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An age-structured epidemic model for the demographic transition*

Hisashi Inaba
Graduate School of Mathematical Science, The University of Tokyo
稲葉 寿 †
東京大学大学院数理科学研究科

Abstract

In this paper, we formulate an age-structured epidemic model for the demographic transition in which we assume that the cultural norms leading to lower fertility are transmitted amongst individuals in the same way as infectious diseases. First, we formulate the basic model as an abstract homogeneous Cauchy problem on a Banach space to prove the existence, uniqueness, and well-posedness of solutions. Next based on the normalization arguments, we investigate the existence of nontrivial exponential solutions and then study the linearized stability at the exponential solutions using the idea of asynchronous exponential growth. For the boundary exponential solutions, we formulate the stability condition using reproduction numbers. We show that bi-unstability of boundary exponential solutions is one of conditions which guarantee the existence of coexistent exponential solutions.

1 Introduction

During the 18^{th} and 19^{th} centuries, the mortality rate in industrialized countries declined along with economic progress and industrial development. The birth rate began to decline somewhat later, and in the 20^{th} century these countries experienced low mortality and low fertility rates to an unprecedented extent, while the growth rates of their populations also sharply declined. The change from "high-birth and high-death" to "high-birth and low-death" and then to "low-birth and low-death" is called the *demographic transition*. Since World War II, the demographic transition has been observed even in developing countries.

The well-known modernization hypothesis insists that the low fertility rate is a result of individual adaptation to general modernized environments, as industrialization, urbanization, educational standards, and families change. On the other hand, diffusion theory for the demographic transition assumes that innovative cultural norms that lower the number of births could be transmitted from individuals with low fertility (infecteds) to traditional individuals with high fertility (susceptibles). In this study, we develop an age-structured epidemic model to explain these demographic transition dynamics based on the diffusion theory ([5], [6]).

Here, to illustrate the basic aspects of the transmission dynamics in a growing or shrinking population, let us consider a simple age-independent case. Let $S(t)$ be the density of individuals with high fertility, i.e. susceptible people, at time t, and let $I(t)$ be the density of individuals with low fertility, i.e. infected people, at time t. The basic unstructured system for the demographic transition is described by a homogeneous Lotka–Volterra system:

$$\begin{aligned} \frac{dS(t)}{dt} &= \lambda_1 S(t) - \frac{\beta S(t)I(t)}{S(t)+I(t)}, \\ \frac{dI(t)}{dt} &= \lambda_2 I(t) + \frac{\beta S(t)I(t)}{S(t)+I(t)}, \end{aligned} \tag{1.1}$$

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†inaba@ms.u-tokyo.ac.jp

where λ_1 and λ_2 are the Malthusian parameters for susceptible and infected people respectively such that $\lambda_1 > \lambda_2$, and $\beta > 0$ is the transmission coefficient. The sign of λ_j does not matter in the following argument. We are assuming that the force of infection is given by the homogeneous law $\beta S/N$ with $N = S + I$, and that the difference between the Malthusian parameters reflects the difference in the crude birth rates (that is, we neglect differential mortality)[1].

For the homogeneous dynamical system, if p^* is an equilibrium solution, then so is cp^* for any $c > 0$. Then there is no possibility of an attracting nontrivial equilibrium except for the origin, which is very different from the standard nonlinear system. Then our interest focuses on the existence and stability of persistent (exponential) solutions, which play a role as stationary solutions in nonhomogeneous nonlinear dynamical systems.

As we see below, there are essentially two cases for all positive initial data:

- if $\beta < \lambda_1 - \lambda_2$, then the susceptible population grows asymptotically like $e^{\lambda_1 t}$, the infected population grows or decays asymptotically like $e^{(\beta+\lambda_2)t}$, so the infected fraction tends to 0 (the demographic transition does not occur and the reverse transition occurs if the susceptibles invade into the infected population);
- if $\beta > \lambda_1 - \lambda_2$, then the susceptible population grows or decays asymptotically like $e^{(\lambda_1-\beta)t}$, the infected population grows or decays asymptotically like $e^{\lambda_2 t}$, so the infected fraction tends to 1 (the demographic transition occurs, and the reverse transition does not occur).

In this studies, we extend the above observations to an age-structured model, which allows more complex behavior. In particular, there is a non-degenerate third case where both S and I are positive, and grow exponentially with the same Malthusian parameter $\lambda \in (\lambda_2, \lambda_1)$, which may be called the *coexistent exponential solution*. For two trivial exponential solutions, there is no bi-unstable case for the age-independent model. However, it is not the case for the age-dependent model, we can show that the bi-unstability of the boundary exponential solutions (where S or I is zero) is one of conditions which guarantee the existence of coexistent exponential solutions, and in fact, by numerical simulations, we can prove that the bi-unstable case is possible.

