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A Lyapunov Function for Constant Equilibria to the Deneubourg Chemotaxis System

シロアリ造巣の走化性モデルに対する空間一様解の大域安定性

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1. Introduction

We study the following three-components chemotaxis system with a forcing term:

$$
\text{(E)}\qquad
\begin{cases}
\dfrac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u) & \text{in } \Omega \times (0,\infty), \\
\delta \dfrac{\partial v}{\partial t} = \Delta v - v + w & \text{in } \Omega \times (0,\infty), \\
\tau \dfrac{\partial w}{\partial t} = -w + u & \text{in } \Omega \times (0,\infty), \\
\dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\
u(x,0) = u_0(x) \geq 0, \quad v(x,0) = v_0(x) \geq 0, \quad w(x,0) = w_0(x) \geq 0 & \text{in } \Omega.
\end{cases}
$$

Here, $\Omega \subset \mathbb{R}^2$ is a two-dimensional bounded domain with smooth boundary $\partial\Omega$. The system (E) was presented by Deneubourg [3] (see also [2, 7]) for modeling the self-organized nest construction process of social insects, specifically, termites. The unknown functions $u(x,t)$, $v(x,t)$, and $w(x,t)$ are the densities of, respectively, worker insects (hereafter workers), nest building material, and a chemical substance at position x and time t. The coefficient χ is a positive constant which indicates the intensity of chemotaxis. The function $f(u)$ consists of the migration into the working area and the resting of workers. The first term of the second equation and the second term of the third equation represent the weathering of deposited materials and the decay of chemical substance, respectively. The coefficients $\delta > 0$ and $\tau > 0$ are the time-scale constants of the reactions in the respective equations.

Deneubourg [3] defined function f as

$$
f(u) = 1 - \mu u, \quad u \geq 0, \tag{1.1}
$$

where μ is a positive constant. Here the migration rate of workers is normalized to 1, and μ denotes the resting rate of workers. We adopt this same linear decaying function (1.1) for the system (E) in the present work. Firstly, we have:

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Theorem 1.1. *If $\chi\cdot\max\{\|u_0\|_{L_1}, |\Omega|/\mu\}^{2/3}$ is sufficiently small, then, for each triplet of nonnegative initial functions $(u_0, v_0, w_0) \in H_N^2(\Omega) \times H^2(\Omega) \times H_N^3(\Omega)$, the system* (E) *admits a unique global-in-time solution (u, v, w) in the function space*

$$\begin{cases} u \in \mathcal{C}^1((0,\infty); H^1(\Omega)) \cap \mathcal{C}([0,\infty); H_N^2(\Omega)) \cap \mathcal{C}((0,\infty); H_N^3(\Omega)), \\ v \in \mathcal{C}^1((0,\infty); H^2(\Omega)) \cap \mathcal{C}([0,\infty); H^2(\Omega)), \\ w \in \mathcal{C}^1((0,\infty); H_N^2(\Omega)) \cap \mathcal{C}([0,\infty); H_N^3(\Omega)) \cap \mathcal{C}((0,\infty); H_{N^2}^4(\Omega)). \end{cases} \tag{1.2}$$

The solution satisfies the uniform estimate by the norm of initial functions such that

$$\|u(t)\|_{H^2} + \|v(t)\|_{H^2} + \|w(t)\|_{H^3} \le \psi\left(\|u_0\|_{H^2} + \|v_0\|_{H^2} + \|w_0\|_{H^3}\right), \quad t \ge 0, \tag{1.3}$$

for some increasing function $\psi(\cdot)$. In addition, the mapping $(u_0, v_0, w_0) \mapsto (u(t), v(t), w(t))$ is continuous in $H_N^2(\Omega) \times H^2(\Omega) \times H_N^3(\Omega)$.

