^k = \frac{m_i m_j}{n} \sum_{l=0}^{d} \frac{1}{k_l^2} p_l(i) p_l(j) \overline{p_l(k)}.$$

3 Scott's Theorem

Let G be a transitive permutation group on a finite set Ω. We may assume the permutation character θ of G decomposes into the sum of distinct irreducible characters: $\theta = \chi_0 + \chi_1 + \cdots + \chi_d$, where χ_i are distinct irreducible characters of G for $i \in \{0, 1, \ldots, d\}$. Note that, this condition is called *multiplicity-free*. Define the action of G on $\Omega \times \Omega$ by $(x, y)^g = (x^g, y^g)$ for $x, y \in \Omega$ and $g \in G$. Then $\mathcal{X} = (\Omega, \{\Lambda_i\}_{i=0}^d)$ is a commutative association scheme, where $\Lambda_0, \Lambda_1, \ldots, \Lambda_d$ are the orbits of G on $\Omega \times \Omega$ and $\Lambda_0 = \{(x, x) \mid x \in \Omega\}$ (see [2, Theorem 1.4 in Chapter II]).

Let $H \leq G$ be the stabilizer of a point $a \in \Omega$. Identifying $H\backslash G$ with Ω, we regard the action of H on Ω as that on $H\backslash G$ by multiplication of G. Then $a \in \Omega$, which H stabilizes, corresponds to $H \in H\backslash G$. Then the permutation character θ decompose into $\theta = \chi_0 + \chi_1 + \cdots + \chi_d$.

Lemma 3.1 ([2, Corollary 11.7 in Chapter II]). *Let $P = (p_j(i))$ be the first eigenmatrix.*

$$p_{i'}(j) = \frac{1}{|H|} \sum_{x \in Ha_iH} \chi_j(x),$$

where $\{a_0, a_1, \ldots, a_d\}$ is the representatives of $H\backslash G/H$.

Lemma 3.2 ([3, Section 3.4]). *Let ρ be the permutation representation of G. Set*

$$E_i = \frac{\deg(\chi_i)}{|G|} \sum_{g \in G} \overline{\chi_i(g)} \rho(g)$$

for every $i = 0, 1, \ldots, d$. Then each E_i is the orthogonal projection onto the irreducible G-module corresponding to χ_i. In other words, $E_0, E_1, \ldots, E_d$ are all primitive idempotents of the adjacency algebra $\mathcal{A}$ of the commutative association scheme $\mathcal{X}$.

Theorem 3.3 (Scott's Theorem, [2, Theorem 8.1 in Chapter II], [5, Theorem 3]). *Let G be a transitive permutation group on a finite set Ω and $\mathcal{X} = (\Omega, \{\Lambda_i\}_{i=0}^d)$ be the association scheme defined by the action of G on $\Omega \times \Omega$, where Λ_i are the orbits of G on $\Omega \times \Omega$. Assume the permutation character θ is multiplicity-free: $\theta = \chi_0 + \chi_1 + \cdots + \chi_d$, where χ_i are irreducible characters of G and $\chi_0 = \mathbf{1}_G$. Then, for any $i, j, k \in \{0, 1, \ldots, d\}$, if $q_{i,j}^k \neq 0$, then $(\chi_i\chi_j, \chi_k) \neq 0$.*

The detailed proof of this theorem is written in the references. In this paper, we mention a sketch of the proof. To prove this theorem, for $i, j, k \in \{0, 1, \ldots, d\}$, we construct $\sigma_{i,j}^k \in \mathrm{Hom}_G(V_i \otimes V_j, V_k)$ such that $\sigma_{i,j}^k = 0$ if and only if $q_{i,j}^k = 0$, where V_l is an irreducible G-module with its irreducible character χ_l for $i \in \{0, 1, \ldots, d\}$. If $q_{i,j}^k \neq 0$, then $\sigma_{i,j}^k \neq 0$ and, by Schur's lemma, it implies that $V_i \otimes V_j$ contains the irreducible G-module which is isomorphic to V_k.