2 Age-structured model

Consider now the following age-dependent epidemic model for the demographic transition:

$$\begin{aligned} \frac{\partial p_1(t,a)}{\partial t} + \frac{\partial p_2(t,a)}{\partial a} &= -(\mu(a) + \pi(t,a))p_1(t,a), \\ \frac{\partial p_2(t,a)}{\partial t} + \frac{\partial p_2(t,a)}{\partial a} &= \pi(t,a)p_1(t,a) - \mu(a)p_2(t,a), \\ p_1(t,0) = \int_0^\infty m_1(a)\, p_1(t,a)\, da, &\quad p_2(t,0) = \int_0^\infty m_2(a)\, p_2(t,a)\, da, \end{aligned} \tag{2.1}$$

where $m_1(a)$ is the age-specific birth rate of the susceptible population (population with high fertility), $m_2(a)$ is the age-specific birth rate of the infected population (population with low fertility), and $\mu(a)$ is the common age-specific death rate. We assume that the force of "cultural" infection $\pi(t,a)$ is given by the homogeneous law

$$\pi(t,a) := \frac{1}{N(t)} \int_0^\infty \beta(a,\sigma)\, p_2(t,\sigma)\, d\sigma,$$

where $N(t) := \int_0^\infty n(t,a)\, da$ is the total size of the population, $n(t,a) := p_1(t,a) + p_2(t,a)$ is the age density of the host population, and $\beta(a,\sigma)$ is the transmission coefficient between susceptible individuals at age a and infected individuals at age σ. Let $\ell(a)$ be the survival probability, defined by $\ell(a) = \exp\left(-\int_0^a \mu(\sigma)d\sigma\right)$

[1]The natural death rate of each population is given by a common value $\mu > 0$ and $\lambda_j = m_j - \mu$ with $m_1 > m_2$, where m_j is the birth rate of j-th population.

and let $\Phi_j(a) := m_j(a)\ell(a)$ be the net reproduction function (or the net maternity function) of the j-th population. We assume that $m_1(a) > m_2(a)$, so that $\mathcal{R}_1 > \mathcal{R}_2$ where $\mathcal{R}_j := \int_0^\infty m_j(a)\ell(a)da$ is the demographic reproduction number (net reproduction rate) for the j-th population. The intrinsic growth rate λ_j $(j = 1, 2)$ is defined as the real dominant root of Lotka's characteristic equation [?] [7]:

$$\int_0^\infty e^{-\lambda_j a}\Phi_j(a)da = 1.$$

Therefore, $\lambda_1 > \lambda_2$. Moreover, we adopt the following technical assumption:

Assumption 1. *1. $\lambda_1 > \lambda_2 > -\underline{\mu}$, where $\underline{\mu} := \inf_{a\in\mathbf{R}_+}\mu(a) > 0$.*

2. Assume that $m_j \in L^1_+(\mathbf{R}_+) \cap L^\infty_+(\mathbf{R}_+)$, $\mu \in L^\infty_+(\mathbf{R}_+)$ and that there exist $\beta_1 \in L^1_+(\mathbf{R}_+) \cap L^\infty_+(\mathbf{R}_+)$ and $\beta_2 \in L^\infty_+(\mathbf{R}_+)$ such that $\beta_1(a)\beta_2(\sigma) \le \beta(a,\sigma) \le \kappa\beta_1(a)\beta_2(\sigma)$ with $\kappa > 1$, and β_1 and β_2 are quasi-interior points of L^1_+. Moreover, we assume that

$$\lim_{h\to 0}\int_0^\infty |\beta(a+h,\sigma) - \beta(a,\sigma)|da = 0 \text{ uniformly for } \sigma \in \mathbf{R}_+.$$

Let $p(t) = (p_1(t,\cdot), p_2(t,\cdot))^{\mathrm{T}}$. Then the basic system is considered to be an abstract homogeneous nonlinear dynamical system on $X := L^1(\mathbf{R}_+) \times L^1(\mathbf{R}_+)$:

$$\frac{dp}{dt} = Ap(t) + F(p(t)), \quad p(0) = p_0, \tag{2.2}$$

where A is the infinitesimal generator of a positive C_0-semigroup defined by

$$(A\phi)(a) = \begin{pmatrix} -\frac{d\phi_1}{da}(a) - \mu(a)\phi_1(a) \\ \\ -\frac{d\phi_2}{da}(a) - \mu(a)\phi_2(a) \end{pmatrix}$$

and having domain $\mathcal{D}(A) := \left\{\phi \in W^{1,1}(\mathbf{R}_+) : \phi_j(0) = \int_0^\infty m_j(a)\phi_j(a)da\right\}$ and $F : X_+ \to X$ is a nonlinear operator given by

$$F(\phi)(a) := \begin{pmatrix} -\pi(a|\phi)\phi_1(a) \\ \pi(a|\phi)\phi_1(a) \end{pmatrix},$$

where $\phi = (\phi_1, \phi_2) \in X_+$, $\|\phi\|_X := \int_0^\infty(|\phi_1(a)| + |\phi_2(a)|)da$ and

$$\pi(a|\phi) := \frac{1}{\|\phi\|_X}\int_0^\infty \beta(a,\sigma)\phi_2(\sigma)d\sigma,$$

denotes a nonlinear operator acting on X. Then F is homogeneous of degree one, that is, $F(\alpha\phi) = \alpha F(\phi)$ for $\alpha > 0$ and $F(0) = 0$.