Secondly, we examine the asymptotic behavior of the global solutions by defining the dynamical system. Uniform estimates for the global solutions derive an absorbing set in the universal space $\mathcal{H} = H^1(\Omega) \times H^2(\Omega) \times H_N^2(\Omega)$. From this, we also construct an invariant set $\mathcal{X}$ for the dynamical system in the universal space $\mathcal{H}$. Under an additional condition, we then construct a global Lyapunov functional for the constant equilibrium:

$$U^* = {}^T[u^*\, v^*\, w^*] := {}^T\left[\frac{1}{\mu}\ \frac{1}{\mu}\ \frac{1}{\mu}\right]. \tag{1.4}$$

For the two-component chemotaxis system with quadratic degradation (the case of $\delta = 0$ and $f(u) = u(1-\mu u)$ in (E)), He and Zheng [5] constructed a Lyapunov functional for constant equilibrium under the condition $\mu > \chi/4$. This is optimal in the sense that the destabilization of the homogeneous state can occur if the opposite inequality $\mu < \chi/4$ holds by taking a suitable spatial domain. He and Zheng [5] also extended the result to a three-dimensional bounded smooth domain with μ sufficiently large (the three-dimensional case requires a large μ for global existence [9]; for the convergence of solutions to the constant equilibrium, see also [4, 10]). On the other hand, for the first equation of (E) with an α-th degradation, $f(u) = u^{\alpha-1}(1-\mu u)$, the same procedure as [5] derives the result that

$$\frac{d}{dt}\int_\Omega [\mu u - 1 - \log(\mu u)]\, dx = -\int_\Omega \frac{|\nabla u|^2}{u^2}\, dx + \chi \int_\Omega \frac{\nabla u \cdot \nabla w}{u}\, dx - \mu^2 \int_\Omega \frac{1}{u^{2-\alpha}}(u - u^*)^2\, dx.$$

This shows that although quadratic degradation $\alpha = 2$ introduces an L_2 absorbing term $-\mu^2\|u - u^*\|_{L_2}^2$, the linear degradation $\alpha = 1$ in (1.1) only introduces an L_1 absorbing $-\mu^2\|u\|_{L_1}$, where $u^* = 1/\mu$ is the first component of the equilibrium. To overcome this difficulty, we use the uniform boundedness of the maximum norm of $u(t)$ in the eventual invariant set $\mathcal{X}$, included in the ball with radius r, that is, $\|u(t)\|_{\mathcal{C}} \le r$, $(u(t), v(t), w(t)) \in \mathcal{X}$. We can then construct an L_2 absorbing term $-(\mu^2/r)\|u - u^*\|_{L_2}^2$, which shows the Lyapunov functional is monotone decreasing (Section 3).

Notation. Let $A = -\Delta + 1$, Δ be the Laplace operator in $L_2(\Omega)$ with the Neumann boundary condition, the domain of which is $H_N^2(\Omega)$ with the norm equivalence

$$\|w\|_{H^2} \le C\|(-\Delta + 1)w\|_{L_2} \le C(\|\Delta w\|_{L_2} + \|w\|_{L_2}) \quad \text{for } w \in H_N^2(\Omega). \tag{1.5}$$

Then, the domains of the fractional powers of A are characterized by

$$\mathcal{D}(A^\theta) = \begin{cases} H^{2\theta}(\Omega) & \text{for } 0 \le \theta < \frac{3}{4}, \\ H^{2\theta}_N(\Omega) & \text{for } \frac{3}{4} < \theta < \frac{7}{4}, \\ H^{2\theta}_{N^2}(\Omega) & \text{for } \frac{7}{4} < \theta \le \frac{5}{2}, \end{cases} \tag{1.6}$$

with norm equivalence. Here, $H^s_N(\Omega)$ for $s > 3/2$ and $H^s_{N^2}(\Omega)$ for $s > 7/2$ denote closed subspaces of $H^s(\Omega)$ such that

$$H^s_N(\Omega) = \left\{ w \in H^s(\Omega);\ \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \right\} \quad \text{for } s > \frac{3}{2},$$

$$H^s_{N^2}(\Omega) = \left\{ w \in H^2_N(\Omega);\ \Delta w \in H^{s-2}_N(\Omega) \right\} \quad \text{for } s > \frac{7}{2}.$$

See [11, Theorems 2.8, 2.9 and 16.7].

2. A priori estimates and global-in-time solutions

In this section, we construct several a priori estimates. Let $m(\varphi)$ be the average value over Ω for the integral of L_1-function $\varphi \in L_1(\Omega)$:

$$m(\varphi) := \frac{1}{|\Omega|} \int_\Omega |\varphi(x)|\, dx, \quad \varphi \in L_1(\Omega),$$

where $|\Omega|$ is the measure of domain Ω.