To investigate the converse of Theorem 3.3, we suppose $q_{i,j}^k = 0$. Then, we obtain $\sigma_{i,j}^k = 0$ by $\sigma_{i,j}^k = 0$ if and only if $q_{i,j}^k = 0$. This does not always imply $(\chi_i\chi_j, \chi_k) = 0$. Indeed, the symmetric group S_{2d} on the Johnson Scheme $J(2d, d)$ is an example (see [2, Section 2.8]).

4 Transitive permutation groups of semidirect product type

Let G be a transitive permutation group on a finite set Ω. We assume that $G = H \ltimes N$, where H is the stabilizer of a point $x \in \Omega$, N is a regular abelian permutation group and $\ltimes$ denotes the semidirect product. I.e.

- $\{1\} \times N$ is a normal subgroup of G, where $1 \in H$ is the identity element of H,
- For any $g \in G$, there exist unique $h \in H$ and unique $n \in N$ such that $g = (h, n)$,
- The multiplication of G is defined as $(h_1, n_1)(h_2, n_2) = (h_1h_2, n_1^{h_2} + n_2)$.

In this case, we may identify N with Ω, where the identity element $0 \in N$ corresponds to the fixed point $x \in \Omega$.

When we regard N as a normal subgroup of G, we should write $\{1\} \times N$ instead of N. However we will identify $\{1\} \times N$ with N for brevity. Similarly, we will identify $H \times \{0\}$ with H.

By the definition of the semidirect product, it is clear that $H \backslash G = \{H(1, n) \mid n \in N\}$.

Let $\mathrm{Irr}(N)$ be the set of all irreducible characters of N. Since N is abelian, irreducible characters of N are also irreducible representations of N. Define an action of H on $\mathrm{Irr}(N)$ by

$$\varphi^h(n) = \varphi(n^{h^{-1}})$$

for $h \in H$, $\varphi \in \mathrm{Irr}(N)$ and $n \in N$. Let

$$H_\varphi = \{h \in H \mid \varphi^h = \varphi\}$$

for $\varphi \in \mathrm{Irr}(N)$. This means that H_φ is the stabilizer of $\varphi \in \mathrm{Irr}(N)$ and $H_\varphi \leq H$. The *extension of* $\varphi \in \mathrm{Irr}(N)$ *to* $H_\varphi \ltimes N$, denoted by $\mathrm{ex}(\varphi)$, is defined as

$$\mathrm{ex}(\varphi)(h, n) = \varphi(n)$$

for $h \in H_\varphi$ and $n \in N$ (see [6, Chapter 17]).

Lemma 4.1. *For each* $\varphi \in \mathrm{Irr}(N)$, $\mathrm{ex}(\varphi)$ *is an irreducible representation of* $H_\varphi \ltimes N$.

Proof. Since

$$\begin{aligned}
\mathrm{ex}(\varphi)((h, n)(h', n')) &= \mathrm{ex}(\varphi)(hh', n^{h'} + n') \\
&= \varphi(n^{h'} + n') \\
&= \varphi^{h'^{-1}}(n)\varphi(n') \\
&= \varphi(n)\varphi(n') \\
&= \mathrm{ex}(\varphi)(h, n)\mathrm{ex}(\varphi)(h', n')
\end{aligned}$$

for $h, h' \in H_\varphi$ and $n, n' \in N$, $\mathrm{ex}(\varphi)$ is a representation of $H_\varphi \ltimes N$. By the definition of the extension, $\deg(\mathrm{ex}(\varphi)) = \deg(\varphi)$, and $\mathrm{ex}(\varphi)$ is irreducible. $\square$

Lemma 4.2. *For $\varphi \in \mathrm{Irr}(N), h \in H$ and $n \in N$,*

$$\mathrm{Ind}_{H_\varphi \ltimes N}^{G} \mathrm{ex}(\varphi)(h,n) = \sum_{\substack{\psi \in \varphi^H \\ \psi^h = \psi}} \psi(n).$$