Lemma 2.1. *The homogeneous operator F is globally Lipschitz continuous on X_+, and there exists $\epsilon > 0$ such that $(I + \epsilon F)(X_+) \subset X_+$.*

Proposition 1. *Let $p_0 \in X_+$. Then the Cauchy problem (2.2) has a unique mild solution $p(t) \in X_+$ that defines a semiflow $S(t)$ such that $p(t) = S(t)p_0$ and $S(t)(X_+) \subset X_+$.*

3 Normalized system and the stability of exponential solutions

The solution of (2.2) is called a persistent (exponential) solution if it has the form as $e^{\lambda^* t}w^*$, where λ^* is a constant and $w^* \in X$. Then it follows from the homogeneous nonlinearity that a (biologically meaningful) persistent solution exists if and only if the nonlinear eigenvalue problem $Aw^* + F(w^*) = \lambda^* w^*$ has a solution, which propose us a fixed point problem. For the homogeneous system, we are mainly interested in persistent (exponential) solutions, In the following, first we introduce the normalized system ([1], [3]). Then using the idea of asynchronous exponential growth (AEG) by Webb ([8], [9], [?]), we shall prove a simple criterion for the orbital stability of exponential solutions.

Let θ be the bounded linear positive functional from X_+ to $\mathbf{R}_+$ defined by

$$\langle \theta, z \rangle := \int_0^\infty (z_1(a) + z_2(a))da, \quad z = (z_1, z_2) \in X,$$

which gives the total population size if $z \in X_+$. So $\langle \theta, z(t) \rangle > 0$ if $z(0) \in X_+ \setminus \{0\}$. Let us introduce some new variables called the *age profile*:

$$w_1(t,a) = \frac{p_1(t,a)}{\langle \theta, p(t) \rangle}, \quad w_2(t,a) = \frac{p_2(t,a)}{\langle \theta, p(t) \rangle},$$

and

$$w(t) := (w_1(t,\cdot), w_2(t,\cdot)) \in \Gamma := \{\phi \in X_+ : \langle \theta, \phi \rangle = 1\},$$

where Γ is a state space of age profiles. Then we can replace the basic system (2.2) by the normalized system as follows

$$\frac{dw}{dt} = Aw + F(w) - \langle \theta, Aw + F(w) \rangle w, \quad w \in \Gamma, \tag{3.1}$$

The normalized system has a unique mild global solution $S(t)w_0$, and Γ is positively invariant with respect to the semiflow $S(t), t \geq 0$. If $w_0 \in \mathcal{D}(A)$, then $S(t)p_0$ becomes a classical solution. It follows from (3.1) that

$$\frac{d}{dt}\langle \theta, w \rangle = \langle \theta, (A+F)w(t) \rangle (1 - \langle \theta, w(t) \rangle).$$

Therefore,

$$\langle \theta, w(t) \rangle = 1 - (1 - \langle \theta, w(0) \rangle) e^{-\int_0^t \langle \theta, (A+F)w(s) \rangle ds},$$

from which we conclude that $S(t)(\Gamma) \subset \Gamma$ for all $t > 0$. If we use the solution $w(t)$ of the normalized system (3.1), the solution $p(t)$ of the original system is given by

$$p(t) = w(t) \exp\left(\int_0^t \langle \theta, (A+F)w(s) \rangle ds \right) \langle \theta, z(0) \rangle.$$

Therefore, the original problem has been reduced to a problem on the space of age profiles Γ.

For our case, notice that $\langle \theta, (A+F)w \rangle$ is a linear functional on X. In fact, if we introduce a linear functional H from $X_+ \to \mathbf{R}$ defined by

$$H(\phi) := \int_0^\infty [(m_1(a) - \mu(a))\phi_1(a) + (m_2(a) - \mu(a))\phi_2(a)]da, \quad \phi = (\phi_1, \phi_2) \in X_+.$$

and if w is a solution of the normalized system, then $H(w(t))$ gives the Malthusian parameter for the total population size:

$$H(w(t)) = \langle \theta, Aw + F(w) \rangle = \frac{1}{\langle \theta, p(t) \rangle} \frac{d\langle \theta, p(t) \rangle}{dt}.$$

Let us consider the problem of existence and stability of (exponentially growing) persistent solutions. Let $w^* \in \Gamma$ be a steady state of the normalized system. Then we have the following nonlinear eigenvalue problem:

$$\lambda^* w^* = Aw^* + F(w^*), \quad w^* \in \mathcal{D}(A) \cap \Gamma, \tag{3.2}$$

where $\lambda^* = \langle \theta, (A+F)w^* \rangle$. It is clear that $e^{\lambda^* t} w^*$ is a persistent solution of the original system. Conversely, if there exists an exponential solution $e^{\lambda^* t} z^*$ for the original homogeneous system (2.2), then λ^* and $w^* = z^* / \langle \theta, z^* \rangle$ must satisfy (3.2), and so w^* is a stationary state of the normalized system.

To demonstrate the linearized stability of the exponential solution, the Euler formula for the homogeneous system is crucial [9].

Proposition 2. *If F is Fréchet differentiable at $x \in X_+$, then the Euler formula $F'[x]x = F(x)$ holds, where $F'[x]$ denotes the Fréchet derivative at x. Moreover, $F'[x] = F'[cx]$ for any $c > 0$.*

Therefore, if F is Fréchet differentiable at a steady state w^* of the normalized system such that $(A + F)w^* = \lambda^* w^*$, then w^* is the positive eigenvector for the linearized operator $B := A + F'[w^*]$ associated with the eigenvalue λ^*, and B is independent of positive scalar multipliers for w^*.