Proposition 2.1. *Let (u, v, w) be a local solution to* (E). *Then, it holds that*

$$\|u(t)\|_{L_1} = \int_\Omega u(x,t)\, dx = e^{-\mu t} \left(\|u_0\|_{L_1} - \frac{|\Omega|}{\mu} \right) + \frac{|\Omega|}{\mu}. \tag{2.1}$$

Proof. Integrating the first equation of (E) over Ω, we have

$$\frac{d}{dt} \int_\Omega u\, dx = \int_\Omega f(u)\, dx = |\Omega| - \mu \int_\Omega u\, dx. \tag{2.2}$$

By solving this in $\|u(t)\|_{L_1}$, we obtain (2.1). □

As a corollary, we have

Corollary 2.2. *Let K_0 be the supremum of $\|u(t)\|_{L_1}$. Then, it holds that*

$$K_0 := \sup_{t \ge 0} \|u(t)\|_{L_1} = \max\left\{ \|u_0\|_{L_1}, \frac{|\Omega|}{\mu} \right\} = |\Omega| \cdot \max\left\{ m(u_0), \frac{1}{\mu} \right\}. \tag{2.3}$$

Proposition 2.3. *Let (u, v, w) be a local solution to* (E) *and assume the smallness condition* (2.8) *below, or equivalently* (2.9). *Then, it holds that*

$$\begin{aligned} &N^1_{\log}(u(t)) + \frac{\chi\delta}{2} \|v(t)\|^2_{L_2} + \frac{\chi\tau}{4} \|w(t)\|^2_{H^1} \\ &\le e^{-d_1 t} \left(N^1_{\log}(u_0) + \frac{\chi\delta}{2} \|v_0\|^2_{L_2} + \frac{\chi\tau}{4} \|w_0\|^2_{H^1} \right) + 5 \int_0^t e^{-d_1(t-s)} (\, \|u(s)\|_{L_1} + |\Omega|\,)\, ds \\ &\le \psi \left(N^1_{\log}(u_0) + \|v_0\|_{L_2} + \|w_0\|_{H^1} \right), \end{aligned} \tag{2.4}$$

where $d_1 = \min\{\mu, 1/(2\delta), 1/\tau\}$ and $\psi(\cdot)$ is some increasing continuous function.

Proof. Multiplying the first equation of (E) by $\log(u+1)$ and integrating over Ω, we have

$$\frac{d}{dt}\int_\Omega [(u+1)\log(u+1)-u]dx \le -\int_\Omega \frac{|\nabla u|^2}{u+1}dx + \chi\int_\Omega u|\Delta w|dx + \int_\Omega (1-\mu u)\log(u+1)dx \tag{2.5}$$
$$\le -\int_\Omega \frac{|\nabla u|^2}{u+1}dx + \chi\int_\Omega u^2dx + \frac{\chi}{4}\int_\Omega |\Delta w|^2dx + \int_\Omega (1-\mu u)\log(u+1)dx.$$

Next, multiplying the second equation of (E) by v and integrating over Ω, we have

$$\frac{\delta}{2}\frac{d}{dt}\int_\Omega v^2dx \le -\frac{1}{2}\int_\Omega v^2dx + \frac{1}{2}\int_\Omega u^2dx. \tag{2.6}$$

Thirdly, multiplying the third equation of (E) by $-\Delta w + w$ and integrating over Ω, we have

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega (w^2+|\nabla w|^2)dx \le -\frac{1}{2}\int_\Omega (w^2+|\nabla w|^2)dx - \frac{1}{2}\int_\Omega |\Delta w|^2dx + \int_\Omega v^2dx. \tag{2.7}$$

Adding (2.6) multiplied by 2χ, (2.7) multiplied by $\chi/2$, and (2.2) to (2.5), we then obtain

$$\frac{d}{dt}\left[N^1_{\log}(u) + \chi\delta\|v\|^2_{L_2} + \frac{\chi\tau}{4}\|w\|^2_{H^1}\right] \le -\int_\Omega \frac{|\nabla u|^2}{u+1}dx + 2\chi\|u\|^2_{L_2} + \int_\Omega (1-\mu u)\log(u+1)dx$$
$$+\int_\Omega (1-\mu u)dx - \frac{\chi}{2}\|v\|^2_{L_2} - \frac{\chi}{4}\|w\|^2_{H^1}.$$