Proof. For $h \in H, n \in N$,

$$\mathrm{Ind}_{H_\varphi \ltimes N}^{G} \mathrm{ex}(\varphi)(h,n) = \sum_{l \in H/H_\varphi} \mathrm{ex}(\varphi)(l^{-1}hl, n^l) = \sum_{\substack{l \in H/H_\varphi \\ l^{-1}hl \in H_\varphi}} \varphi(n^l) = \sum_{\substack{\psi \in \varphi^H \\ \psi^h = \psi}} \psi(n).$$

Since there is a bijection from H/H_φ to $G/(H_\varphi \ltimes N)$ defined by $lH_\varphi \mapsto (l,0)H_\varphi \ltimes N$, we can choose $l \in H/H_\varphi$ instead of $G/(H_\varphi \ltimes N)$. $\square$

Lemma 4.3.

(i) Induced characters $\mathrm{Ind}_{H_\varphi \ltimes N}^{G} \mathrm{ex}(\varphi)$ are irreducible characters of G for all $\varphi \in \mathrm{Irr}(N)$.

(ii) For $\varphi, \varphi' \in \mathrm{Irr}(N)$, $\mathrm{Ind}_{H_\varphi \ltimes N}^{G} \mathrm{ex}(\varphi) = \mathrm{Ind}_{H_{\varphi'} \ltimes N}^{G} \mathrm{ex}(\varphi')$ if and only if there exists $h \in H$ such that $\varphi^h = \varphi'$.

Proof. By Lemma 4.2,

$$\begin{aligned}
(\mathrm{Ind}_{H_\varphi \ltimes N}^{G} \mathrm{ex}(\varphi), \mathrm{Ind}_{H_{\varphi'} \ltimes N}^{G} \mathrm{ex}(\varphi'))_G &= \frac{1}{|H||N|} \sum_{h \in H} \sum_{n \in N} \left(\sum_{\substack{\psi \in \varphi^H \\ \psi^h = \psi}} \psi(n) \right) \left(\sum_{\substack{\psi' \in \varphi'^H \\ \psi'^h = \psi'}} \overline{\psi'(n)} \right) \\
&= \frac{1}{|H|} \sum_{\psi \in \varphi^H} \sum_{\psi' \in \varphi'^H} |H_\psi \cap H_{\psi'}| \left(\frac{1}{|N|} \sum_{n \in N} \psi(n)\overline{\psi'(n)} \right) \\
&= \frac{1}{|H|} \sum_{\psi \in \varphi^H} \sum_{\psi' \in \varphi'^H} |H_\psi \cap H_{\psi'}| (\psi, \psi')_N \\
&= \frac{1}{|H|} \delta_{\varphi^H, \varphi'^H} \sum_{\psi, \psi' \in \varphi^H} |H_\psi \cap H_{\psi'}| \delta_{\psi, \psi'} \\
&= \delta_{\varphi^H, \varphi'^H}.
\end{aligned}$$

$\square$

Remark that Lemma 4.3(i) is also given in [4, (6.11)Theorem].

Let $\varphi_0, \varphi_1, \ldots, \varphi_d$ be representatives of the orbit partition of H on $\mathrm{Irr}(N)$, where $\varphi_0 = \mathbf{1}_N$. In other words,

$$\mathrm{Irr}(N) = \coprod_{i=0}^{d} \varphi_i^H.$$

Lemma 4.4. *Let $\mathbf{1}_H^G$ be the permutation character of G. Then*

$$\mathrm{Ind}_H^G \mathbf{1}_H = \sum_{i=0}^{d} \mathrm{Ind}_{H_{\varphi_i} \ltimes N}^G \mathrm{ex}(\varphi_i).$$