Let $\zeta(t) := w(t) - w^* \in X_\theta := \{\zeta \in X : \langle \theta, \zeta \rangle = 0\}$. Then the normalized system can be rewritten as an equation on X_θ.

$$\begin{aligned} \frac{d\zeta(t)}{dt} &= B\zeta(t) - \langle \theta, B\zeta(t) \rangle w^* - \lambda^* \zeta(t) + \mathcal{G}(\zeta(t)) \\ &= C\zeta(t) + \mathcal{G}(\zeta(t)), \end{aligned} \tag{3.3}$$

where $C\zeta := B\zeta - \langle \theta, B\zeta \rangle w^* - \lambda^* \zeta$ is a linear operator and $\mathcal{G}(\zeta)$ is the second order term as $\|\mathcal{G}(\zeta)\|_X / \|\zeta\|_X \to 0$ when $\zeta \to 0$.

Lemma 3.1. *The subspace X_θ is invariant with respect to C, e^{tC} and $\mathcal{G}$. Then if $\zeta(0) \in X_\theta$, the mild solution of* (3.3) *is given as the solution of the integral equation in X_θ:*

$$\zeta(t) = e^{Ct}\zeta(0) + \int_0^t e^{C(t-s)} \mathcal{G}(\zeta(s)) ds. \tag{3.4}$$

Definition 3.2. *If for any $\epsilon > 0$, there exists $\delta > 0$ such that $\|\zeta(t)\|_X < \epsilon$ for all $t > 0$ when $\|\zeta(0)\|_X < \delta$, then w^* is called locally stable. If w^* is locally stable and $\lim_{t \to \infty} \|\zeta(t)\|_X = 0$ when $\|\zeta(0)\|_X < \delta'$ for some $\delta' > 0$, then w^* is called locally asymptotically stable, while w^* is called unstable if it is not stable. An exponential solution of the original system is called stable (in the sense of Hadeler [1], [2]) if the corresponding steady state of the normalized system is locally stable, while it is unstable if the corresponding steady state of the normalized system is unstable.*

Let C_θ be the *part* of C in X_θ, that is, $C_\theta = C$ on $\mathcal{D}(C_\theta) = \{\phi \in \mathcal{D}(C) \cap X_\theta : C\phi \in X_\theta\}$. Then C_θ is an infinitesimal generator of a strongly continuous semigroup e^{tC_θ} on X_θ. The linearized equation of (3.3) is $\zeta' = C_\theta \zeta$ on X_θ, so we can apply Proposition 4.13 of [7] to obtain the following stability result:

Proposition 3. *If $\omega_0(C_\theta) < 0$, then w^* is locally asymptotically stable. If there exists $\lambda_\dagger \in \sigma(C_\theta)$ such that $\Re \lambda_\dagger > 0$ and $\max\{\omega_1(C_\theta), \sup_{\lambda \in \sigma(C_\theta) \setminus (E\sigma(C_\theta) \cup \{\lambda_\dagger\})} \Re \lambda\} < \Re \lambda_\dagger$, then w^* is unstable.*

On the other hand, if we use the concept of *asynchronous exponential growth* (AEG) by Webb ([8], [9]), we can state a more simple stability condition.

Definition 3.3. *Let $T(t), t \geq 0$, be a strongly continuous semigroup of bounded linear operators in the Banach space X. Then $T(t)$, $t \geq 0$, has asynchronous exponential growth (AEG) with intrinsic growth constant $\lambda \in \mathbf{R}$ if there exists a nonzero rank-one operator P in X such that $\lim_{t \to \infty} e^{-\lambda t} T(t) = P$, where the limit is in the operator norm topology.*

Applying the result of [9], we obtain

Proposition 4. *Suppose that the linearized operator $B = A + F'[w^*]$ is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$, $t \geq 0$, and it has AEG such that $\lim_{t \to \infty} e^{-\lambda^* t} T(t) = P$, where P is a nonzero rank-one operator in X. Then there exists $\delta > 0$ such that, if $x \in U_\delta := \{x \in X_+ \backslash \{0\} : \|(I-P)x\|_X / \|Px\|_X < \delta\}$, then $Qx := \lim_{t \to \infty} e^{-\lambda^* t} p(t)$ exists, where $Qx \in \mathrm{Range}(P)$ and $Qx \neq 0$.*

Proposition 5. *An equilibrium $w^* \in \Gamma$ is locally asymptotically stable in the sense of Hadeler if $B = A + F'[w^*]$ has AEG with intrinsic growth constant λ^*.*

4 Existence of nontrivial exponential solutions

The basic system (2.1) has two boundary exponential solutions corresponding to boundary equilibrium points of the normalized system. In fact, if $p_2 = 0$, then there exists a trivial exponential solution such that $p_1^*(t,a) = e^{\lambda_1 t}w_1(a)$, $w_1(a) := (c_1(a), 0)$, where $c_1(a) := \frac{e^{-\lambda_1 a}\ell(a)}{\int_0^\infty e^{-\lambda_1 x}\ell(x)dx}$ is the stable age profile of the susceptible population with high fertility. On the other hand, if $p_1 = 0$, then there exists another trivial exponential solution such that $p_2^*(t,a) = e^{\lambda_2 t}w_2(a)$, $w_2(a) := (0, c_2(a))$, where $c_2(a) := \frac{e^{-\lambda_2 a}\ell(a)}{\int_0^\infty e^{-\lambda_2 x}\ell(x)dx}$ is the stable age profile of the infected population with low fertility.