Here, we have

$$\|u\|^2_{L_2} \le \|u\|^{\frac{2}{3}}_{L_1}\|u\|^{\frac{4}{3}}_{L_4} \le \|u\|^{\frac{2}{3}}_{L_1}\|\sqrt{1+u}\|^{\frac{8}{3}}_{L_8} \le \|u\|^{\frac{2}{3}}_{L_1}\cdot C_G\|\sqrt{1+u}\|^2_{H^1}\|\sqrt{1+u}\|^{\frac{2}{3}}_{L_2}$$
$$\le C_G\|u\|^{\frac{2}{3}}_{L_1}\|1+u\|^{\frac{1}{3}}_{L_1}\cdot\left(\|\nabla\sqrt{1+u}\|^2_{L_2} + \|1+u\|_{L_1}\right)$$
$$\le C_G\|u\|^{\frac{2}{3}}_{L_1}\|1+u\|^{\frac{1}{3}}_{L_1}\cdot\left(\frac{1}{4}\int_\Omega \frac{|\nabla u|^2}{1+u}dx + \|1+u\|_{L_1}\right),$$

where $C_G = (C_{2,8})^{\frac{8}{3}}$ with an embedding constant $C_{2,8}$ of the Gagliardo-Nirenberg inequality: $\|\varphi\|_{L_8} \le C_{2,8}\|\varphi\|^{\frac{3}{4}}_{H^1}\|\varphi\|^{\frac{1}{4}}_{L_2}$, $\varphi \in H^1(\Omega)$ (e.g., see [11, p.424]). This leads to

$$\frac{d}{dt}\left[N^1_{\log}(u) + \chi\delta\|v\|^2_{L_2} + \frac{\chi\tau}{4}\|w\|^2_{H^1}\right]$$
$$\le -\int_\Omega \frac{|\nabla u|^2}{u+1}dx + 2\chi C_G K_0^{\frac{2}{3}}(|\Omega|+K_0)^{\frac{1}{3}}\left(\frac{1}{4}\int_\Omega \frac{|\nabla u|^2}{1+u}dx + \|1+u\|_{L_1}\right)$$
$$+\int_\Omega (1-\mu u)\log(u+1)dx + \int_\Omega (1-\mu u)dx - \frac{\chi}{2}\|v\|^2_{L_2} - \frac{\chi}{4}\|w\|^2_{H^1}.$$

Therefore, by choosing χ, $\|u_0\|_{L_1}$ and $1/\mu$ sufficiently small as

$$\zeta = \frac{C_G}{2}\chi K_0^{\frac{2}{3}}(|\Omega|+K_0)^{\frac{1}{3}} = \frac{C_G}{2}\chi\cdot\max\left\{\|u_0\|_{L_1}, \frac{|\Omega|}{\mu}\right\}^{\frac{2}{3}}\left[|\Omega| + \max\left\{\|u_0\|_{L_1}, \frac{|\Omega|}{\mu}\right\}\right]^{\frac{1}{3}} \tag{2.8}$$
$$= \frac{C_G|\Omega|}{2}\chi\cdot\max\left\{m(u_0), \frac{1}{\mu}\right\}^{\frac{2}{3}}\left[1+\max\left\{m(u_0), \frac{1}{\mu}\right\}\right]^{\frac{1}{3}} < 1,$$

we have (noting that $\zeta < 1$)

$$
\begin{aligned}
\frac{d}{dt}&\left[N^1_{\log}(u) + \chi\delta\|v\|^2_{L_2} + \frac{\chi\tau}{4}\|w\|^2_{H^1}\right] \\
&\le 4\zeta\|1+u\|_{L_1} + \int_\Omega (1-\mu u)\log(u+1)dx + \int_\Omega (1-\mu u)dx - \frac{\chi}{2}\|v\|^2_{L_2} - \frac{\chi}{4}\|w\|^2_{H^1} \\
&\le 5\int_\Omega (1+u)dx - \mu N^1_{\log}(u) - \frac{\chi}{2}\|v\|^2_{L_2} - \frac{\chi}{4}\|w\|^2_{H^1} \\
&\le 5(|\Omega| + \|u\|_{L_1}) - d_1\left[N^1_{\log}(u) + \chi\delta\|v\|^2_{L_2} + \frac{\chi\tau}{4}\|w\|^2_{H^1}\right]
\end{aligned}
$$

with $d_1 = \min\{\mu, 1/(2\delta), 1/\tau\}$. By solving this differential inequality and using the equality (2.1), we obtain the estimate (2.4). □