Proof. It suffices to prove $(\mathrm{Ind}_H^G \mathbf{1}_H, \mathrm{Ind}_{H_{\varphi_i} \ltimes N}^G \mathrm{ex}(\varphi_i))_G = 1$ and $\sum_{i=0}^d \deg(\mathrm{Ind}_{H_\varphi \ltimes N}^G \mathrm{ex}(\varphi_i)) = |G/H|$. Since N is abelian, $\deg(\psi) = 1$ for any $\psi \in \mathrm{Irr}(N)$. By Lemma 4.2 and Frobenius reciprocity theorem (see [1, Theorem 2 in Section 6.16]),

$$\begin{aligned}
(\mathrm{Ind}_H^G \mathbf{1}_H, \mathrm{Ind}_{H_{\varphi_i} \ltimes N}^G \mathrm{ex}(\varphi_i))_G &= (\mathbf{1}_H, \mathrm{Res}_H^G \mathrm{Ind}_{H_{\varphi_i} \ltimes N}^G \mathrm{ex}(\varphi_i))_H \\
&= \frac{1}{|H|} \sum_{h \in H} \sum_{\substack{\psi \in \varphi^H \\ \psi^h = \psi}} \psi(0) = \frac{1}{|H|} \sum_{\psi \in \varphi_i^H} |H_\psi| \deg(\psi) = 1.
\end{aligned}$$

Moreover,

$$\sum_{i=0}^{d} \deg(\mathrm{Ind}_{H_{\varphi_i} \ltimes N}^G \mathrm{ex}(\varphi_i)) = \sum_{i=0}^{d} \frac{|H|}{|H_{\varphi_i}|} \deg(\mathrm{ex}(\varphi_i)) = \sum_{i=0}^{d} \frac{|H|}{|H_{\varphi_i}|} = \sum_{i=0}^{d} |\varphi_i^H| = |N|. \qquad \square$$

5 The converse of Scott's Theorem

In this section, we assume that a transitive permutation group $G = H \ltimes N$ on Ω is same as that of the previous section. By Lemma 4.4, $\mathbf{1}_H^G$ is multiplicity-free: $\mathbf{1}_H^G = \chi_0 + \chi_1 + \cdots + \chi_d$, where $\chi_i = \mathrm{ex}(\varphi_i)^G$. Then we can construct the association scheme $\mathcal{X} = (\Omega, \{\Lambda_i\}_{i=0}^d)$, where Λ_i are orbits of G on $\Omega \times \Omega$ and $\Lambda_0 = \{(x,x) \mid x \in \Omega\}$. Let $\{n_0, n_1, \dots, n_d\}$ be a set of representatives of $H \backslash G / H$. We can assume $n_i \in N$ for all $i \in \{0, 1, \dots, d\}$ and $n_0 = 0$, i.e.

$$H \backslash G / H = \coprod H \times n_i^H.$$

Then the association scheme $\mathcal{X}$ has valencies $k_i = |H| / |\mathrm{Stab}_H(n_i)|$ and multiplicities $m_i = |H| / |H_{\varphi_i}|$.

Lemma 5.1. *For any $i, j, k \in \{0, 1, \dots, d\}$,*

$$q_{i,j}^k = \frac{|H_{\varphi_k}||R_{i,j}^k|}{|H|},$$

where

$$R_{i,j}^k = \left\{ (\psi_1, \psi_2, \psi_3) \in \varphi_i^H \times \varphi_j^H \times \varphi_k^H \mid \psi_1 \psi_2 = \psi_3 \right\}.$$

Proof. By Lemma 3.1 and Lemma 4.2,

$$
\begin{aligned}
p_{s'}(i) &= \frac{1}{|H|} \sum_{g \in H \times n_s^H} \operatorname{Ind}_{H_{\varphi_\imath} \ltimes N}^{G} \operatorname{ex}(\varphi_\imath)(g) \\
&= \frac{1}{|H|} \frac{1}{|\operatorname{Stab}_H(n_s)|} \sum_{h_1, h_2 \in H} \operatorname{Ind}_{H_{\varphi_\imath} \ltimes N}^{G} \operatorname{ex}(\varphi_i)(h_1, n_s^{h_2}) \\
&= \frac{1}{|H||\operatorname{Stab}_H(n_s)|} \sum_{h_1, h_2 \in H} \sum_{\substack{\psi \in \varphi_i^H \\ \psi^{h_1} = \psi}} \psi(n_s^{h_2}) \\
&= \frac{|H_{\varphi_\imath}||H|}{|H||\operatorname{Stab}_H(n_s)|} \sum_{\psi \in \varphi_i^H} \psi(n_s) \\
&= \frac{|H_{\varphi_\imath}|}{|\operatorname{Stab}_H(n_s)|} \sum_{\psi \in \varphi_\imath^H} \psi(n_s).
\end{aligned}
$$

Remark that $|H_{\varphi_\imath}| = |H_\psi|$ holds for all $\psi \in \varphi_i^H$.