Our fundamental problem is now to determine whether a positive steady state $(u^*, v^*) > 0$ exists for the normalized system (3.1). Let $w^* = (u^*, v^*) \in \Gamma$ be a positive steady state solution of (3.1). Then we have

$$\begin{aligned} \frac{du^*(a)}{da} &= -H^*u^*(a) - (\mu(a) + \pi^*(a))u^*(a), \\ \frac{dv^*(a)}{da} &= -H^*v^*(a) - \mu(a)v^*(a) + \pi^*(a)u^*(a), \\ u^*(0) &= \int_0^\infty m_1(a)u^*(a)da, \quad v^*(0) = \int_0^\infty m_2(a)v^*(a)da, \end{aligned}$$

where $\pi^*(a) := \int_0^\infty \beta(a,\sigma)v^*(\sigma)d\sigma$ and $H^* = H(w^*)$. Thus we have

$$\begin{aligned} u^*(a) &= u^*(0)\ell(a)e^{-H^*a - \int_0^a \pi^*(z)dz}, \\ v^*(a) &= v^*(0)\ell(a)e^{-H^*a} + e^{-H^*a}\ell(a)(1 - e^{-\int_0^a \pi^*(z)dz})u^*(0). \end{aligned} \tag{4.1}$$

From $u^*(a) + v^*(a) = (u^*(0) + v^*(0))\ell(a)e^{-H^*a}$ and the condition $|w^*|_1 = 1$, we have

$$(u^*(0) + v^*(0)) \int_0^\infty \ell(a)e^{-H^*a}da = 1. \tag{4.2}$$

Since $u^*(0) = \int_0^\infty m_1(a)u^*(a)da$, it follows from the boundary condition for u^* that

$$1 = \int_0^\infty m_1(a)\ell(a)e^{-H^*a - \int_0^a \pi^*(\sigma)d\sigma}da. \tag{4.3}$$

Because (4.3) has a unique real root H^* for a given π^*, we can define a continuous functional $\Psi : L^1 \to \mathbf{R}$ such that $H^* = \Psi(\pi^*)$ satisfies (4.3). That is, for any $\phi \in L^1_+(\mathbf{R}_+)$, it follows that

$$1 = \int_0^\infty m_1(a)\ell(a)e^{-\Psi(\phi)a - \int_0^a \phi(\sigma)d\sigma}da,$$

from which $\Psi(\phi) \le \lambda_1$ and $\Psi(0) = \lambda_1$.

From (4.1) and (4.2), we get

$$\begin{aligned} u^*(0) &= \frac{1}{\int_0^\infty \ell(a)e^{-\Psi(\pi^*)a}da} \frac{1 - \int_0^\infty m_2(a)\ell(a)e^{-\Psi(\pi^*)a}da}{1 - \int_0^\infty m_2(a)\ell(a)e^{-\Psi(\pi^*)a - \int_0^a \pi^*(z)dz}da}, \\ v^*(0) &= \frac{1}{\int_0^\infty \ell(a)e^{-\Psi(\pi^*)a}da} \frac{\int_0^\infty m_2(a)\ell(a)e^{-\Psi(\pi^*)a}(1 - e^{-\int_0^a \pi^*(z)dz})da}{1 - \int_0^\infty m_2(a)\ell(a)e^{-\Psi(\pi^*)a - \int_0^a \pi^*(z)dz}da}. \end{aligned}$$

Using (4.1) and the definition of π^*, we obtain a fixed-point equation for the unknown force of infection π^* as $\pi^*(a) = G(\pi^*)(a)$, where G is a nonlinear operator from $L^1(\mathbf{R}_+)$ into itself defined by

$$G(\phi)(a) := \int_0^\infty \beta(a,\sigma)b_0(\phi)\ell(\sigma)e^{-\Psi(\phi)\sigma}\left[g_2(\phi) + (1 - e^{-\int_0^\sigma \phi(z)dz})g_1(\phi)\right]d\sigma.$$

The functionals g_j $(j = 1, 2)$ from $L^1_+(\mathbf{R}_+)$ to $\mathbf{R}$ are defined by

$$g_1(\phi) := \frac{1 - \int_0^\infty m_2(a)\ell(a)e^{-\Psi(\phi)a}da}{1 - \int_0^\infty m_2(a)\ell(a)e^{-\Psi(\phi)a - \int_0^a \phi(z)dz}da},$$

$$g_2(\phi) := \frac{\int_0^\infty m_2(a)\ell(a)e^{-\Psi(\phi)a}(1 - e^{-\int_0^a \phi(z)dz})da}{1 - \int_0^\infty m_2(a)\ell(a)e^{-\Psi(\phi)a - \int_0^a \phi(z)dz}da},$$

with $\phi \in L^1_+$ and $b_0(\phi) := (\int_0^\infty \ell(a)e^{-\Psi(\phi)a}da)^{-1}$. Note that $g_1(\phi) + g_2(\phi) = 1$.