Remark 2.4. *The smallness condition* (2.8) *has another equivalent expression for the chemotactic intensity* χ :

$$
(2.9)\quad \chi < \frac{2}{C_G \max\left\{\|u_0\|_{L_1}, \frac{|\Omega|}{\mu}\right\}^{\frac{2}{3}}\left[|\Omega| + \max\left\{\|u_0\|_{L_1}, \frac{|\Omega|}{\mu}\right\}\right]^{\frac{1}{3}}} = \frac{2\mu}{C_G|\Omega| \max\{\mu m(u_0), 1\}^{\frac{2}{3}}\left[\mu + \max\{\mu m(u_0), 1\}\right]^{\frac{1}{3}}}.
$$

Here, the constant C_G is bounded below from $|\Omega|^{-1}$, as we see in [12, Corollary22]. Hence, the measure $|\Omega|$ of the domain Ω cannot be a control parameter for reducing the value $C_G|\Omega|$ to 0, which implies a wider valid region of (μ, χ) contained in $\mathbb{R}^2_+$ of (2.9).

Similarly, we obtain the following three propositions (see [12, Propositions 9, 10 and 12]).

Proposition 2.5. *Let (u, v, w) be a local solution to* (E). *Then, under the smallness condition* (2.8), *it holds that*

$$
\begin{aligned}
(2.10)\quad &\|u(t)\|^2_{L_2} + \delta\|v(t)\|^2_{H^1} + \frac{\tau}{4}\|(-\Delta+1)w(t)\|^2_{L_2} \\
&\le e^{-d_2 t}\left(\|u_0\|^2_{L_2} + \delta\|v_0\|^2_{H^1} + \frac{\tau}{4}\|(-\Delta+1)w_0\|^2_{L_2}\right) + \int_0^t e^{-d_2(t-s)}\psi(\,N^1_{\log}(u(s)) + \|w(s)\|_{H^1})\,ds \\
&\le \psi\left(\|u_0\|_{L_2} + \|v_0\|_{H^1} + \|w_0\|_{H^2}\right),
\end{aligned}
$$

where $d_2 = \min\{2\mu, 1/(2\delta), 1/\tau\}$ and $\psi(\cdot)$ is some increasing continuous function.

Proposition 2.6. *Let (u, v, w) be a local solution to* (E). *Then, under the smallness condition* (2.8), *it holds that*

$$
\begin{aligned}
(2.11)\quad \|\nabla u\|^2_{L_2} &\le e^{-d_3 t}\|\nabla u_0\|^2_{L_2} + \int_0^t e^{-d_3(t-s)}\psi(\,\|u(s)\|_{L_2} + \|w(s)\|_{H^2})\,ds \\
&\le \psi\left(\,\|u_0\|_{H^1} + \|v_0\|_{H^1} + \|w_0\|_{H^2}\right),
\end{aligned}
$$

where $d_3 = 2\mu$ and $\psi(\cdot)$ is some increasing continuous function.

Proposition 2.7. *Let (u, v, w) be a local solution to* (E). *Then, under the smallness condition* (2.8), *it holds that*

$$
\begin{aligned}
(2.12)\quad & \|u\|_{H^2}^2 + \delta\|v\|_{H^2}^2 + \tau\|w\|_{H^3}^2 \\
&\le e^{-d_5 t}(\|u_0\|_{H^2}^2 + \delta\|v_0\|_{H^2}^2 + \tau\|w_0\|_{H^3}^2) + \int_0^t e^{-d_5(t-s)}\psi(\|u(s)\|_{H^1} + \|w(s)\|_{H^2})\,ds \\
&\le \psi(\|u_0\|_{H^2} + \|v_0\|_{H^2} + \|w_0\|_{H^3}),
\end{aligned}
$$

where $d_5 = \min\{2\mu, 1/(2\delta), 2/\tau\}$ *and* $\psi(\cdot)$ *is some increasing continuous function.*