Since $k_\imath = |H|/|\operatorname{Stab}_H(n_i)|$ and $m_i = |H|/|H_{\varphi_\imath}|$, by Lemma 2.3,

$$
\begin{aligned}
q_{i,j}^{k} &= \frac{m_\imath m_\jmath}{|X|} \sum_{s=0}^{d} \frac{1}{k_s^2} p_{s'}(i) p_{s'}(j) \overline{p_{s'}(k)} \\
&= \frac{|H|^2}{|H_{\varphi_\imath}||H_{\varphi_\jmath}||N|} \sum_{s=0}^{d} \frac{|\operatorname{Stab}_H(n_s)|^2}{|H|^2} \left(\frac{|H_{\varphi_\imath}|}{|\operatorname{Stab}_H(n_s)|} \sum_{\psi_1 \in \varphi_i^H} \psi_1(n_s) \right) \\
&\qquad \left(\frac{|H_{\varphi_\jmath}|}{|\operatorname{Stab}_H(n_s)|} \sum_{\psi_2 \in \varphi_\jmath^H} \psi_2(n_s) \right) \left(\frac{|H_{\varphi_k}|}{|\operatorname{Stab}_H(n_s)|} \sum_{\psi_3 \in \varphi_k^H} \overline{\psi_3(n_s)} \right) \\
&= \frac{|H_{\varphi_k}|}{|N|} \sum_{s=0}^{d} \frac{1}{|\operatorname{Stab}_H(n_s)|} \sum_{\psi_1 \in \varphi_i^H} \sum_{\psi_2 \in \varphi_j^H} \sum_{\psi_3 \in \varphi_k^H} \psi_1(n_s) \psi_2(n_s) \overline{\psi(n_s)} \\
&= \frac{|H_{\varphi_k}|}{|H||N|} \sum_{s=0}^{d} \sum_{l \in H/\operatorname{Stab}_H(n_s)} \sum_{\psi_1 \in \varphi_i^H} \sum_{\psi_2 \in \varphi_j^H} \sum_{\psi_3 \in \varphi_k^H} \psi_1(n_s) \psi_2(n_s) \overline{\psi_3(n_s)} \\
&= \frac{|H_{\varphi_k}|}{|H||N|} \sum_{s=0}^{d} \sum_{l \in H/\operatorname{Stab}_H(n_s)} \sum_{\psi_1 \in \varphi_i^H} \sum_{\psi_2 \in \varphi_j^H} \sum_{\psi_3 \in \varphi_k^H} \psi_1(n_s^{l^{-1}}) \psi_2(n_s^{l^{-1}}) \overline{\psi_3(n_s^{l^{-1}})} \\
&= \frac{|H_{\varphi_k}|}{|H|} \sum_{\psi_1 \in \varphi_i^H} \sum_{\psi_2 \in \varphi_j^H} \sum_{\psi_3 \in \varphi_k^H} \left(\frac{1}{|N|} \sum_{n \in N} \psi_1 \psi_2 \overline{\psi_3}(n) \right)
\end{aligned}
$$

$$= \frac{|H_{\varphi_k}|}{|H|}|R_{i,j}^k|.$$

□

Lemma 5.2. *For any $i, j, k \in \{0, 1, \ldots, d\}$,*

$$(\chi_i\chi_j, \chi_k)_G = \frac{1}{|H|} \sum_{(\psi_1,\psi_2,\psi_3)\in R_{i,j}^k} |H_{\psi_1} \cap H_{\psi_2} \cap H_{\psi_3}|.$$