Lemma 4.1. *The operator G is nonnegative and defined for all $\phi \in L^1_+(\mathbf{R}_+)$. Moreover, it holds that $G(U) \subset M_0$, where $U := \{\phi \in L^1_+(\mathbf{R}) : \Psi(\phi) \geq \lambda_2\}$ and $M_0 := \{\phi \in L^1_+(\mathbf{R}) : |\phi|_\infty \leq |\beta|_\infty, |\phi|_1 \leq \kappa|\beta_1|_1|\beta_2|_\infty\}$.*

Let $G'[0]$ be the Fréchet derivative at the origin. Then, for $\phi \in L^1_+$,

$$(G'[0]\phi)(a) = b_0(0)\int_0^\infty \beta(a,\sigma)\ell(\sigma)e^{-\lambda_1\sigma}d\sigma \frac{\int_0^\infty m_2(a)\ell(a)e^{-\lambda_1 a}\int_0^a \phi(z)dzda}{1 - \int_0^\infty m_2(a)\ell(a)e^{-\lambda_1 a}da}$$
$$+ b_0(0)\int_0^\infty \beta(a,\sigma)\ell(\sigma)e^{-\lambda_1\sigma}\int_0^\sigma \phi(z)dzd\sigma,$$

where $b_0(0) = (\int_0^\infty \ell(a)e^{-\lambda_1 a}da)^{-1}$.

To show existence of a positive steady state, we apply a fixed point argument similar to that in Krasnoselskii ([4], Theorem 4.11).

Proposition 6. *Suppose that $\lambda_1 - \lambda_2 > |\beta|_\infty := \sup_{(a,\sigma)\in\mathbf{R}_+\times\mathbf{R}_+} |\beta(a,\sigma)|$ and $r(G'[0]) > 1$. Let*

$$M_\epsilon := \{\phi \in L^1_+ : |\phi|_1 \leq (1+\epsilon)C, |\phi|_\infty \leq (1+\epsilon)|\beta|_\infty\},$$

where $C := \kappa|\beta_1|_1|\beta_2|_\infty$ and $\epsilon > 0$ is chosen so that $\lambda_1 - \lambda_2 > (1+\epsilon)|\beta|_\infty$. Then G has at least one positive fixed point ϕ^* in M_0 such that $g_j(\phi^*) > 0$ for $j = 1, 2$.

As is shown in Proposition 8, the spectral radius of $G'[0]$ equals that of the next generation operator for the low-fertility population in the normalized system, so the assumption of Proposition 6 is not satisfied if all parameters are age-independent. However, we found numerical examples such that the assumption of 6 is satisfied and there exists a coexistent exponential growth orbit.

Once π^* is given as a fixed point of G with $g_j(\pi^*) > 0$, the corresponding positive stationary solution is given by $u^*(a) = g_1(\pi^*)\ell(a)e^{-\Psi(\pi^*)a - \int_0^a \pi^*(z)dz}$ and $v^*(a) = \ell(a)e^{-\Psi(\pi^*)a}[g_2(\pi^*) + g_1(\pi^*)(1 - e^{-\int_0^a \pi^*(z)dz})]$.

Proposition 7. *Let $w^* = (u^*, v^*) > 0$ be the positive steady state. Then $\lambda_2 < H^* = H(w^*) < \lambda_1$.*

5 Basic reproduction number

Suppose that a small number of infected individuals appear in the population p_1^*. Then the initial dynamics of the low-fertility population in the normalized system is described by the following linearized equation in the state space $X_2 := L^1(\mathbf{R}_+)$:

$$\frac{dv}{dt} = (A_2 - \lambda_1)v + Pv, \tag{5.1}$$

where A_2 is a standard population operator defined by $(A_2\phi)(a) := -\frac{d\phi(a)}{da} - \mu(a)\phi(a)$ with domain $D(A_2) = \{\phi \in W^{1,1}(\mathbf{R}_+) : \phi(0) = \int_0^\infty m_2(a)\phi(a)da\}$, and P is a bounded perturbation given by

$$(P\phi)(a) := c_1(a)\int_0^\infty \beta(a,\sigma)\phi(\sigma)d\sigma.$$

In other words, the perturbation $v(t,a)$ satisfies

$$\begin{aligned} \frac{\partial v}{\partial t}+\frac{\partial v}{\partial a} &= -(\lambda_1+\mu(a))v(t,a)+c_1(a)\int_0^\infty \beta(a,b)v(t,b)db, \\ v(t,0) &= \int_0^\infty m_2(a)v(t,a)da. \end{aligned} \tag{5.2}$$

The mild solution of (5.2) satisfies

$$v(t)=e^{(A_2-\lambda_1)t}v(0)+\int_0^t e^{(A_2-\lambda_1)(t-s)}Pv(s)ds.$$

Applying P to both sides of this equation, we obtain

$$Pv(t)=Pe^{(A_2-\lambda_1)t}v(0)+P\int_0^t e^{(A_2-\lambda_1)(t-s)}Pv(s)ds,$$

where $Pv(t)$ gives the prevalence of newly horizontally transmitted individuals. This equation of $Pv(t)$ describes the renewal process for the horizontally transmitted population, but it also takes into account the effect of vertical transmission through the boundary condition of the generator A_2.