3. Global attractors and Lyapunov function for uniform equilibrium

Throughout this section, we assume the smallness condition (2.8). We here examine the asymptotic behavior of the global-in-time solutions constructed in Theorem 1.1. Let

$$\mathcal{H} = H^1(\Omega) \times H^2(\Omega) \times H_N^2(\Omega)$$

be the universal space of a dynamical system. We set the initial function space with the smallness condition (2.8), such as

$$
\begin{aligned}
\mathcal{K} = \{(u, v, w) \in H_N^2(\Omega) \times H^2(\Omega) \times H_N^3(\Omega);\ u, v, w > 0,\ u \text{ satisfies } (2.8)\}, \\
\|U\|_{\mathcal{K}} = \|u\|_{H^2} + \|v\|_{H^2} + \|w\|_{H^3}, \quad U = {}^T[u\ v\ w].
\end{aligned}
$$

An additional positivity condition is used in the proof of Theorem 3.4. Theorem 1.1 with the strong comparison principle defines a continuous semigroup of the solution operator $S(t) : \mathcal{K} \to \mathcal{K}$. Let us consider the dynamical system $(S(t), \mathcal{K}, \mathcal{H})$ hereafter.

Combining the a priori estimates in Section 2, we can construct an absorbing set $\mathcal{B}$ for the dynamical system $(S(t), \mathcal{K}, \mathcal{H})$, where we say that a set $\mathcal{B} \subset \mathcal{H}$ is absorbing if, for every bounded set $B \subset \mathcal{K}$, there exists a time t_B such that $\bigcup_{t \ge t_B} S(t)B \subset \mathcal{B}$.

Theorem 3.1. *A ball $\mathcal{B}$ in $\mathcal{K}$ with sufficiently large radius r :*

$$
\begin{aligned}
\mathcal{B} = \{(u, v, w) \in H_N^2(\Omega) \times H^2(\Omega) \times H_N^3(\Omega);\ \|u\|_{H^2} + \|v\|_{H^2} + \|w\|_{H^3} \le r, \\
u, v, w > 0,\ u \text{ satisfies } (2.8)\} \subset \mathcal{K}
\end{aligned}
$$

is an absorbing set for the dynamical system $(S(t), \mathcal{K}, \mathcal{H})$.

Proof. The theorem can be shown by inductively applying the uniform Gronwall lemma [8, p.91] (or [11, Section 1-10]). For the above a priori estimates, however, the absorbing set $\mathcal{B}$ can be constructed by using the following lemma, which is simplified for the estimates. Specifically, applying Lemma 3.3 with $k = 0, 1, 2, 3$ and 4 for (2.1), (2.4), (2.10), (2.11) and (2.12), respectively, proves the theorem. □

Remark 3.2. *The radius r of the absorbing set $\mathcal{B}$ is suitably determined by the above a priori estimates. In particular, r is of order $O(1)$ for large μ.*

Lemma 3.3. *Let $f_k \in C([0,\infty);\mathbb{R})$, $k = 0,1,2,\dots,n$, be nonnegative continuous functions. Assume that the following inequalities hold for each $k = 1,2,\dots,n$:*

$$f_k(t) \le e^{-d_k t} f_k(0) + \int_0^t e^{-d_k(t-s)} \varphi_k(f_{k-1}(s))\, ds, \quad t \ge 0,$$

with a positive constant $d_k > 0$ and an increasing continuous function $\varphi_k(\cdot)$. Assume also a uniform estimate for $k = 0$ such that $f_0(t) \le r_0$ for all $t \ge t_0$ with a positive constant $r_0 > 0$ and a time $t_0 > 0$. Then, there exist a positive constant $r > 0$ and a time $t_* > 0$ such that $f_k(t) \le r$ for all $t \ge t_*$, uniformly in k.