Proof. By Lemma 4.2,

$$\begin{aligned}
(\chi_i\chi_j, \chi_k)_G &= (\mathrm{Ind}_{H_{\varphi_i}\ltimes N}^G \mathrm{ex}(\varphi_i)\mathrm{Ind}_{H_{\varphi_j}\ltimes N}^G \mathrm{ex}(\varphi_j), \mathrm{Ind}_{H_{\varphi_k}\ltimes N}^G \mathrm{ex}(\varphi_k))_G \\
&= \frac{1}{|G|}\sum_{h\in H}\sum_{n\in N}\left(\sum_{\substack{\psi_1\in\varphi_i^H\\ \psi_1{}^h=\psi_1}}\psi_1(n)\right)\left(\sum_{\substack{\psi_2\in\varphi_j^H\\ \psi_2{}^h=\psi_2}}\psi_2(n)\right)\left(\sum_{\substack{\psi_3\in\varphi_k^H\\ \psi_3{}^h=\psi_3}}\overline{\psi_3(n)}\right) \\
&= \frac{1}{|G|}\sum_{\psi_1\in\varphi_i^H}\sum_{\psi_2\in\varphi_j^H}\sum_{\psi_3\in\varphi_k^H}|H_{\psi_1}\cap H_{\psi_2}\cap H_{\psi_3}|\sum_{n\in N}\psi_1\psi_2\overline{\psi_3}(n) \\
&= \frac{1}{|H|}\sum_{\psi_1\in\varphi_i^H}\sum_{\psi_2\in\varphi_j^H}\sum_{\psi_3\in\varphi_k^H}\left(\frac{1}{|N|}\sum_{n\in N}\psi_1\psi_2\overline{\psi_3}(n)\right)|H_{\psi_1}\cap H_{\psi_2}\cap H_{\psi_3}| \\
&= \frac{1}{|H|}\sum_{(\psi_1,\psi_2,\psi_3)\in R_{i,j}^k}|H_{\psi_1}\cap H_{\psi_2}\cap H_{\psi_3}|.
\end{aligned}$$

□

Lemma 5.3 ([4, (6.32)Theorem]). *Let G be a group and H be a group which acts on* $\mathrm{Irr}(G)$ *and the set of conjugacy classes of G, where* $\mathrm{Irr}(G)$ *is the set of irreducible characters of G. If $\chi^h(g^h) = \chi(g)$ for all $g \in G, h \in H, \chi \in \mathrm{Irr}(G)$, then the number of irreducible characters which h fixes is equal to the number of conjugacy classes which h fixes for any $h \in H$.*

Theorem 5.4. *Let $G = H \ltimes N$ be a transitive permutation group on Ω, H be the stabilizer of a point $x \in \Omega$ and N be an abelian normal subgroup. Then the permutation character is multiplicity-free:* $\mathrm{Ind}_H^G 1_H = \chi_0 + \chi_1 + \cdots + \chi_d$, *where χ_0 is the identity character. Moreover, the following hold.*

(i) For any $i, j, k \in \{0, 1, \ldots, d\}$, if $(\chi_i\chi_j, \chi_k)_G \neq 0$, then $q_{i,j}^k \neq 0$.

(ii) If any $g \in G$ $(g \neq 1_G)$ fixes less than 3 points in Ω, then

$$q_{i,j}^k = |H_{\varphi_k}|(\chi_i\chi_j, \chi_k)_G$$

for distinct $i, j, k \in \{0, 1, \ldots, d\}$.

Proof. (i) By Lemma 5.1, if $q^k_{i,j} = 0$, then $R^k_{i,j} = \emptyset$. By Lemma 5.2,

$$(\chi_i\chi_j, \chi_k)_G = \frac{1}{|H|} \sum_{(\psi_1,\psi_2,\psi_3)\in R^k_{i,j}} |H_{\psi_1} \cap H_{\psi_2} \cap H_{\psi_3}| = 0.$$

This means that we proved the contrapositive of the assertion.

(ii) By Lemma 5.3, if any $g \in G$ ($g \neq 1$) fixes less than 3 points in Ω, then $|H_{\psi_1} \cap H_{\psi_2} \cap H_{\psi_3}| = 1$ for distinct $\psi_1, \psi_2, \psi_3 \in \mathrm{Irr}(N)$. Thus we obtain the result. $\square$

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