Because $\int_0^\infty m_2(a)\ell(a)e^{-\lambda_1 a}da<1$, the next generation operator for the horizontal transmission in the normalized system at $w=w_1$ is calculated as

$$\mathcal{K}_2=P\int_0^\infty e^{(A_2-\lambda_1)t}dt=P(\lambda_1-A_2)^{-1}, \tag{5.3}$$

It is easy to see that $\mathcal{K}_2=K_2(\lambda_1)$, which may also be written as

$$(\mathcal{K}_2\phi)(a)=\int_0^\infty k(a,b)\phi(b)db \tag{5.4}$$

with the kernel

$$k(a,b)=c_1(a)\int_b^\infty \beta(a,\sigma)\frac{\ell(\sigma)}{\ell(b)}e^{-\lambda_1(\sigma-b)}d\sigma+\frac{c_1(a)\int_0^\infty \beta(a,\sigma)\ell(\sigma)e^{-\lambda_1\sigma}d\sigma}{1-\int_0^\infty m_2(\sigma)\ell(\sigma)e^{-\lambda_1\sigma}d\sigma}\int_b^\infty m_2(\sigma)e^{-\lambda_1(\sigma-b)}\frac{\ell(\sigma)}{\ell(b)}d\sigma.$$

$P_\sigma(A_2+P)\cap\Lambda=\{z\in\Lambda\setminus\Pi_2 : 1\in P_\sigma(K_2(z))\}$ has the dominant eigenvalue λ_d such that $\lambda_d\in(\lambda_2,\lambda_1)$ if $\mathbb{R}_2<1$ and $\lambda_d\in(\lambda_1,\infty)$ if $\mathbb{R}_2>1$. Therefore the Malthusian parameter of the newly horizontally transmitted population prevalence Pv is positive if $\mathbb{R}_2=r(\mathcal{K}_2)>1$, while it is negative if $\mathbb{R}_2=r(\mathcal{K}_2)<1$. So $\mathbb{R}_2=r(K_2(\lambda_1))=r(\mathcal{K}_2)$ gives the reproduction number of the normalized low-fertility population prevalence. Notice that $\mathbb{R}_2$ is an increasing function of $\beta(\cdot,\cdot)$ and that $\mathbb{R}_2\to 0$ if $|\beta|_\infty\to 0$. When all coefficients are age-independent, one gets

$$k(a,b)=\frac{\beta m_1}{m_1-m_2}e^{-m_1 a},\quad \mathbb{R}_2=\int_0^\infty k(a,a)\,da=\frac{\beta}{m_1-m_2},$$

which is given in section 2 because $\lambda_j=m_j-\mu$.

Moreover, the invasion condition $\mathbb{R}_2>1$ is one of the conditions for the existence of non-trivial exponential solutions (endemicity of the low-fertility population). In fact, we can show that the following.

Proposition 8. *Suppose that $\mathcal{K}_2$ and $G'[0]$ are nonsupporting compact positive operators. Then it follows that $r(G'[0])=r(\mathcal{K}_2)=\mathbb{R}_2$.*

Therefore, instability of the stable growth orbit for a high fertility population suggests the existence of non-trivial (coexistent) exponential solutions.

6 Stability of the boundary exponential solution with low fertility

We next consider the stability of the boundary exponential solution for a low-fertility population, which corresponds to the steady state $w^* = w_2 = (0, c_2)$ of the normalized system. In this case, the linearized system $d\zeta(t)/dt = (A + F'[w^*])\zeta(t)$ can be reduced to the following system:

$$\frac{\partial \zeta_1(t,a)}{\partial t} + \frac{\zeta_1(t,a)}{\partial a} = -\mu(a)\zeta_1(t,a) - \epsilon(a)\zeta_1(t,a),$$
$$\frac{\partial \zeta_2(t,a)}{\partial t} + \frac{\zeta_2(t,a)}{\partial a} = -\mu(a)\zeta_2(t,a) + \epsilon(a)\zeta_1(t,a),$$
$$\zeta_1(t,0) = \int_0^\infty m_1(a)\zeta_1(t,a)da, \quad \zeta_2(t,0) = \int_0^\infty m_2(a)\zeta_2(t,a)da, \tag{6.1}$$

where $\epsilon(a) := \int_0^\infty \beta(a,\sigma)c_2(\sigma)d\sigma$.

System (6.1) is a well-known multistate stable population model [3] that has the trivial exponential solution $e^{\lambda_2 t}(0, c_2(a))$. From the homogeneous stability theory in section 4, to show the local stability of the boundary exponential solution $e^{\lambda_2 t}(0, c_2(a))$, it is sufficient to check that the linearized generator $B_2 := A + F'[w_2]$ has AEG with the intrinsic growth constant λ_2 such that $\lim_{t\to\infty} e^{-\lambda_2 t}T(t) = P$ is a rank-one operator. If λ_2 is the dominant, simple characteristic root of the multistate system (6.1), this strong ergodicity result has already been shown using multistate stable population theory [3].