We then construct a global Lyapunov functional for the unique constant equilibrium with suitably large μ. In the construction of the Lyapunov functional, the uniform boundedness of the maximum norm of u plays a crucial role. Let us introduce a positively invariant set

$$\mathcal{X} = \bigcup_{t \ge t_{\mathcal{B}}} S(t)\mathcal{B} \subset \mathcal{B}.$$

The asymptotic behavior of the solutions thereby reduces to the eventual dynamical system $(S(t), \mathcal{X}, \mathcal{H})$. From the existence of the absorbing set $\mathcal{B}$ (Theorem 3.1), there exists a uniform constant M_r for $\|u(t)\|_C$, of order $O(1)$ in large μ, that is,

$$\|u(t)\|_C \le C\|U(t)\|_{H^2\times H^2\times H^3} \le C\cdot r := M_r \text{ for all } U(t) = {}^T[u(t)\, v(t)\, w(t)] \in \mathcal{X}, \tag{3.1}$$

where C is a constant for the embedding inequality $\|u\|_C \le C\|u\|_{H^2}$, and r is the radius of absorbing set $\mathcal{B}$. We then show the following:

Theorem 3.4. *Assume another largeness condition for μ: $\mu > \chi\sqrt{M_r}/4$, where M_r is a constant in* (3.1). *Then, a functional*

$$\Phi(U(t)) = \int_\Omega \left[\mu u - 1 - \log \mu u + \frac{\delta\mu^2}{M_r}(v - v^*)^2 + \frac{\tau\chi^2}{8}(w - w^*)^2\right] dx$$

satisfies $\frac{d}{dt}\Phi(U(t)) \le 0$, $\Phi(U) > 0$ $(U \ne U^*)$, and $\Phi(U^*) = 0$, that is, Φ is a Lyapunov functional for the trivial fixed point U^* of the dynamical system $(S(t), \mathcal{X}, \mathcal{H})$.

Remark 3.5. *Because $M_r = O(1)$ for sufficiently large μ, the region of (χ, μ) contained in $\mathbb{R}_+^2$ satisfying the inequality $\mu > \chi\sqrt{M_r}/4$ is non-empty.*

Proof. It is clear that $\Phi(U) > 0$ $(U \ne U^*)$ and $\Phi(U^*) = 0$. We can show $\frac{d}{dt}\Phi(U(t)) \le 0$ in a similar manner to [5, 6] except for the need to construct an L_2 absorbing $-\|u - u^*\|_{L_2}^2$. By noting $\|u(t)\|_C \le M_r$, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (\mu u - \log \mu u)\, dx &= \int_\Omega \left(\mu - \frac{1}{u}\right)[\nabla\cdot(\nabla u - \chi u \nabla w) + (1 - \mu u)]\, dx \\
&= -\int_\Omega \nabla\left(\mu - \frac{1}{u}\right)\cdot(\nabla u - \chi u \nabla w)\, dx - \int_\Omega \frac{1}{u}(1 - \mu u)^2\, dx \\
&\le -\int_\Omega \frac{|\nabla u|^2}{u^2}\, dx + \chi\int_\Omega \frac{\nabla u\cdot\nabla w}{u}\, dx - \frac{\mu^2}{M_r}\int_\Omega (u - u^*)^2\, dx.
\end{aligned}$$

Similarly, we have

$$\frac{d}{dt}\int_\Omega \frac{\delta\mu^2}{M_r}(v-v^*)^2\,dx = -\frac{2\mu^2}{M_r}\int_\Omega (v-v^*)^2\,dx + \frac{2\mu^2}{M_r}\int_\Omega (v-v^*)(u-u^*)\,dx,$$

and also

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{\tau\chi^2}{8}(w-w^*)^2\,dx &= \int_\Omega (w-w^*)(\tau w_t)\,dx = \int_\Omega (w-w^*)(\Delta w + v - w)\,dx \\ &= -\frac{\chi^2}{4}\int_\Omega |\nabla w|^2\,dx + \frac{\chi^2}{4}\int_\Omega (v-v^*)(w-w^*)\,dx - \frac{\chi^2}{4}\int_\Omega (w-w^*)^2\,dx.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{d}{dt}\varPhi(U(t)) &= \frac{d}{dt}\int_\Omega \left[\mu u - 1 - \log \mu u + \delta\mu^2(v-v^*)^2 + \frac{\tau\chi^2}{8}(w-w^*)^2\right] dx \\ &= -\int_\Omega \frac{|\nabla u|^2}{u^2}\,dx + \chi\int_\Omega \frac{\nabla u\cdot\nabla w}{u}\,dx - \frac{\chi^2}{4}\int_\Omega |\nabla w|^2\,dx \\ &\quad - \frac{\mu^2}{M_r}\int_\Omega (u-u^*)^2\,dx + \frac{2\mu^2}{M_r}\int_\Omega (v-v^*)(u-u^*)\,dx - \frac{2\mu^2}{M_r}\int_\Omega (v-v^*)^2\,dx \\ &\quad - \frac{\chi^2}{4}\int_\Omega (w-w^*)^2\,dx + \frac{\chi^2}{4}\int_\Omega (v-v^*)(w-w^*)\,dx \\ &= -\int_\Omega \left|\frac{\nabla u}{u} - \frac{\chi}{2}\nabla w\right|^2 dx - \frac{\mu^2}{M_r}\int_\Omega [(u-u^*)-(v-v^*)]^2\,dx \\ &\quad - \frac{\mu^2}{M_r}\int_\Omega \left[(v-v^*) - \frac{\chi^2 M_r}{8\mu^2}(w-w^*)\right]^2 dx \\ &\quad - \frac{\chi^2}{4}\left(1-\frac{\chi^2 M_r}{16\mu^2}\right)\int_\Omega (w-w^*)^2\,dx.\end{aligned}$$

Therefore, we have $\frac{d}{dt}\varPhi(U(t)) \le 0$ under the condition $\mu > \chi\sqrt{M_r}/4$. □

Proposition 3.6. *Under the conditions* (2.8) *and* $\mu > \chi\sqrt{M_r}/4$, *the convergence of* $U(t)$ *to* U^* *in* $\mathcal{K}$ *is uniform:*

$$\|u(t)-u^*\|_{C^1} \to 0,\ \ \|v(t)-v^*\|_C \to 0,\ \ \|w(t)-w^*\|_{C^2} \to 0, \quad t\to\infty.$$

Proof. By referring to, e.g., [1, 5, 6], we can show the convergence. From the proof of Theorem 3.4, we have

$$\frac{d}{dt}\varPhi(U(t)) \le -\eta\int_\Omega \left[[(u-u^*)-(v-v^*)]^2 + \left[(v-v^*) - \frac{\chi^2 M_r}{8\mu^2}(w-w^*)\right]^2 + (w-w^*)^2\right] dx, \tag{3.2}$$

where $\eta := \min\{\frac{\mu^2}{M_r}, \frac{\chi^2}{4}\left(1-\frac{\chi^2 M_R}{16\mu^2}\right)\}$. We set $\varphi(t) := \int_\Omega \left[[(u-u^*)-(v-v^*)]^2 + \left[(v-v^*) - \frac{\chi^2 M_r}{8\mu^2}(w-w^*)\right]^2 + (w-w^*)^2\right] dx$. Then, by integrating (3.2) from 1 to t, we have $\int_1^\infty \varphi(s)\,ds \le \frac{1}{\eta}\Phi(U(1)) < \infty$. The positivity of $\varphi(t)$ indicates that $\varphi(t) \to 0$ $(t\to\infty)$. We then have the convergence to the constant solution U^* in L_2-norm. Since the solution U belongs to the functional space (1.2), the convergence in $\mathcal{K}$ and maximum norms is proved from the Gagliardo-Nirenberg inequality, e.g., $\|u\|_{C^1} \le C\|u\|_{H^{\frac{5}{2}}} \le C\|u\|_{H^3}^{\frac{5}{6}}\|u\|_{L_2}^{\frac{1}{6}}$, $\|v\|_{C} \le C\|v\|_{H^{\frac{3}{2}}} \le C\|v\|_{H^2}^{\frac{3}{4}}\|v\|_{L_2}^{\frac{1}{4}}$, and $\|w\|_{C^2} \le C\|w\|_{H^{\frac{7}{2}}} \le C\|w\|_{H^4}^{\frac{7}{8}}\|w\|_{L_2}^{\frac{1}{8}}$. □

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