We define the demographic reproduction number R_d by the spectral radius of the net reproduction matrix $\mathcal{K} := \int_0^\infty \Psi(a)da$, where $\Psi(a) := M(a)L(a)$, M is a fertility matrix given by

$$M(a) := \begin{pmatrix} m_1(a) & 0 \\ 0 & m_2(a) \end{pmatrix},$$

and $L(a)$ is the survival matrix given by the solution matrix of $L'(a) = Q(a)L(a)$, with $L(0) = I$ (I denotes the identity matrix) and

$$Q(a) := \begin{pmatrix} -\mu(a) - \epsilon(a) & 0 \\ \epsilon(a) & -\mu(a) \end{pmatrix}.$$

Then the characteristic equation is

$$\left(I - \int_0^\infty e^{-\lambda a}\Psi(a)da\right) = \left(1 - \int_0^\infty e^{-\lambda a} m_1(a)\ell(a)e^{-\int_0^a \epsilon(z)dz}da\right)\left(1 - \int_0^\infty e^{-\lambda a} m_2(a)\ell(a)da\right).$$

Let

$$\Sigma_{L,2} := \left\{\lambda \in \mathbb{C} : 1 = \int_0^\infty e^{-\lambda a} m_1(a)\ell(a)e^{-\int_0^a \epsilon(z)dz}da\right\} \cap \Lambda,$$

and let

$$\Sigma_L := \left\{\lambda \in \mathbb{C} : \det\left(I - \int_0^\infty e^{-\lambda a}\Psi(a)da\right) = 0\right\} = (\Pi_2 \cap \Lambda) \cup \Sigma_{L,2},$$

be the set of characteristic roots of the multistate stable population model (6.1). Then the intrinsic growth rate λ_d (Malthusian parameter) of the multistate stable model is given by the dominant characteristic root of Σ_L, and the sign relation $\text{sign}(\lambda_d) = \text{sign}(R_d - 1)$ holds.

From the standard argument for Lotka's characteristic equation, there exists a dominant real root $\lambda_3 \in \Sigma_{L,2}$ such that $\Re z < \lambda_3$ for any $z \in \Sigma_{L,2} \setminus \{\lambda_3\}$.

Lemma 6.1. *Suppose that $\lambda_3 > -\underline{\mu}$. It follows that $\lambda_3 \in [\lambda_1 - |\beta|_\infty, \lambda_1)$ and $\lambda_d = \max\{\lambda_3, \lambda_2\}$.*

It is not difficult to show that $\sigma(B_2) \cap \Lambda = P_\sigma(B_2) \cap \Lambda = \Sigma_L \cap \Lambda$. Define the invasion index for the high-fertility population $\mathbb{R}_1$ by

$$\mathbb{R}_1 := \int_0^\infty m_1(a)\ell(a)e^{-\lambda_2 a - \int_0^a \epsilon(\sigma)d\sigma}da. \tag{6.2}$$

In fact, now the susceptible individuals are invaders into the completely low-fertility population, so its linearized (prevalence) dynamics is given by

$$\begin{aligned} \frac{\partial \zeta_1(t,a)}{\partial t} + \frac{\zeta_1(t,a)}{\partial a} &= -(\lambda_2 + \mu(a))\zeta_1(t,a) - \epsilon(a)\zeta_1(t,a), \\ \zeta_1(t,0) &= \int_0^\infty m_1(a)\zeta_1(t,a)da, \end{aligned} \tag{6.3}$$

Since (6.3) is a well-known stable population model, its basic reproduction number is given by (6.2). Notice that $\mathbb{R}_1$ is a decreasing function of β. The characteristic roots of (6.3) is given by

$$\left\{\lambda \in \mathbb{C} : 1 = \int_0^\infty e^{-(\lambda+\lambda_2)a} m_1(a)\ell(a)e^{-\int_0^a \epsilon(z)dz} da\right\},$$

then its Malthusian parameter is given by $\lambda_3 - \lambda_2$. Then it is positive if $\mathbb{R}_1 > 1$, while it is negative if $\mathbb{R}_1 < 1$.

Then we have the following result.

Proposition 9. *If $\mathbb{R}_1 < 1$, then the low-fertility exponential solution $e^{\lambda_2 t}w_2$ is locally stable, while it is unstable if $\mathbb{R}_1 > 1$ in the sense of Hadeler.*

It is remarked that the condition $\lambda_1 - \lambda_2 > |\beta|_\infty$ is sufficient for $\mathbb{R}_1 > 1$. In fact, if $\lambda_1 - \lambda_2 > |\beta|_\infty$, then

$$\mathbb{R}_1 \geq \int_0^\infty m_1(a)\ell(a)e^{-(\lambda_2+|\beta|_\infty)a}da > \int_0^\infty m_1(a)\ell(a)e^{-\lambda_1 a}da = 1.$$

Therefore, if the difference between the birth rates for the high fertility and low fertility populations is large enough under common mortality, then the stable population with low fertility can be destabilized by the invasion of high-fertility individuals and simultaneous exponential growth could be realized, although the uniqueness and stability for the coexistence exponential solution is an open problem.

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